A NOTE ON THE GROWTH OF BETTI NUMBERS AND RANKS OF 3-MANIFOLD GROUPS

STEFAN FRIEDL

Abstract. Let N be an irreducible, compact 3-manifold with empty or toroidal boundary which is not a closed graph manifold. We show that it follows from the work of Agol, Kahn-Markovic and Przytycki-Wise that $\pi_1(N)$ admits a cofinal filtration with 'fast' growth of Betti numbers as well as a cofinal filtration of $\pi_1(N)$ with 'slow' growth of ranks.

1. Introduction

A *filtration of a group* π is a sequence $\{\pi_i\}_{i\in\mathbb{N}}$ of finite index subgroups of π such that $\pi_{i+1} \subset \pi_i$ for every i. We say that a filtration is *cofinal* if $\cap_{i\in\mathbb{N}}\pi_i$ is trivial, we call it *normal* if $\pi_i \triangleleft \pi$ for every i, and we say it is *almost normal* if there exists a k such that $\pi_i \triangleleft \pi_k$ for every $i \geq k$. A group which admits a cofinal normal filtration is called *residually finite*.

Given a filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of a group π it is of interest to study how the following measures of 'complexity' grow:

(1) the first Betti number $b_1(\pi_i) = \dim H_1(\pi_i;\mathbb{Q})$,
(2) the $\mathbb{F}_p$-Betti numbers $b_1(\pi_i,\mathbb{F}_p) = \dim H_1(\pi_i;\mathbb{F}_p)$,
(3) the rank $d(\pi_i)$, i.e. the minimal size of a generating set,
(4) the order of Tor $H_1(\pi_i;\mathbb{Z})$.

Such growth functions have been studied for 3-manifold groups by many authors over the years. We refer to [CE10, CW03, De10, EL12, Gi10, GS91, KMT03, La09, La11, Le10, Lü94, LL95, KS12, Ra10, Ri90, ShW92, SiW02a, SiW02b, Wa09] for a sample of results in this direction. It is clear that given any group π we have $d(\pi) \geq b_1(\pi)$, i.e. given a filtration the ranks grow at least as fast as the Betti numbers.

Now let N be a 3-manifold. Throughout this paper we will use the following convention: a 3-manifold will always be assumed to be connected, compact, orientable and irreducible with empty or toroidal boundary. By [He87] the group $\pi_1(N)$ is residually finite. In this paper we are interested in how fast Betti numbers can grow in a cofinal filtration of $\pi_1(N)$ and how slowly the ranks can grow in a cofinal filtration of $\pi_1(N)$.

First note that given any cofinal normal filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of $\pi = \pi_1(N)$ it follows from the work of Lück [Lü94, Theorem 0.1] and Lott and Lück [LL95, Theorem 0.1] that

$$\lim_{i\to\infty} \frac{1}{[\pi:\pi_i]} b_1(\pi_i) = 0, \tag{1.1}$$

i.e. the first Betti number grows sublinearly. The same equality also holds for almost normal cofinal filtrations of $\pi_1(N)$ if we apply the aforementioned results to an appropriate finite cover of N.

Remark. Note that (1.1) does not necessarily hold for cofinal filtrations of $\pi_1(N)$ which are not almost normal. In fact Girão [Gi10] (see proof of [Gi10, Theorem 3.1]) gives an example of a cusped hyperbolic 3-manifold together with a cofinal filtration of $\{\pi_i\}_{i\in\mathbb{N}}$ of $\pi = \pi_1(N)$ such that

$$\lim_{i\to\infty} \frac{1}{[\pi:\pi_i]} b_1(\pi_i) > 0.$$

It is an interesting question how quickly $\frac{1}{[\pi:\pi_i]} b_1(\pi_i)$ converges to zero, and to what degree the convergence depends on the choice of normal cofinal filtration of $\pi = \pi_1(N)$. This question for example was recently studied by Kionke and Schwermer [KS12].

We will use recent work of Agol [Ag12] (which in turn builds on work of Kahn-Markovic [KM12] and Wise [Wi12]) to prove the following theorem which says that 'most' 3-manifolds admit cofinal filtrations with 'fast' sublinear growth of first Betti numbers.

Theorem 1.1. *Let $N \neq S^1\times D^2$ and $N \neq T^2\times I$ be a 3-manifold which is neither spherical nor covered by a torus bundle. Then the following hold:*

(1) *Given any function $f\colon \mathbb{N}\to\mathbb{R}_{\geq 0}$ such that*
$$\lim_{n\to\infty}\frac{f(n)}{n} = 0$$
there exists an almost normal cofinal filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of π such that
$$b_1(\pi_i) \geq f([\pi:\pi_i]) \text{ for every } i\in\mathbb{N}.$$

(2) *There exists a normal cofinal filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of $\pi = \pi_1(N)$ and an $\varepsilon\in(0,1)$ such that*
$$b_1(\pi_i) \geq [\pi:\pi_i]^{\varepsilon} \text{ for every } i\in\mathbb{N}.$$

We now turn to the construction of cofinal filtrations with 'slow' growth of ranks. First note that if H is a finite index subgroup of a finitely generated group G, then it follows from the Reidemeister-Schreier method (see e.g. [MKS76, Corollary 2.7.1]) that

$$d(H) \leq [G:H] \cdot (d(G)-1)+1 \leq [G:H] \cdot d(G).$$

In particular if $\{\pi_i\}_{i\in\mathbb{N}}$ is a cofinal filtration of a group π, then

$$\frac{1}{[\pi:\pi_i]} d(\pi_i) \leq d(\pi) \text{ for every } i.$$

Put differently, the rank grows at most linearly with the degree.

We will again use the recent work of Agol, Kahn-Markovic and Wise together with work of Przytycki-Wise [PW12] to prove the following theorem which says that 'most' 3-manifolds admit cofinal filtrations with 'slow' growth of ranks.

Theorem 1.2. *Let N be a 3-manifold which is not a closed graph manifold.*

(1) *Given any function $f\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ with*

$$\lim_{n\to\infty} f(n) = \infty$$

there exists an almost normal cofinal filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of π such that

$$d(\pi_i) \leq f([\pi:\pi_i]) \text{ for every } i \in \mathbb{N}.$$

(2) *There exists a normal cofinal filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of $\pi_1(N)$ and an $\varepsilon \in (0,1)$ such that*

$$d(\pi_i) \leq [\pi:\pi_i]^{\varepsilon} \text{ for every } i \in \mathbb{N}.$$

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2. Proofs

2.1. **3-manifold groups.**

The world of 3-manifold topology was shaken up considerably by the recent breakthroughs due to Agol, Kahn-Markovic, Przytycki-Wise and Wise. In particular the following is a consequence of these recent results:

Theorem 2.1. *Let N be a 3-manifold.*

(1) *Suppose that $N \neq S^1\times D^2$ and $N \neq T^2\times I$ and suppose that N is neither spherical nor covered by a torus bundle. Then $\pi_1(N)$ is large, i.e. $\pi_1(N)$ contains a finite index subgroup which admits an epimorphism onto a non-cyclic free group.*
(2) *Suppose that N is not a closed graph manifold. Then N is virtually fibered, i.e. N admits a finite index cover which fibers over S^1.*

The first statement is a consequence of the 'Virtually Compact Special Theorem' of Agol [Ag12] (building on work of Kahn-Markovic [KM12] and Wise [Wi12]) and older work of Kojima [Ko87] and Luecke [Lu88]. The second statement is also a consequence of the 'Virtually Compact Special Theorem' together with further work of Agol [Ag08] and Przytycki-Wise [PW12]. The fact that graph manifolds with boundary are fibered follows from earlier work of Wang–Yu [WY97] (see also [Li11, PW11]). We refer to the survey paper [AFW12] for details and how this theorem follows precisely from the aforementioned papers.

2.2. **Growth of the first Betti number of large groups.** In this section we will several times make use of the basic fact that if $\varphi\colon G \to H$ is a group homomorphism with finite cokernel, then a transfer argument shows that $H_1(G;\mathbb{Q}) \to H_1(H;\mathbb{Q})$ is surjective, and therefore $b_1(G) \geq b_1(H)$. We start out with the following lemma.

Lemma 2.2. *Let Γ be a residually finite group which admits an epimorphism $\alpha\colon \Gamma \to F$ onto a non-cyclic free group. Let $g\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function such that*

$$\lim_{n\to\infty} \frac{g(n)}{n} = 0.$$

Then there exists a normal cofinal filtration $\{\Gamma_i\}_{i\in\mathbb{N}}$ of Γ such that

$$b_1(\Gamma_i) \geq g([\Gamma : \Gamma_i]) \text{ for every } i \in \mathbb{N}.$$

Proof. Let Γ be a residually finite group which admits an epimorphism $\alpha\colon \Gamma \to F$ onto a non-cyclic free group. Let $g\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function such that $\lim_{n\to\infty} \frac{g(n)}{n} = 0$. After possibly replacing g by

$$n \mapsto \max\{g(1), \dots, g(n)\}$$

we can and will assume that g is monotonically increasing. [1]

Let $\{G_i\}_{i\in\mathbb{N}}$ be any normal cofinal filtration of Γ. We denote the projection maps $\Gamma \to \Gamma/G_i$, $i \in \mathbb{N}$, by ρ_i. We write $d_i := [\Gamma : G_i]$,

[1] Note that if $\lim_{n\to\infty} \frac{g(n)}{n} = 0$ and if we set $f(n) := \max\{g(1), \dots, g(n)\}$, then $\lim_{n\to\infty} \frac{f(n)}{n} = 0$ as well. Indeed, let $\varepsilon > 0$. By assumption there exists an N such that $\frac{g(n)}{n} < \varepsilon$ for all $n \geq N$. We now let M be any integer greater than

$i \in \mathbb{N}$. We pick an epimorphism $\phi\colon F \to \mathbb{Z}$ and given $n \in \mathbb{N}$ we denote by $\phi_n\colon F \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}/n$ the canonical projection. We also write $\psi_n = \phi_n \circ \alpha$.

Since $\lim_{n\to\infty} \frac{g(n)}{n} = 0$ we can iteratively pick $n_i \in \mathbb{N}$ with

$$\frac{g(n_i d_i)}{n_i d_i} < \frac{1}{d_i}, \text{ i.e. such that } g(n_i d_i) < n_i$$

and such that $n_{i+1} | n_i$ if $i > 1$. We now define

$$\Gamma_i := \operatorname{Ker}\{\rho_i \times \psi_{n_i}\colon \Gamma \to \Gamma/G_i \times \mathbb{Z}/n_i\}.$$

Note that $n_i d_i \geq [\Gamma : \Gamma_i]$ and note that $\{\Gamma_i\}_{i\in\mathbb{N}}$ is a cofinal normal filtration of Γ. Given any $i \in \mathbb{N}$ we then have

$$\begin{array}{rcl} \frac{1}{g([\Gamma:\Gamma_i])} b_1(\Gamma_i) & \geq & \frac{1}{g(n_i d_i)} b_1(\Gamma_i) \\ & \geq & \frac{1}{n_i} b_1(\operatorname{Ker}\{\rho_i \times \psi_{n_i}\colon \Gamma \to \Gamma/G_i \times \mathbb{Z}/n_i\}) \\ & \geq & \frac{1}{n_i} b_1(\operatorname{Ker}\{\phi_{n_i} : F \to \mathbb{Z}/n_i\}) \\ & = & \frac{1}{n_i}(n_i b_1(F) - 1) \geq 1. \end{array}$$

□

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $N \neq S^1 \times D^2$ and $N \neq T^2 \times I$ be a 3-manifold which is neither spherical nor covered by a torus bundle. By Theorem 2.1 (1) the group $\pi = \pi_1(N)$ is large, i.e. it admits a finite index subgroup Γ which surjects onto a non-cyclic free group. Since this property is preserved by going to finite index subgroups we can assume that Γ is a normal subgroup of π. We write $k = [\pi : \Gamma]$.

(1) Let $f\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function with $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. After possibly replacing f by

$$n \mapsto n \sup\left\{\frac{f(n)}{n}, \frac{f(n+1)}{n+1}, \dots\right\}$$

we can and will assume that $\frac{f(n)}{n}$ is monotonically decreasing.

We apply Lemma 2.2 to Γ and the function $g(n) = kf(n)$ and we denote by $\{\Gamma_i\}_{i\in\mathbb{N}}$ the resulting cofinal normal filtration

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$N, \frac{2}{\varepsilon}g(1), \dots, \frac{2}{\varepsilon}g(N-1)$. For every $n \geq M$ we then have

$$\begin{array}{rcl} \frac{1}{n}f(n) & = & \max\{\frac{1}{n}g(1), \dots, \frac{1}{n}g(N-1), \frac{1}{n}g(N), \dots, \frac{1}{n}g(M)\} \\ & \leq & \max\{\frac{1}{M}g(1), \dots, \frac{1}{M}g(N-1), \frac{1}{N}g(N), \dots, \frac{1}{M}g(M)\} < \varepsilon. \end{array}$$

of Γ. Note that $\{\Gamma_i\}_{i\in\mathbb{N}}$ is a cofinal almost normal filtration of π, and that

$$\begin{array}{rcl} b_1(\Gamma_i) & \geq & f([\Gamma:\Gamma_i])[\pi:\Gamma] \\ & = & \frac{f([\Gamma:\Gamma_i])}{[\Gamma:\Gamma_i]}[\Gamma:\Gamma_i][\pi:\Gamma] \\ & \geq & \frac{f([\pi:\Gamma_i])}{[\pi:\Gamma_i]}[\Gamma:\Gamma_i][\pi:\Gamma] = f([\pi:\Gamma_i]). \end{array}$$

(2) By Lemma 2.2 there exists a cofinal normal filtration $\{\Gamma_i\}_{i\in\mathbb{N}}$ of Γ such that

$$b_1(\Gamma_i) \geq k^{\frac{1}{2k}}\sqrt{[\Gamma:\Gamma_i]} \text{ for every } i\in\mathbb{N}.$$

We pick a complete set of representatives $a_1,\dots,a_k$ for π/Γ. Given $i\in\mathbb{N}$ we define

$$\pi_i := \bigcap_{j=1}^{k} a_j\Gamma_i a_j^{-1}.$$

Note that $\{\pi_i\}_{i\in\mathbb{N}}$ is a normal cofinal filtration of π. Also note that

$$\pi_i = \operatorname{Ker}\{\Gamma\to\Gamma/a_1\Gamma_i a_1^{-1}\times\cdots\times\Gamma/a_k\Gamma_i a_k^{-1}\}.$$

It thus follows that

$$[\pi:\pi_i] = [\pi:\Gamma]\cdot[\Gamma:\pi_i] \leq [\pi:\Gamma]\cdot[\Gamma:\Gamma_i]^k = k\cdot[\Gamma:\Gamma_i]^k.$$

Finally note that $b_1(\pi_i)\geq b_1(\Gamma_i)$, we thus see that for every i we have

$$b_1(\pi_i)\geq b_1(\Gamma_i)\geq k^{\frac{1}{2k}}\sqrt{[\Gamma:\Gamma_i]}\geq[\pi:\pi_i]^{\frac{1}{2k}}.$$

□

Remark. It seems unlikely that one can turn the almost normal sequence of Theorem 1.1 (1) into a normal sequence without paying a price. For example consider the group

$$\pi = \mathbb{Z}/2\ltimes(F\times F)$$

where F is a free non-cyclic group and $1\in\mathbb{Z}/2$ acts by commuting the two copies of F. If we apply the principle of the proof of Theorem 1.1 (1) to $\Gamma = F\times F$ and $\alpha\colon F\times F\to F$ the projection on the first factor and $\Gamma_n := \operatorname{Ker}\{F\times F\to F\to\mathbb{Z}/n\}$, then if we normalize these groups we really take the kernel $\operatorname{Ker}\{F\times F\to F\to\mathbb{Z}/n\times\mathbb{Z}/n\}$ but now the growth of the Betti numbers is sublinear (in fact it grows with the square root of the index).

2.3. **Growth of the rank of virtually fibered 3-manifolds.** In the following we mean by a surface group G the fundamental group of a compact orientable surface. We will make use of the following two facts:

(1) For any surface group G we have $b_1(G) = d(G)$.
(2) If H is a finite index subgroup of a surface group G, then an Euler characteristic argument shows that $b_1(H) \leq l \cdot b_1(G)$.

We can now formulate and prove the following lemma.

Lemma 2.3. *Let $\Gamma = \mathbb{Z} \ltimes G$ be the semidirect product of $\mathbb{Z}$ with a surface group G. Let $f\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function with $\lim_{n\to\infty} f(n) = \infty$. Then there exists a normal cofinal filtration $\{\Gamma_i\}_{i\in\mathbb{N}}$ of Γ such that*

$$d(\Gamma_i) \leq f([\Gamma : \Gamma_i]) \text{ for every } i \in \mathbb{N}.$$

Proof. Let G be a surface group. We write $r = b_1(G)$. Note that surface groups are residually finite, in particular there exists a cofinal filtration $\{G_i\}_{i\in\mathbb{N}}$ of G by characteristic finite index subgroups of G. (Recall that a subgroup of G is called characteristic if it is preserved by every automorphism of G.) We write $d_i := [G : G_i]$, $i \in \mathbb{N}$.

We denote by $\phi\colon \Gamma = \mathbb{Z} \ltimes G \to \mathbb{Z}$ the projection onto the first factor and given $n \in \mathbb{N}$ we denote by $\phi_n\colon \Gamma = \mathbb{Z} \ltimes G \to \mathbb{Z}/n$ the composition of ϕ with the surjection onto $\mathbb{Z}/n$. Since $\lim_{n\to\infty} f(n) = \infty$ we can iteratively pick $n_i \in \mathbb{N}$ such that

$$f(n_i d_i) \geq 1 + d_i r,$$

such that n_i divides the order of $\mathrm{Aut}(G/G_i)$ and such that $n_i | n_{i+1}$ for $i > 1$. We then define $\Gamma_i := n_i\mathbb{Z} \ltimes G_i$. Note that $\Gamma_i, i \in \mathbb{N}$ is normal in $\Gamma = \mathbb{Z} \ltimes G$ since $G_i \subset G$ is characteristic and since n_i divides the order of $\mathrm{Aut}(G/G_i)$. In particular the $\{\Gamma_i\}_{i\in\mathbb{N}}$ form a normal cofinal filtration of Γ. It now follows that

$$\begin{aligned} d(\Gamma_i) = d(n_i\mathbb{Z} \ltimes G_i) \leq 1 + d(G_i) &= 1 + b_1(G_i) \\ &\leq 1 + d_i r \\ &\leq f(n_i d_i) = f([\Gamma : \Gamma_i]). \end{aligned}$$

$\square$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let N be a 3-manifold which is not a closed graph manifold. We write $\pi = \pi_1(N)$. By Theorem 2.1 (2) there exists a finite cover $\tilde{N}$ which fibers over S^1, i.e. $\Gamma := \pi_1(\tilde{N}) \cong \mathbb{Z} \ltimes G$, where G is a surface group. Since finite covers of fibered 3-manifolds are again fibered, we can assume that $\Gamma := \pi_1(\tilde{N})$ is a normal subgroup of π.

(1) Let $g\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function with $\lim_{n\to\infty} g(n) = \infty$. We then apply Lemma 2.3 to $\Gamma = \mathbb{Z} \ltimes G$ and $f(n) := \frac{1}{[\pi:\Gamma]} g(n)$. The resulting filtration is an almost normal cofinal filtration of π with the desired property.

(2) By Lemma 2.3 there exists a normal cofinal filtration $\{\Gamma_i\}_{i\in\mathbb{N}}$ of Γ such that
$$d(\Gamma_i) \leq [\Gamma : \Gamma_i]^{\frac{1}{2}} \text{ for all } i.$$
Given $i \in \mathbb{N}$ we write $n_i := [\Gamma : \Gamma_i]$. We now denote by $a_1, \dots, a_k$ a complete set of representatives of π/Γ. Given any $i \in \mathbb{N}$ we define
$$\pi_i := \bigcap_{j=1}^{k} a_j \Gamma_i a_j^{-1} \subset \Gamma_i.$$
Note that $\{\pi_i\}_{i\in\mathbb{N}}$ is now a normal cofinal filtration of π. Given $i \in \mathbb{N}$ we write $s_i := [\Gamma_i : \pi_i]$. Note that $n_i \cdot s_i = [\Gamma : \Gamma_i] \cdot [\Gamma_i : \pi_i] \leq n_i^k$. We thus see that $s_i \leq n_i^{k-1}$. Using this observation we obtain that
$$\begin{array}{rcl} d(\pi_i) \leq [\Gamma_i : \pi_i] \cdot d(\Gamma_i) & = & s_i \cdot n_i^{\frac{1}{2}} \\ & = & s_i^{\frac{2k-1}{2k}} s_i^{\frac{1}{2k}} \cdot n_i^{\frac{1}{2}} \leq s_i^{\frac{2k-1}{2k}} \cdot n_i^{\frac{k-1}{2k}} n_i^{\frac{1}{2}} \\ & = & s_i^{\frac{2k-1}{2k}} n_i^{\frac{2k-1}{2k}} = k^{-\frac{2k-1}{2k}} (s_i n_i k)^{\frac{2k-1}{2k}} \\ & = & k^{-\frac{2k-1}{2k}} \cdot [\pi : \pi_i]^{\frac{2k-1}{2k}}. \end{array}$$
It follows that the sequence $\{\pi_i\}_{i\in\mathbb{N}}$ together with $\varepsilon = \frac{2k-1}{2k}$ has the desired properties.

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Mathematisches Institut, Universität zu Köln, Germany
E-mail address: sfriedl@gmail.com