REALIZATIONS OF SEIFERT MATRICES BY HYPERBOLIC KNOTS

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Abstract. Recently Kearton showed that any Seifert matrix of a knot is S–equivalent to the Seifert matrix of a prime knot. We show in this note that such a matrix is in fact S–equivalent to the Seifert matrix of a hyperbolic knot. This result follows from reinterpreting this problem in terms of Blanchfield pairings and by applying results of Kawauchi.

1. Introduction

We say that a square integral matrix A is of *Seifert type* if $\det(A - A^t) = 1$. Let A be a square integral matrix, then for any column vector v the matrices

$$\begin{pmatrix} A & 0 & 0 \\ v^t & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} A & v & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

are called elementary enlargements of A. We also say that A is an elementary reduction of any of its elementary enlargements. Two matrices are *S–equivalent* if they can be connected by a chain of elementary enlargements, elementary reductions and unimodular congruences.

Let $K \subset S^3$ be a knot and F a Seifert surface. Given a basis for $H_1(F)$ we can then define the Seifert matrix A of K. It is well–known that A is of Seifert type. It is shown in [Mu65, Theorem 3.1] (cf. also [Le70, Theorem 1]) that the S–equivalence class of the Seifert matrix is a knot invariant.

It is well–known that any matrix of Seifert type is the Seifert matrix of a knot. In [Ke04] Kearton showed that any matrix of Seifert type is S–equivalent to the Seifert matrix of a prime knot.

In this note we prove the following:

Theorem 1.1. *Let A be a matrix of Seifert type, then there exist infinitely many hyperbolic knots $K_i, i \in \mathbb{N}$ such that A is S–equivalent to a Seifert matrix of K_i.*

The proof relies on a reformulation of the S–equivalence class in terms of Blanchfield pairings and on realization results of Kawauchi.

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Added in proof: This theorem also follows for links from combining Theorem 2.2 with [Ka94, Theorem A.1].

2. Proof of the theorem

2.1. **S–equivalence and Blanchfield forms.** Given a knot $K \subset S^3$ we write $X(K) = S^3 \setminus \nu K$, the knot exterior. In the following we let $\Lambda = \mathbb{Z}[t, t^{-1}]$ and $Q(\Lambda) = \mathbb{Q}(t)$ the quotient field of Λ. We view $\Lambda = \mathbb{Z}[t, t^{-1}]$ with the involution $p \mapsto \overline{p}$ induced by $t \mapsto t^{-1}$.

Consider the following sequence of Λ–homomorphisms

$$\begin{array}{rcl} H_1(X(K);\Lambda) & \xrightarrow{\cong} & H_1(X(K), \partial X(K);\Lambda) \xrightarrow{\cong} H^2(X(K);\Lambda) \xrightarrow{\cong} \mathrm{Ext}^1_\Lambda(H_1(X(K);\Lambda),\Lambda) \\ & \xleftarrow{\cong} & \mathrm{Hom}(H_1(X(K);\Lambda), Q(\Lambda)/\Lambda). \end{array}$$

Here the first map comes from the long exact sequence of the pair $(X(K), \partial X(K))$, and is easily seen to be an isomorphism. The second homomorphism is Poincaré duality, the third homomorphism comes from the universal coefficient spectral sequence (and is an isomorphism by [Le77, Proposition 3.2]) and finally the last homomorphism comes from the long exact Ext–sequence corresponding to the short exact sequence of coefficients

$$0 \to \Lambda \to Q(\Lambda) \to Q(\Lambda)/\Lambda \to 0.$$

This sequence of homomorphisms defines the Blanchfield pairing

$$\lambda(K) : H_1(X(K);\Lambda) \times H_1(X(K);\Lambda) \to Q(\Lambda)/\Lambda.$$

This pairing is non–singular and Λ–hermitian. Furthermore if A is a Seifert matrix for K of size $k \times k$, then the Blanchfield pairing is isometric to the pairing

$$\begin{array}{rcl} \Lambda^k/(At - A^t)\Lambda^k \times \Lambda^k/(At - A^t)\Lambda^k & \to & Q(\Lambda)/\Lambda \\ (v, w) & \mapsto & \overline{v}^t(t-1)(At - A^t)^{-1}w. \end{array}$$

In particular the (S–equivalence class of a) Seifert matrix determines the Blanchfield pairing of a knot. By [Tr73] (and also by comparing [Ke75] with [Le70]) the converse holds as well, more precisely, the following theorem holds true.

Theorem 2.1. *Let $K_1, K_2 \subset S^3$ be knot. Then K_1 and K_2 have S–equivalent Seifert matrices if and only if the Blanchfield pairings $\lambda(K_1)$ and $\lambda(K_2)$ are isometric.*

2.2. **Kawauchi's realization results.** Before we continue we recall that the derived series $G^{(n)}, n \in \mathbb{N}$ of a group G is defined inductively by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$, the commutator of $G^{(n)}$. We recall the following hyperbolic realization result by Kawauchi.

Theorem 2.2. *Let $L \subset S^3$ be any link, then for any $V \in \mathbb{R}$ there exists a hyperbolic link $\tilde{L} \subset S^3$ together with a map $f : (S^3, \tilde{L}) \to (S^3, L)$ such that the following hold:*

(1) *$Vol(S^3 \setminus \tilde{L}) > V$,*

(2) *the induced map* $\pi_1(S^3 \setminus \tilde{L})/\pi_1(S^3 \setminus \tilde{L})^{(n)} \to \pi_1(S^3 \setminus L)/\pi_1(S^3 \setminus L)^{(n)}$ *is an isomorphism for any* n.

The theorem follows from the theory of almost identical imitations of Kawauchi. More precisely, the theorem follows from combining [Ka89b, Theorem 1.1] with [Ka89a, Properties I and V, p. 450] (cf. also [Ka89c]).

2.3. **Conclusion of the proof of the theorem.** Let $K \subset S^3$ be a knot and $V \in \mathbb{R}$. Let $\tilde{K}$ be as in Theorem 2.2. Since we can choose V arbitrarily large it follows from Theorem 2.1 that it is enough to show that the Blanchfield pairings $\lambda(K)$ and $\lambda(\tilde{K})$ are isometric.

First note that by Theorem 2.2 (2), applied to $n = 1$, we have a commutative diagram

$$\begin{tikzcd} \pi_1(X(\tilde{K})) \arrow[rr, "f_*"] \arrow[dr] & & \pi_1(X(K)) \arrow[dl] \\ & \mathbb{Z}. & \end{tikzcd}$$

In particular we get induced maps $H_i(X(\tilde{K});\Lambda) \to H_i(X(K);\Lambda)$. Write $X = X(K)$ and $\tilde{X} = X(\tilde{K})$. We then get the following commutative diagram

$$\begin{array}{ccccccccc} H_1(\tilde{X};\Lambda) & \to & H_1(\tilde{X},\partial\tilde{X};\Lambda) & \to & H^2(\tilde{X};\Lambda) & \to & \mathrm{Ext}^1_\Lambda(H_1(\tilde{X};\Lambda),\Lambda) & \xleftarrow{\cong} & \mathrm{Hom}(H_1(\tilde{X};\Lambda),Q(\Lambda)/\Lambda) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_1(X;\Lambda) & \to & H_1(X,\partial X;\Lambda) & \to & H^2(X;\Lambda) & \to & \mathrm{Ext}^1_\Lambda(H_1(X;\Lambda),\Lambda) & \xleftarrow{\cong} & \mathrm{Hom}(H_1(X;\Lambda),Q(\Lambda)/\Lambda). \end{array}$$

This means that we get a commutative diagram

$$\begin{array}{ccccc} H_1(X(\tilde{K});\Lambda) & \times & H_1(X(\tilde{K});\Lambda) & \to & Q(\Lambda)/\Lambda \\ \downarrow & & \downarrow & & \downarrow = \\ H_1(X(K);\Lambda) & \times & H_1(X(K);\Lambda) & \to & Q(\Lambda)/\Lambda. \end{array}$$

But it follows from Theorem 2.2 (2), applied to $n = 2$, that the induced map $H_1(X(\tilde{K});\Lambda) \to H_1(X(K);\Lambda)$ is an isomorphism of Λ–modules. In particular $\lambda(\tilde{K})$ is isometric to $\lambda(K)$.

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