Twisted Alexander polynomials of hyperbolic knots

Stefan Friedl
joint with N. Dunfield, N. Jackson and S. Vidussi

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The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:

1. $\Delta_K(t) \in \mathbb{Z}[t^\pm 1]$,
2. $\Delta_K(1) = 1$,
3. $\Delta_K(t^{-1}) = \Delta_K(t)$.

In fact any polynomial satisfying (1), (2) and (3) appears as the Alexander polynomial of a knot $K$.

Note that (1), (2) and (3) also imply (2') $\Delta_K(\xi) \neq 0$ for any prime power root of unity $\xi$.

The Alexander polynomial $\Delta_K(t)$ also contains topological information:

4. $\Delta_K(t)$ does not depend on the orientation,
5. $\Delta_K^*(t) = \Delta_K(t)$ where $K^*$ is the mirror image,
6. $\Delta_K(t)$ is invariant under mutation,
7. $\deg(\Delta_K(t)) \leq 2\text{genus of } K$,
8. if $K$ is fibered, then $\Delta_K(t)$ is monic.
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\[ \pi = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_{k-1} \rangle \]
be a presentation of \( \pi_1(S^3 \setminus K) \). Denote by \( \phi : \pi \to \langle t \rangle \) the epimorphism.
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\Delta_K(t) \over t - 1 = \frac{\det(\phi(\text{matrix } \left( \frac{\partial r_k}{\partial g_i} \right) \text{ with } i\text{-th column removed})))}{\phi(g_i) - 1}.
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Note that the formula on the right hand side really computes the Reidemeister torsion of \(C_*(S^3 \setminus K, \mathbb{Q}(t))\).
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$$\tau(K, \alpha) := \frac{\det((\alpha \otimes \phi)(\text{matrix } \left( \frac{\partial r_k}{\partial g_i} \right) \text{ with } i\text{-th column removed}))}{(\alpha \otimes \phi)(g_i) - 1}.$$
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This invariant is sometimes referred to as:

(1) twisted Alexander polynomial,
(2) Wada's invariant,
(3) twisted Reidemeister torsion.

Theorem. (Wada, Goda-Kitano-Morifuji) If $\alpha$ is an even dimensional representation, then $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i$, $i \in \mathbb{Z}$. 

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**Theorem.** (Wada, Goda-Kitano-Morifuji) If $\alpha$ is an even dimensional representation, then $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i$, $i \in \mathbb{Z}$. 
Let $K \subset S^3$ be an oriented hyperbolic knot. We denote by $\mu, \lambda$ its meridian and longitude. There exists a discrete and faithful repr. 

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which is unique up to conjugation if we demand that
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Hyperbolic knots

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(1) $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i$, $i \in \mathbb{Z}$. (Wada, Goda-Kitano-Morifuji)

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3. $\tau(K, \alpha)(t = 1)$ is non-zero (Menal-Ferrer and Porti)
4. $\tau(K, \alpha)$ is a polynomial in $\mathbb{C}[t^{\pm 1}]$ of even degree.
The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K, \alpha).$$
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\[ T_K(t) := \text{symmetrization of } \tau(K, \alpha). \]

**Theorem.** (1) \( T_K(t) \) is a well-defined polynomial in \( \mathbb{C}[t^{\pm 1}] \),

(2) \( T_K(1) \neq 0 \), in fact \( T_K(\xi) \neq 0 \) for any root of unity \( \xi \),

(follows from results of Menal-Ferrer and Porti)

(3) \( T_K(t-1) = T_K(t) \),

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**Example.** \( T_{\text{Figure 8 knot}}(t) = t - 4 + t - 1 \).

**Remark.** If \( K \) is a hyperbolic knot with at most thirteen crossings, then \( K \) is amphichiral if and only if \( T_K(t) \) is a real polynomial.

**Remark.** If \( K \) is an amphichiral hyperbolic knot with at most thirteen crossings, then the top coefficient of \( T_K(t) \) is at least one.
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**Example.** $\mathcal{T}$ Figure 8 knot ($t$) = $t - 4 + t^{-1}$. 

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**Example.**  
$$\mathcal{T}_{\text{Figure 8 knot}}(t) = t - 4 + t^{-1}.$$

**Remark.** If $K$ is a hyperbolic knot with at most thirteen crossings, then $K$ is amphichiral if and only if $\mathcal{T}_K(t)$ is a real polynomial.
The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K, \alpha).$$

**Theorem.** (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$, 
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity $\xi$,
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**Remark.** If $K$ is an amphichiral hyperbolic knot with at most thirteen crossings, then the top coefficient of $\mathcal{T}_K(t)$ is at least one.
The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.

\[ T_{\text{Conway}}(t) \approx (4.895 - 0.099i)t^5 + (-15.686 + 0.298i)t^4 + \ldots \]

\[ T_{\text{KT}}(t) \approx (4.418 + 0.376i)t^3 + (-22.942 - 4.845i)t^2 + \ldots + (-22.926 - 4.845i)t^{-2} + (4.418 + 0.376i)t^{-3} \]

This shows that \( T_{\text{KT}}(t) \) detects mutation.

The evaluations \( T_{\text{KT}}(1) \) and \( T_{\text{KT}}(-1) \) conjecturally do not detect mutation. E.g. for both knots evaluation at \( t = 1 \) gives \( 4.1860 - 4.2286i \) and for \( t = -1 \) we get \( 261.34 + 102.13i \).

Questions.

(1) Is \( |T_{\text{KT}}(1)| \) always less than \( |T_{\text{KT}}(-1)| \)?

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$T_K(t)$ and the knot genus

We now write $x(K) = 2\text{genus}(K) - 1$. 

Theorem (F-Taehee Kim) For any hyperbolic knot $K$ we have $\deg(T_K(t)) \leq 2x(K)$.

E.g. if $K$ is the Conway knot, then the genus is three and we calculated $\deg(T_{\text{Conway}}(t)) = 10$.

Similarly the genus of the Kinoshita-Terasaka knot is two and $\deg(T_{\text{Kinoshita-Terasaka}}(t)) = 6$.

We show that $\deg(T_K(t)) = 2x(K)$ for all knots with at most thirteen crossings.

The genera of all thirteen crossings were also independently determined by Stoimenow.

Conjecture. For any hyperbolic knot $K$ we have $\deg(T_K(t)) = 2x(K)$. 

Stefan Friedl joint with N. Dunfield, N. Jackson and S. Vidussi 

Twisted Alexander polynomials of hyperbolic knots
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**Theorem (F-Taehee Kim)** For any hyperbolic knot $K$ we have
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**Conjecture.** For any hyperbolic knot $K$ we have

$$\text{deg}(\mathcal{T}_K(t)) = 2x(K).$$
Theorem (Goda-Kitano-Morifuji) If $K$ is hyperbolic and fibered, then
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We show that $\mathcal{T}_K(t)$ detects all fibered knots with at most thirteen crossings.
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Note that it is known that the set of twisted Alexander polynomials corresponding to *all* finite representations detects whether $K$ is fibered.
The adjoint representation

Let $K$ be a hyperbolic knot and $\alpha : \pi \to SL(2, \mathbb{C})$ the canonical representation.

Then consider the adjoint representation $\alpha_{\text{adj}} : \pi \to \text{sl}(2, \mathbb{C})$. This is a 3-dimensional, irreducible and faithful representation. In particular $\deg \tau(K, \alpha_{\text{adj}}) \leq \frac{1}{3} \times (K)$. For the Conway knot (which has genus 3) we compute $\deg \tau(\text{Conway knot}, \alpha_{\text{adj}}) = 13$. I.e. here the inequality is a strict inequality.
Let $K$ be a hyperbolic knot and $\alpha : \pi \to \text{SL}(2, \mathbb{C})$ the canonical representation. Then consider the \textit{adjoint} representation $\alpha_{\text{adj}} : \pi \to \mathfrak{sl}(2, \mathbb{C})$. This is a 3-dimensional, irreducible and faithful representation. In particular $\deg \tau(K, \alpha_{\text{adj}}) \leq \frac{1}{3} \chi(K)$.

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The character variety is defined as

\[ X(K) := \text{Hom}(\pi_1(S^3 \setminus K), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})\text{-conjugation}. \]
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**Theorem.**

\[ \{ \chi \in X(K) \mid \deg(\tau(K, \chi)) = 2x(K) \} \text{ is Zariski open} \]

\[ \{ \chi \in X(K) \mid \tau(K, \chi) \text{ is monic} \} \text{ is Zariski closed} \]