Twisted Alexander polynomials of hyperbolic knots

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In fact any polynomial satisfying (1), (2) and (3) appears as the Alexander polynomial of a knot K.

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(8) if K is fibered, then $\Delta_K(t)$ is monic.

The classical Alexander polynomial and Fox calculus

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$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_{k-1} \rangle$$

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$$\frac{\Delta_K(t)}{t-1} = \frac{\det(\phi(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{ with } i\text{-th column removed}))}{\phi(g_i) - 1}.$$

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Note that the formula on the right hand side really computes the Reidemeister torsion of $C_*(S^3 \setminus K, \mathbb{Q}(t))$.

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Theorem. (Wada, Goda-Kitano-Morifuji) If α is an even dimensional representation, then $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i, i \in \mathbb{Z}$.

Hyperbolic knots

Let $K \subset S^3$ be an oriented hyperbolic knot. We denote by μ, λ its meridian and longitude. There exists a discrete and faithful repr.

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which is unique up to conjugation if we demand that

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(4) $\tau(K, \alpha)$ is a polynomial in $\mathbb{C}[t^{\pm 1}]$ of even degree.

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$$\mathcal{T}_{\text{Figure 8 knot}}(t) = t - 4 + t^{-1}.$$

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This shows that $\mathcal{T}_K(t)$ detects mutation.

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Questions. (1) Is $|\mathcal{T}_K(1)|$ always less than $|\mathcal{T}_K(-1)|$?
(2) Does $\mathcal{T}_K(t)$ distinguish hyperbolic knots?

$\mathcal{T}_K(t)$ and the knot genus

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Conjecture. For any hyperbolic knot K we have

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$\mathcal{T}_K(t)$ and fibered knots

Theorem (Goda-Kitano-Morifuji) If K is hyperbolic and fibered, then

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Note that it is known that the set of twisted Alexander polynomials corresponding to *all* finite representations detects whether K is fibered.

The adjoint representation

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For the Conway knot (which has genus 3) we compute

$$\deg \tau(\text{Conway knot}, \alpha_{adj}) = 13.$$

I.e. here the inequality is a strict inequality.

The character variety

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Theorem.

$$\{\chi \in X(K) \,|\, \deg(\tau(K, \chi)) = 2x(K)\} \text{ is Zariski open}$$

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