L^2–ETA–INVARIANTS AND THEIR APPROXIMATION BY UNITARY ETA–INVARIANTS

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Abstract. Cochran, Orr and Teichner introduced L^2–eta–invariants to detect highly non–trivial examples of non slice knots. Using a recent theorem by Lück and Schick we show that their metabelian L^2–eta–invariants can be viewed as the limit of finite dimensional unitary representations. We recall a ribbon obstruction theorem proved by the author using finite dimensional unitary eta–invariants. We show that if for a knot K this ribbon obstruction vanishes then the metabelian L^2–eta–invariant vanishes too. The converse has been shown by the author not to be true.

1. Introduction

A knot $K \subset S^{n+2}$ is a smooth submanifold homeomorphic to S^n. A knot is called slice if it bounds a smooth disk in D^4. We say that a knot K is algebraically slice if K has a Seifert matrix of the form $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ where B, C, D are square matrices of the same size. It is a well-known fact that any slice knot is algebraically slice. Levine showed that in higher odd dimensions the converse is true, i.e. if a knot is algebraically slice it is also geometrically slice (cf. [L69]). In the classical dimension $n = 1$ this no longer holds as was shown by Cassan and Gordon [CG86].

A knot $K \subset S^3$ is called ribbon if there exists a smooth disk D in $S^3 \times [0,1] \subset D^4$ ($S^3 = S \times 0$) bounding K such that the projection map $S^3 \times [0,1] \to [0,1]$ is a Morse map and has no local minima. Such a slice disk is called a ribbon disk. Fox [F61] conjectured that all slice knots are ribbon.

In [F03] the author studies metabelian unitary eta–invariants of M_K, the result of zero framed surgery along a knot $K \subset S^3$. These can be used to detect knots which are not slice respectively, not ribbon.

For a pair $(M^3, \varphi : \pi_1(M) \to G)$ Cheeger and Gromov [CG85] introduced the L^2–eta–invariant $\eta^{(2)}(M, \varphi)$. Cochran, Orr and Teichner [COT01] gave examples of knots which look slice 'up to a certain level' but can be shown to be not slice using L^2–eta–invariants.

Lück and Schick [LS01] showed that L^2–eta–invariants can be viewed as a limit of ordinary unitary eta–invariants if G is residually finite. We show that the metabelian groups used by Cochran, Orr and Teichner are residually finite. Sorting out several

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technical problems we can show that if for a knot K the metabelian eta–invariant ribbonness obstruction vanishes then the metabelian L^2–eta–invariant sliceness obstruction vanishes as well. In [F03] we show that the converse is not true.

The structure of the paper is as follows. In section 2 we recall the eta–invariant sliceness and ribbonness obstruction theorems of [F03]. In section 3 we give the definition of (n)–solvability for a knot $n \in \frac{1}{2}\mathbb{N}$, and quote some results of [COT01]. Furthermore we state the metabelian L^2–eta–invariant sliceness obstruction theorem of Cochran, Orr and Teichner. We state and prove the main theorem in section 4.

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2. Unitary eta–invariants as knot invariants

Let M^{2q+1} be a closed odd-dimensional smooth manifold and $\alpha : \pi_1(M) \to U(k)$ a unitary representation. Atiyah, Patodi, Singer [APS75] associated to (M, α) a number $\eta(M, \alpha)$ called the (reduced) eta–invariant of (M, α). This invariant has the property that if $\partial(W^{2q+2}, \beta) = (M^{2q+1}, \alpha)$ then

$$\eta(M, \alpha) = \text{sign}_\beta(W) - k\text{sign}(W)$$

where $\text{sign}_\beta(W)$ denotes the signature of W twisted by β.

2.1. **Abelian eta–invariants.** Let K be knot, μ a meridian and A a Seifert matrix for K. Let $\alpha : \pi_1(M_K) \to U(1)$ be a representation, then

$$\eta(M_K, \alpha) = \sigma_z(K) := \text{sign}(A(1 - z) + A^t(1 - \bar{z}))$$

where $z := \alpha(\mu)$ (cf. [L84]).

The following proposition follows immediately from the definitions and the explicit computation of the abelian eta–invariant.

Proposition 2.1. *Let K be an algebraically slice knot, then $\eta(M_K, \alpha) = 0$ for any representation $\alpha : \pi_1(M_K) \to U(1)$ which sends the meridian to a transcendental number.*

If a knot satisfies the conclusion of this proposition we say that K has zero abelian eta–invariant sliceness obstruction.

2.2. **Metabelian eta–invariants.** There exists a canonical map $\epsilon : \pi_1(M_K) \to H_1(M_K) = \mathbb{Z}$ sending the meridian to 1. Denote the k–fold cover of M_K by M_k. If k is a prime power, then Casson and Gordon [CG86] showed that $H_1(M_k) = \mathbb{Z} \oplus TH_1(M_k)$ where $TH_1(M_k)$ denotes the $\mathbb{Z}$–torsion part of $H_1(M_k)$. Furthermore there exists a non–singular symmetric linking pairing

$$\lambda_{lk} : TH_1(M_k) \times TH_1(M_k) \to \mathbb{Q}/\mathbb{Z}$$

We say that $P_k \subset TH_1(M_k)$ is a Λ–metabolizer for λ_{lk} if P_k is a Λ–submodule and if

$$P_k = P_k^\perp := \{x \in TH_1(M_k) | \lambda_{lk}(x, y) = 0 \text{ for all } y \in TH_1(M_k)\}$$

Denote by $\tilde{M}_K$ the universal abelian cover corresponding to ϵ. $H_1(\tilde{M}_K)$ carries a $\Lambda := \mathbb{Z}[t,t^{-1}]$–module structure, we will henceforth write $H_1(M_K,\Lambda)$ for $H_1(\tilde{M}_K)$. Blanchfield [B57] shows that there exists a non-singular Λ-hermitian pairing

$$\lambda_{Bl} : H_1(M_K,\Lambda) \times H_1(M_K,\Lambda) \to \mathbb{Q}(t)/\Lambda$$

For a Λ-submodule $P \subset H_1(M_K,\Lambda)$ define

$$P^\perp := \{v \in H_1(M_K,\Lambda) | \lambda_{Bl}(v,w) = 0 \text{ for all } w \in P\}$$

If $P \subset H_1(M_K,\Lambda)$ is such that $P = P^\perp$, then we say that P is a metabolizer for λ_{Bl} and that λ_{Bl} is metabolic. Note that Kearton [K75] showed that a knot is algebraically slice if and only if λ_{Bl} is metabolic.

Recall that for a group G the central series is defined inductively by $G^{(0)} := G$ and $G^{(i)} := [G^{(i-1)}, G^{(i-1)}]$. Let $\pi := \pi_1(M_K)$. We study metabelian representations, i.e. representations that factor through $\pi/\pi^{(2)}$. Consider

$$1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1$$

Note that $\pi^{(1)}/\pi^{(2)} \cong H_1(\tilde{M}_K)$ and $\pi/\pi^{(1)} = H_1(M_K) = \mathbb{Z}$, in particular this sequence splits and we get an isomorphism

$$\pi/\pi^{(2)} \cong \mathbb{Z} \ltimes H_1(M_K,\Lambda)$$

where $1 \in \mathbb{Z}$ acts by conjugating with μ respectively by multiplying by t. Eta invariants corresponding to metabelian representations in the context of knot theory were first studied by Letsche [L00].

For a group G denote by $R_k^{irr}(G)$ (resp. $R_k^{irr,met}(G)$) the set of irreducible, k-dimensional, unitary (metabelian) representations of G. By $\hat{R}$ we denote the conjugacy classes of such representations. The above discussion shows that for a knot K we can identify

$$R_k^{irr,met}(\pi_1(M_K)) = R_k^{irr}(\mathbb{Z} \ltimes H_1(M_K,\Lambda))$$

Lemma 2.2. [F03] *Let $z \in S^1$ and $\chi : H_1(M,\Lambda) \to H_1(M,\Lambda)/(t^k-1) \to S^1$ a character. Then*

$$\begin{array}{rcl} \alpha_{(k,z,\chi)} = \alpha_{(z,\chi)} : \mathbb{Z} \ltimes H_1(M,\Lambda) & \to & U(k) \\ (n,h) & \mapsto & z^n \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{k-1}h) \end{pmatrix} \end{array}$$

defines a representation.

Conversely any irreducible representation $\alpha \in R_k^{irr}(\mathbb{Z} \ltimes H_1(M,\Lambda))$ is (unitary) conjugate to $\alpha_{(z,\chi)}$ for some $z \in S^1$ and a character $\chi : H_1(M,\Lambda) \to H_1(M,\Lambda)/(t^k - 1) \to S^1$ which does not factor through $H_1(M,\Lambda)/(t^l - 1)$ for some $l < k$.

We denote by $P_k^{met}(\pi_1(M_K))$ the set of metabelian representations of $\pi_1(M_K)$ that are conjugate to $\alpha_{(z,\chi)}$ with z transcendental and χ of prime power order. Furthermore for p a prime we write $P_{k,p}^{irr,met}(\pi_1(M_K))$ for the set of representations where χ has order a power of p. In [F03] we prove the following sliceness obstruction theorem which is the strongest theorem detecting non–torsion knots which is not based on L^2–eta–invariants.

Theorem 2.3. *Let K be a slice knot, $k_1, \dots, k_r$ pairwise coprime prime powers, then there exist Λ–metabolizers $P_{k_i} \subset TH_1(M_{k_i}), i = 1, \dots, r$ for the linking pairings λ_{k_i}, such that for any prime number p and any choice of irreducible representations $\alpha_i : \pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M_K, \Lambda)/(t^{k_i} - 1) \to U(k)$ vanishing on $0 \times P_{k_i}$ and lying in $P_{k_i,p}^{irr,met}(\pi_1(M_K))$ we get $\eta(M_K, \alpha_1 \otimes \dots \otimes \alpha_r) = 0$.*

If a knot K satisfies the conclusion of this theorem we say that K has zero metabelian eta–invariant sliceness obstruction.

In [F03] we prove the following ribbon obstruction theorem. In the proof we only use the well-known fact that if K is ribbon then K has a slice disk D such that $\pi_1(S^3 \setminus K) \to \pi_1(D^4 \setminus D)$ is surjective.

Theorem 2.4. [F03] *Let $K \subset S^3$ be a ribbon knot. Then there exists a metabolizer P for the Blanchfield pairing such that for any $\alpha_{(z,\chi)}$ with z transcendental and χ of prime power order, vanishing on $0 \times P$ we get $\eta(M_K, \alpha_{(z,\chi)}) = 0$.*

We say that K has zero metabelian eta–invariant ribbonness obstruction if the conclusion of the theorem holds for K.

3. The Cochran–Orr–Teichner sliceness obstruction

3.1. **The Cochran–Orr–Teichner sliceness filtration.** We give a short introduction to the sliceness filtration introduced by Cochran, Orr and Teichner [COT01]. For a manifold W denote by $W^{(n)}$ the cover corresponding to $\pi_1(W)^{(n)}$. Denote the equivariant intersection form

$$H_2(W^{(n)}) \times H_2(W^{(n)}) \to \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}]$$

by λ_n, and the self-intersection form by μ_n. An (n)-Lagrangian is a submodule $L \subset H_2(W^{(n)})$ on which λ_n and μ_n vanish and which maps onto a Lagrangian of $\lambda_0 : H_2(W) \times H_2(W) \to \mathbb{Z}$.

Definition. [COT01, def. 8.5] A knot K is called (n)-solvable if M_K bounds a spin 4-manifold W such that $H_1(M_K) \to H_1(W)$ is an isomorphism and such that W admits two dual (n)-Lagrangians. This means that λ_n pairs the two Lagrangians non-singularly and that the projections freely generate $H_2(W)$.

A knot K is called $(n.5)$-solvable if M_K bounds a spin 4-manifold W such that $H_1(M_K) \to H_1(W)$ is an isomorphism and such that W admits an (n)-Lagrangian and a dual $(n+1)$-Lagrangian.

We call W an (n)-solution respectively $(n.5)$-solution for K.

Remark. 1. The size of an (n)-Lagrangian depends only on the size of $H_2(W)$, in particular if K is slice, D a slice disk, then $\overline{D^4 \setminus N(D)}$ is an (n)-solution for K for all n, since $H_2(D^4 \setminus N(D)) = 0$.
2. By the naturality of covering spaces and homology with twisted coefficients it follows that if K is (h)-solvable, then it is (k)-solvable for all $k < h$.

Theorem 3.1.

$$\begin{array}{rcl} K \text{ is (0)-solvable} & \Leftrightarrow & \mathit{Arf}(K) = 0 \\ K \text{ is (0.5)-solvable} & \Leftrightarrow & K \text{ is algebraically slice} \\ K \text{ is (1.5)-solvable} & \Rightarrow & \text{Casson-Gordon invariants vanish and } K \text{ algebraically slice} \end{array}$$

The converse of the last statement is not true, i.e. there exist algebraically slice knots which have zero Casson-Gordon invariants but are not (1.5)*-solvable.*

The first part, the third part and the $\Leftarrow$ direction of the second part have been shown by Cochran, Orr and Teichner [COT01, p. 6, p. 72, p. 66, p. 73]. Cochran, Orr and Teichner [COT01, p. 6] showed that a knot is (0.5) solvable if and only if the Cappell-Shaneson surgery obstruction in $\Gamma_0(\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z})$ vanishes. This is equivalent to a knot being algebraically slice (cf. [K89]). Taehee Kim [K02] showed that there exist (1.0)-solvable knots which have zero Casson-Gordon invariants, but are not (1.5)-solvable. Cochran, Orr and Teichner [COT01] also showed that there exist (2)–solvable knots which are not (2.5)–solvable.

3.2. **L^2–eta–invariants as sliceness-obstructions.** In this section we'll very quickly summarize some L^2–eta–invariant theory.

Let M^3 be a smooth manifold and $\varphi : \pi_1(M) \to G$ a homomorphism, then Cheeger and Gromov [CG85] defined an invariant $\eta^{(2)}(M, \varphi) \in \mathbf{R}$, the (reduced) L^2–eta–invariant. When it's clear which homomorphism we mean, we'll write $\eta^{(2)}(M, G)$ for $\eta^{(2)}(M, \varphi)$.

Remark. If $\partial(W, \psi) = (M^3, \varphi)$, then (cf. [COT01, lemma 5.9 and remark 5.10])

$$\eta^{(2)}(M, \varphi) = \mathrm{sign}^{(2)}(W, \psi) - \mathrm{sign}(W)$$

where $\mathrm{sign}^{(2)}(W, \psi)$ denotes Atiyah's L^2-signature (cf. [A76]).

Cochran, Orr and Teichner study when L^2–eta–invariants vanish for homomorphisms $\pi_1(M_K) \to G$, where G is a PTFA-group. PTFA stands for poly-torsion-free-abelian, and means that there exists a normal subsequence where each quotient is torsion-free-abelian.

Theorem 3.2. [COT02, p. 5] *Let G be a PTFA-group with $G^{(n)} = 1$. If K is a knot, and $\varphi : \pi_1(M_K) \to G$ a homomorphism which extends over a $(n.5)$-solution of M_K, then $\eta^{(2)}(M_K, \varphi) = 0$. In particular if K is slice and φ extends over $D^4 \setminus D$ for some slice disk D, then $\eta^{(2)}(M_K, \varphi) = 0$.*

Remark. It's a crucial ingredient in the proposition that the group G is a PTFA-group, for example it's not true in general that $\eta^{(2)}(M_K,\mathbb{Z}/k)=0$ for a slice knot K. Corollary 4.3 shows that $\eta^{(2)}(M_K,\mathbb{Z}/k)=\sum_{j=1}^k \sigma_{e^{2\pi ij/k}}(K)$, but this can be non-zero for some slice knot K, e.g. take a slice knot with Seifert matrix

$$A=\begin{pmatrix}0&0&1&1\\0&0&0&1\\1&1&0&1\\0&1&0&0\end{pmatrix}$$

Then $\eta^{(2)}(M_K,\mathbb{Z}/6)=-2$.

We use this theorem only in the abelian and the metabelian setting. Let $\mathbb{Q}\Lambda := \mathbb{Q}[t,t^{-1}]$.

Theorem 3.3. [COT01]

1. *If K is (0.5)–solvable, then $\eta^{(2)}(M_K,\mathbb{Z})=0$.*
2. *If K is (1.5)–solvable, then there exists a metabolizer $P_{\mathbb{Q}}\subset H_1(M_K,\mathbb{Q}\Lambda)$ for the rational Blanchfield pairing*
$$\lambda_{Bl,\mathbb{Q}}: H_1(M_K,\mathbb{Q}\Lambda)\times H_1(M_K,\mathbb{Q}\Lambda)\to \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]$$
such that for all $x\in P_{\mathbb{Q}}$ we get $\eta^{(2)}(M_K,\beta_x)=0$ where β_x denotes the map
$$\pi_1(M_K)\to \mathbb{Z}\ltimes H_1(M_K,\Lambda)\to \mathbb{Z}\ltimes H_1(M_K,\mathbb{Q}\Lambda)\xrightarrow{id\times\lambda_{Bl,\mathbb{Q}}(x,-)}\mathbb{Z}\ltimes \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]$$

Proof. Let D be a slice disk for K, write $N_D := \overline{D^4\setminus N(D)}$. Then the statement follows from proposition 3.2 and work by Letsche [L00] who showed that for $P_{\mathbb{Q}} := \operatorname{Ker}\{H_1(M_K,\mathbb{Q}\Lambda)\to H_1(N_D,\Lambda\mathbb{Q})\}$ the map β_x extends over $\pi_1(N_D)$. □

We say that K has zero abelian L^2–eta–invariant sliceness obstruction if $\eta^{(2)}(M_K,\mathbb{Z})=0$. We say that K has zero metabelian L^2–eta–invariant sliceness obstruction if there exists a metabolizer $P_{\mathbb{Q}}\subset H_1(M_K,\mathbb{Q}\Lambda)$ for $\lambda_{Bl,\mathbb{Q}}$ such that for all $x\in P_{\mathbb{Q}}$ we get $\eta^{(2)}(M_K,\beta_x)=0$.

4. Relation between eta–invariants and L^2–eta–invariants

If a knot K has zero abelian eta–invariant sliceness obstruction, then a multiple of K is algebraically slice (cf. Levine [L69b] and Matumuto [M77]), in particular K has zero abelian L^2–eta–invariant sliceness obstruction. This fact will also follow immediately from corollary 4.3. Conversely, if K has zero abelian L^2–eta–invariant, then it is not necessarily true that K has zero abelian eta–invariant, as was shown in [F03].

In [K02] Taehee Kim gave examples of knots where the metabelian eta–invariant sliceness obstruction is zero, but where the metabelian L^2–eta–invariant obstruction is non–zero. This shows that more eta–invariants have to vanish to get zero L^2–eta–invariants.

Our main theorem is the following.

Theorem 4.1. *Let K be a knot with zero metabelian eta–invariant ribbonness obstruction, then K has zero metabelian L^2–eta–invariant sliceness obstruction.*

The proof of the theorem will be done in the next two sections. In [F03] we showed that the converse is not true, i.e. there exists a knot with zero metabelian L^2–eta–invariant but non–zero metabelian eta–invariant ribbonness obstruction

4.1. **Approximation of L^2–eta–invariants.**

Definition. We say that G is residually finite it there exists a sequence of normal subgroups $G \supset G_1 \supset G_2 \supset \dots$ of finite index $[G : G_i]$ such that $\cap_i G_i = \{1\}$. We call the sequence $\{G_i\}_{i \geq 1}$ a resolution of G.

If $\varphi : \pi_1(M) \to G$ is a homomorphism to a finite group, then define $\eta(M, G) = \eta(M, \alpha_G)$ where $\alpha_G : \pi_1(M) \xrightarrow{\varphi} G \to U(\mathbb{C}G)$ is the canonical induced unitary representation given by left multiplication.

Theorem 4.2. *Let $\varphi : \pi_1(M) \to G$ be a homomorphism.*

1. *If G is finite, then*

$$\begin{aligned} \eta(M, G) &= \textstyle\sum_{\alpha \in \hat{R}^{irr}(G)} \dim(\alpha) \eta(M,_{\alpha \circ \varphi}(M) \\ \eta^{(2)}(M, G) &= \tfrac{\eta(M,G)}{|G|} \end{aligned}$$

2. *If G is residually finite group then the above equality "holds in the limit", i.e. if $\{G_i\}_{i \geq 1}$ is a resolution of G, then*

$$\eta^{(2)}(M, G) = \lim_{i \to \infty} \frac{\eta(M, G/G_i)}{|G/G_i|}$$

Proof. The first statement follows immediately from the well-known fact of the representation theory of finite groups that

$$\mathbb{C}G = \sum_{\alpha \in \hat{R}^{irr}(G)} V_\alpha^{\dim(\alpha)}$$

The second statement is shown in [A76], Lück and Schick proved the last parts (cf. [LS01, remark 1.23]).

□

Corollary 4.3. *Let K be a knot, then*

$$\begin{aligned} \eta^{(2)}(M_K, \mathbb{Z}/k) &= \tfrac{1}{k} \eta(M_K, \mathbb{Z}/k) = \tfrac{1}{k} \textstyle\sum_{j=1}^{k} \sigma_{e^{2\pi i j/k}}(K) \\ \eta^{(2)}(M_K, \mathbb{Z}) &= \textstyle\int_{S^1} \sigma_z(K) \end{aligned}$$

This corollary was also proven by Cochran, Orr and Teichner (cf. [COT02]), using a different approach.

Proof. The first part is immediate from the decomposition of $\mathbb{C}[\mathbb{Z}/k]$ into one-dimensional $\mathbb{C}[\mathbb{Z}/k]$–modules. For the second part consider the sequence $\mathbb{Z} \supset 2!\mathbb{Z} \supset 3!\mathbb{Z} \supset 4!\mathbb{Z} \supset \dots$, by theorem 4.2 and corollary 4.3

$$\eta^{(2)}(M_K,\mathbb{Z}) = \lim_{k\to\infty} \frac{\eta(M_K,\mathbb{Z}/k!)}{k!} = \lim_{k\to\infty} \frac{\sum_{j=0}^{k!-1} \sigma_{e^{2\pi i j/k!}}(K)}{k!} = \int_{S^1} \sigma_z(K)$$

The last equality follows from the fact that $\sigma_z(K)$ is a step function with only finitely many break points. $\square$

4.2. **Proof of theorem 4.1.** Assume that K has zero metabelian eta–invariant ribbon obstruction. Let P be a metabolizer such that $\eta(M_K,\alpha(z,\chi)) = 0$ for all $\alpha_{(z,\chi)} \in P_k(\pi_1(M_K))$ with $\chi(P) \equiv 0$. Let $P_{\mathbb{Q}} := P \otimes \mathbb{Q}$, this is a metabolizer for the rational Blanchfield pairing $\lambda_{Bl,\mathbb{Q}}$. We will show that for any $x \in P_{\mathbb{Q}}$ $\eta^{(2)}(M_K,\beta_x) = 0$, where β_x denotes the map

$$\pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M_K,\Lambda) \to \mathbb{Z} \ltimes H_1(M_K,\mathbb{Q}\Lambda) \xrightarrow{id\times\lambda_{Bl,\mathbb{Q}}(x,-)} \mathbb{Z} \ltimes \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]$$

This implies the theorem.

So let $x \in P_{\mathbb{Q}}$. Note that $nx \in P$ for some $n \in \mathbb{N}$. The map β_{nx} factors through $\mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda$, hence β_x factors through $\mathbb{Z} \ltimes n^{-1}\Delta_K(t)^{-1}\Lambda/\Lambda$.

Claim. There exists an isomorphism

$$\mathrm{Im}\{\mathbb{Z} \ltimes n^{-1}\Delta_K(t)^{-1}\Lambda/\Lambda \to \mathbb{Z} \ltimes \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]\} \to \mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda$$

Proof. Consider the short exact sequence

$$0 \to \Delta_K(t)^{-1}\Lambda/\Lambda \to n^{-1}\Delta_K(t)^{-1}\Lambda/\Lambda \to \Delta_K(t)^{-1}\Lambda/n^{-1}\Delta_K(t)^{-1}\Lambda \cong \Lambda/n \to 0$$

since tensoring with $\mathbb{Q}$ is exact and since Λ/n is $\mathbb{Z}$–torsion we see that

$$\mathrm{Im}\{n^{-1}\Delta_K(t)^{-1}\Lambda/\Lambda \to \Delta_K(t)^{-1}\mathbb{Q}\Lambda/\mathbb{Q}\Lambda\} \cong \mathrm{Im}\{\Delta_K(t)^{-1}\Lambda/\Lambda \to \Delta_K(t)^{-1}\mathbb{Q}\Lambda/\mathbb{Q}\Lambda\}$$

But $\Delta_K(t)^{-1}\Lambda/\Lambda \to \Delta_K(t)^{-1}\mathbb{Q}\Lambda/\mathbb{Q}\Lambda \to \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]$ is injective, since $\Delta_K(t)^{-1}\Lambda/\Lambda$ is $\mathbb{Z}$–torsion free. This shows that

$$\mathrm{Im}\{n^{-1}\Delta_K(t)^{-1}\Lambda/\Lambda \to \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]\} \cong \Delta_K(t)^{-1}\Lambda/\Lambda$$

Since all maps preserve the $\mathbb{Z}$–action the claim follows. $\square$

Lemma 4.4. *Let K be a knot, then $\mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda$ is residually finite.*

Proof. Write $\Delta_K(t) = a_{2g}t^{2g} + \dots + a_1 t + a_0$ with $a_{2g} \neq 0, a_{2g-i} = a_i$. Let p be a prime number coprime to a_{2g}. Write $H := \Delta_K(t)^{-1}\Lambda/\Lambda$ and $H_i := p^i H$. Then $\{H_i\}_{i\geq 1}$ forms a resolution for H since there exists an embedding $\Delta_K(t)^{-1}\Lambda/\Lambda \cong \Lambda/\Delta_K(t)\Lambda \to \mathbb{Z}[1/a_{2g}]^{2g}$ of $\mathbb{Z}$-modules.

Since the Λ-modules H/H_i are finite there exists for each i a number k_i such that $t^{k_i}v = v$ for all $v \in H/H_i$ where t denotes a generator of $\mathbb{Z}$. Note that $\mathbb{Z}/k_i \ltimes H/H_i$

and the map $\mathbb{Z} \ltimes H \to \mathbb{Z} \ltimes H/H_i$ are well-defined. We can in fact pick k_i with the extra properties that $k_i > i$ and $k_i | k_{i+1}$, then it is clear that the kernels of the maps

$$\mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda \to \mathbb{Z}/k_i \ltimes H/H_i$$

define a resolution for $\mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda$.

□

Let

$$G := \mathrm{Im}\{\beta_x : \mathbb{Z} \ltimes H_1(M,\Lambda) \to \mathbb{Z} \ltimes n^{-1}\Delta_K(t)^{-1}\Lambda/\Lambda \to \mathbb{Z} \ltimes \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]\}$$

Note that $G := \mathbb{Z} \ltimes H$ for some $H \subset \Delta_K(t)^{-1}\Lambda/\Lambda$. It follows from the proof of lemma 4.4 that we can find $H_i \subset H$ and k_i such that H/H_i is a p-group and such that the kernels G_i of

$$\mathbb{Z} \ltimes H \to \mathbb{Z}/k_i^{s_i} \ltimes H/H_i$$

form a resolution for any exponents $s_i \in \mathbb{N}$ with $1 \le s_1 \le s_2 \le \dots$. We will specify the s_i later. Using the fact that in general $\eta^{(2)}(M, \varphi : \pi_1(M) \to J) = \eta^{(2)}(M, \varphi : \pi_1(M) \to \mathrm{Im}(J))$ (cf. [COT02]) we get

$$\eta^{(2)}(M_K, \beta_x : \pi_1(M_K) \to \mathbb{Z} \ltimes \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]) = \eta^{(2)}(M_K, \beta_x : \pi_1(M_K) \to G)$$

The groups G_i are a resolution for G, hence by theorem 4.2

$$\eta^{(2)}(M_K, \beta_x : \pi_1(M_K) \to G) = \lim_{i\to\infty} \frac{\eta(M_K, G/G_i)}{|G/G_i|} = \lim_{i\to\infty} \frac{\sum_{\alpha \in \hat{R}^{irr}(G/G_i)} \dim(\alpha)\eta(M_K,\alpha)}{|G/G_i|}$$

To continue we have to understand the irreducible representations of $G/G_i \cong \mathbb{Z}/k_i^{s_i} \ltimes H/H_i$. The proof of the following lemma is the same as the proof of lemma 2.2 in [F03].

Lemma 4.5. *Let F be a finite module over $\Lambda_k := \mathbb{Z}[t]/(t^k - 1)$. Then any irreducible representation $\mathbb{Z}/k^s \ltimes F \to U(l)$ is conjugate to*

$$\alpha_{(l,z,\chi)}(n,h) = \alpha_{(z,\chi)}(n,h) := z^n \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{l-1}h) \end{pmatrix}$$

for some $z \in S^1$ with $z^k = 1$ and $\chi : F \to F/(t^l - 1) \to S^1$ a character which does not factor through $F/(t^r - 1)$ for some $r < 1$. In particular there are no irreducible representations of dimension greater than k.

Remark. Note that k_i is in general a composite number since the order of a p–group is always composite. In particular $\eta^{(2)}(M_K, \beta_x)$ is the limit of eta–invariants which are in general not of prime power dimension. This explains why the vanishing of the metabelian eta–invariant sliceness obstruction, which involves only prime power dimensional eta–invariants, does not imply the vanishing of the L^2–eta–invariant sliceness obstruction.

This lemma shows that all irreducible representations $\mathbb{Z} \ltimes H_1(M_K, \Lambda) \to G \to G/G_i \cong \mathbb{Z}/k_i^{s_i} \ltimes H/H_i \to U(l)$ are of the type $\alpha_{(z,\chi)}$ where $z^{k_i^{s_i}} = 1$ and χ is of prime power order since H/H_i is a p–group. Furthermore, since $x \in P_{\mathbb{Q}}$ and $P_{\mathbb{Q}} = P_{\mathbb{Q}}^{\perp}$ we have $\chi(P) \equiv 0$. If the z's had been transcendental our proof would be complete by now since we assumed that $\eta(M_K, \alpha_{(z,\chi)}) = 0$ for all χ of prime power order with $\chi(P) \equiv 0$ and all transcendental z.

The next two propositions show that $\eta(M_K, \alpha_{(z,\chi)}) = 0$ for almost all z. We will see that the non–zero contributions in $\frac{1}{|G/G_i|} \sum_{\alpha \in \hat{R}^{irr}(G/G_i)} \dim(\alpha)\eta(M_K, \alpha)$ vanish in the limit.

Proposition 4.6. *There exists a number C such that for any $\chi : H_1(M_K, \Lambda)/(t^k - 1) \to S^1$ of prime power order the map*

$$\begin{array}{rcl} S^1 & \to & \mathbb{Z} \\ z & \mapsto & \eta(M_K, \alpha(k, z, \chi)) \end{array}$$

has at most Ck discontinuities.

For the proof we need the following lemma.

Lemma 4.7. [L94, p. 92] *Let M^3 be a manifold, then for any $r \in \mathbb{N}$ the map*

$$\begin{array}{rcl} \eta_k : R_k(\pi_1(M)) & \to & \mathbf{R} \\ \alpha & \mapsto & \eta(M, \alpha) \end{array}$$

is continuous on $\Sigma_r := \{\alpha \in R_k(\pi_1(M)) | \sum_{i=0}^{3} \dim(H_i^{\alpha}(M, \mathbb{C}^k)) = r\}$.

Let $J := \mathbb{Z} \ltimes H_1(M_K, \Lambda)$. Denote the J-fold cover of M_K by $\hat{M}$. After triangulating M we can view

$$0 \to C_3(\hat{M}) \xrightarrow{\partial_3} C_2(\hat{M}) \xrightarrow{\partial_2} C_1(\hat{M}) \xrightarrow{\partial_1} C_0(\hat{M}) \to 0$$

as a complex of free $\mathbb{Z}J$–modules where $\operatorname{rank}(C_0(\hat{M})) = \operatorname{rank}(C_3(\hat{M})) = 1$ and $\operatorname{rank}(C_1(\hat{M})) = \operatorname{rank}(C_2(\hat{M})) = m$ for some m. Represent ∂_2 by an $m \times m$-matrix R over $\mathbb{Z}J$. Then for $\alpha \in R_k(\pi_1(M_K))$ we get

$$\det(\alpha(R)) \neq 0 \Rightarrow \alpha \in \Sigma_{2k}$$

since $H_*^{\alpha}(M, \mathbb{C}^k) = H_*(C_*(\hat{M}) \otimes_{\mathbb{Z}J} \mathbb{C}^k)$.

For a character $\chi : H_1(M_K, \Lambda) \to H_1(M_K, \Lambda)/(t^k - 1) \to S^1$ define

$$S_{k,\chi} := \{z \in S^1 | \det(\alpha_{(k,z,\chi)}(R)) = 0\}$$

Lemma 4.8. *There exists a number C such that $|S_{k,\chi}| \leq Ck$ for all χ of prime power order.*

Proof. Denote by $f : \mathbb{Z}[J] \to \mathbb{Z}[t, t^{-1}]$ the map induced by $(n, v) \mapsto t^n$. For $g = \sum_{i=n_0}^{n_1} a_i t^i, a_{n_0} \neq 0, a_{n_1} \neq 0$ define $\deg(g) = n_1 - n_0$. Let $C := m \max\{\deg(f(R_{ij}))\}$. Given a character χ denote by z a variable, then $D(z) := \alpha_{(z,\chi)}(R)$ is a $km \times km$–matrix over $\mathbb{C}[z, z^{-1}]$. It's clear that $\deg(\det(D(z))) \leq \frac{C}{m} km = Ck$, hence either

$\det(D(z)) \equiv 0$ or there are at most Ck z's which are zeroes of $\det(D(z))$. Letsche [L00, cor. 3.10] showed that for any χ of prime power order $S_{k,\chi}$ does not contain any transcendental number, in particular $\det(D(z))$ is not identically zero. □

This lemma proves proposition 4.6.

Proposition 4.9. *For each k there exists $D_k \in \mathbf{R}$ such that*

$$|\eta(M_K, \alpha)| \leq D_k$$

for all $\alpha \in R_l(\pi_1(M_K))$ and all $l \leq k$.

Proof. Let

$$\tilde{\Sigma}_r := \{\alpha \in R_k(\pi_1(M)) | \sum_{i=0}^{3} \dim(H_i^\alpha(M, \mathbb{C}^k)) \geq r\}$$

Levine [L94, p. 92] shows that these are subvarieties of $R_k(\pi_1(M))$, that $\tilde{\Sigma}_N = \emptyset$ for some N and that η_k is continuous on $\tilde{\Sigma}_r \setminus \tilde{\Sigma}_{r+1}$ for all r.

We claim that η_k is bounded on each $\tilde{\Sigma}_r$. Note that $\tilde{\Sigma}_r \setminus \tilde{\Sigma}_{r+1}$ has only finitely many components since $\tilde{\Sigma}_{r+1}$ is a subvariety. If η_k is not bounded on $\tilde{\Sigma}$ then it is therefore not bounded on at least one component C of $\tilde{\Sigma}_r \setminus \tilde{\Sigma}_{r+1}$.

Since $\pi_1(M_K)$ is finitely generated it follows that $R_l(\pi_1(M_K))$ compact, hence $\bar{C} \subset \tilde{\Sigma}_r$ is compact too. We can therefore find a sequence $p_i \in C$ such that p_i converges to some point $p \in \bar{C}$ and such that $\lim_{i\to\infty} \eta_k(p_i) = \infty$. Since C is path connected and locally path connected we can find a curve $\gamma : [0,1] \to C$ such that $\gamma(1 - \frac{1}{2^i}) = p_i$. Note that $\gamma(p[0,1]) = [D, \infty)$ for some D. In particular we can find sequences q_i and r_i in $\tilde{\Sigma}_r \setminus \tilde{\Sigma}_{r+1}$ converging to point p with $\eta(q_i) = i + \frac{1}{2}$ and $\eta(r_i) = i$. But this is a contradiction to the fact, established by Levine [L94, p. 92], that $\eta_k \mod \mathbb{Z} : R_k(\pi_1(M)) \to \mathbf{R}/\mathbb{Z}$ is continuous. □

We are now ready to show that $\eta^{(2)}(M_K, \beta_x) = 0$ for any $x \in P_{\mathbb{Q}}$ which proves of theorem 4.1. Recall that we have to show that

$$\lim_{i\to\infty} \frac{\eta(M_K, G/G_i)}{|G/G_i|} = 0$$

We pick s_i with the extra property $k_i^{s_i - 4} \geq D_{k_i}$ for all i. Using lemma 4.2 we get

$$|\eta(M_K, G/G_i)| \leq \sum_{\alpha \in \hat{R}^{irr}(G/G_i)} \dim(\alpha) |\eta(M, \alpha)|$$

Recall that $G/G_i \cong \mathbb{Z}/k_i^{s_i} \ltimes H/H_i$ and that H/H_i is a p-group. By definition of k_i any character actually factors through $(H/H_i)/(t^{k_i} - 1)$. In particular by lemma 4.5 there are no irreducible representations of dimension bigger than k_i. It now follows that the above term is in fact less or equal than

$$\sum_{j=1}^{k_i} j \sum_{\alpha \in \hat{R}_j^{irr}(G/G_i)} |\eta(M_K, \alpha)| \leq \sum_{j=1}^{k_i} j \sum_{\chi:(H/H_i)/(t^j-1)\to S^1} \sum_{z\in S^1, z^{k_i^{s_i}}=1} |\eta(M_K, \alpha(j,z,\chi))|$$

From corollary 4.6 and using that $\eta(M_K, \alpha_{(z,\chi)})$ for all transcendental z and all χ of prime power order with $\chi(P) \equiv 0$, it follows that $\eta(M_K, \alpha(z,\chi)) = 0$ for all but at most Ck_i values of z. Using this observation and using proposition 4.9 we get that the above term is less or equal than

$$\sum_{j=1}^{k_i} j \sum_{\chi:(H/H_i)/(t^j-1)\to S^1} CjD_{k_i} \leq k_i^3 C|H/H_i|D_{k_i}$$

Therefore

$$|\eta^{(2)}(M_K, \beta_x)| = \left| \lim_{i\to\infty} \frac{\eta(M_K, G/G_i)}{|G/G_i|} \right| \leq \lim_{i\to\infty} \frac{k_i^3 C D_{k_i}|H/H_i|}{k_i^{s_i}|H/H_i|} = \lim_{i\to\infty} \frac{k_i^3}{k_i^4}\frac{CD_{k_i}}{k_i^{s_i-4}} = 0$$

since $\lim_{i\to\infty} k_i = \infty$ and by the choice of s_i. This concludes the proof of theorem 4.1.

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