Twisted Alexander polynomials - an overview

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Goal: determine the genus $g(K)$ of a given knot

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Goal: determine the free genus $g_{free}(K)$ of a knot.

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Goal: determine which knots are slice.

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This approach is very effective for knots but does not generalize well to 3-manifolds.

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The Alexander polynomial of the trefoil knot equals $t^{-1} - 1 + t$.

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so the Alexander polynomial is not a complete invariant of knots

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(For K a null-homologous knot in a homology sphere Σ we have $\Delta_K(1) = |H_1(\Sigma)|$)

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(This is a consequence of $\Delta_K(t) = \det(At - A^t)$ where A can be a Seifert matrix of size $2g(K) \times 2g(K)$).

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(If $D \subset D^4$ is a slice disk, this follows from Poincaré duality applied to the pair $(D^4 \setminus D, S^3 \setminus K)$)

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i.e. the Alexander polynomial does not distinguish between mirror images

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(12) The Alexander polynomial of a periodic knot has a special form

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(There are more refined definitions with smaller indeterminacy.)

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(this was shown by Silver-Williams 2005 and F-Vidussi 2005)

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(5b) A refinement of TAPs can detect mirror images
(examples are given by Kirk-Livingston 1996)

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(6) If the representation is unitary, then the TAP is symmetric
(a consequence of Poincaré duality, shown by Kitano 1996)

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(shown by Cha 2001, Goda-Kitano-Morifuji 2001, F-Kim 2004)

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(1) Δ_K^α lies in $R[t^{\pm 1}]$ and is well-defined up to a unit in $R[t^{\pm 1}]$.
(2) TAP can be computed from Fox calculus or torsion
(3) The twisted Alexander polynomial (TAP) is one for the unknot
(4) Any non-trivial knot admits a representation with $\Delta_K^\alpha \neq 1$
(5) TAPs detect mutation and chirality
(6) If the representation is unitary, then the TAP is symmetric
(7) TAP gives lower bounds on the genus and free genus
(8) The TAP of a fibered knot is monic
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(15) More work done by: Cochran, Cogolludo, Dubois, Florens, Harvey, Hirasawa, Horie, Huynh, Le, Matsumoto, Murasugi, Pajitnov, Tamulis, Turaev, Yamaguchi.

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(3) The Alexander polynomial of a knot or 3-manifold corresponds to Seiberg-Witten invariants, and TAPs corresponding to regular representations correspond to Seiberg-Witten invariants of finite covers.