Fibered 3–manifolds and twisted Alexander polynomials

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The Alexander polynomial of fibered knots

Let $K \subset S^3$ a knot, then consider the Alexander module $H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$. Pretending that $\mathbb{Z}[t^{\pm 1}]$ is a PID we have

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The converse does not hold, e.g. fails for the Pretzel knot $P(5, -3, 5)$.

Fibered manifolds

Throughout N will be closed or with toroidal boundary.

Definition For $\phi \in H_2(N, \partial N; \mathbb{Z}) = H^1(N; \mathbb{Z}) = \mathrm{Hom}(\pi_1(N), \mathbb{Z})$ we say that (N, ϕ) fibers if there exists a fibration $p : N \to S^1$ with $p_* = \phi : \mathrm{Hom}(\pi_1(N), \pi_1(S^1))$.

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Remark. Let $p : \tilde{N} \to N$ a finite cover. Then

$$(N, \phi) \text{ fibers} \Leftrightarrow (\tilde{N}, p^{-1}(\phi)) \text{ fibers.}$$

Definition. Let N a 3–manifold, $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ and $\alpha : \pi_1(N) \to G$ an epimorphism onto a finite group.

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$$H_1(N;\mathbb{Z}[G][t^{\pm 1}]) = H_1(C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[G][t^{\pm 1}]).$$

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Example. If $N = S^3 \setminus K$, $\phi \in H^1(S^3 \setminus K;\mathbb{Z}) \cong \mathbb{Z}$ a generator and α the trivial map, then $\Delta^{\alpha}_{N,\phi} = \Delta_K$ is the ordinary Alexander polynomial of the knot K.

Twisted Alexander polynomials and fibered manifolds.

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The following is due to Cha, Goda–Kitano–Morifuji, F–Taehee Kim, Kitayama:

Theorem. If (N,ϕ) fibers, then for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group we have that
(M) $\Delta^{\alpha}_{N,\phi}$ is monic,
(D) the degree of $\Delta^{\alpha}_{N,\phi}$ is determined by $||\phi||_T$.

The main theorem

Our main theorem says that twisted Alexander polynomials detect fibered 3–manifolds.

Theorem (F–Vidussi). Let N be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ such that for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group we have that
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Before we outline the proof we give a corollary.

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Question. Does the converse hold?

The following generalizes a result of Kronheimer.

Theorem (F–Vidussi). If $S^1 \times N$ is symplectic, then there exists $\phi \in H^1(N;\mathbb{Z})$ such that for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group we have that
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$$\begin{aligned} SW(S^1 \times \tilde{N}) &= \text{Alexander polynomial of } \tilde{N} \\ &= \text{twisted Alexander polynomial of } N. \end{aligned}$$

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So combining our results we get:

Theorem (F–Vidussi 2008). Let N be a closed 3–manifold. If $S^1 \times N$ is symplectic, then N is fibered.

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In fact using constructions of symplectic forms in an earlier paper (generalizing Thurston and Fernandez–Gray–Morgan) we can completely determine the symplectic cone:

Theorem. $a \in H^2(S^1 \times N; \mathbb{R}) \cong H^1(N; \mathbb{R}) \oplus H^2(N; \mathbb{R})$ can be represented by a symplectic form if and only if
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(2) the Künneth component of a in $H^1(N; \mathbb{R})$ lies in the interior of a fibered cone.

The main theorem

Recall that our main theorem says that twisted Alexander polynomials detect fibered 3–manifolds.

Theorem (F–Vidussi). Let N be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ such that for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group we have that
(M) $\Delta^{\alpha}_{N,\phi}$ is monic,
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then (N,ϕ) fibers over S^1.

The three main ingredients of the proof.

Throughout assume we have a 3–manifold N and $\phi \in H^1(N;\mathbb{Z})$ primitive. Let $\Sigma \subset N$ be a connected minimal genus surface dual to ϕ.

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Theorem A. If (N, ϕ) satisfies (M) and (D) for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group, then for either inclusion $\iota_\pm$ the map $\pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ induces an isomorphism of prosolvable completions.

Note that $\varphi : A \to B$ induces an isomorphism of prosolvable completions if and only if for any finite solvable group S we have a bijection $\varphi^* : \mathrm{Hom}(B, S) \to \mathrm{Hom}(A, S)$, and if for any $\beta : B \to S$ we have $\mathrm{Im}(A \to B \to S) = \mathrm{Im}(B \to S)$.

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Building on a result of Agol we show:

Theorem C. If $\pi_1(N \setminus \nu\Sigma)$ is residually finite solvable and if $\iota_\pm : \pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ induce an isomorphism of prosolvable completions, then $\pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ is an isomorphism.

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The combination of the theorems shows that (M) and (D) imply that (N,ϕ) fibers over S^1.

Proof of Theorem A.

Assume we have (N, ϕ) such that for any $\alpha : \pi_1(N) \to G$ we have that
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We claim that $\iota_\pm : \pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ induces an isomorphism of prosolvable completions,i.e. for any finite solvable group S the map $\iota^*_\pm : \mathrm{Hom}(\pi_1(N \setminus \nu\Sigma), S) \to \mathrm{Hom}(\pi_1(\Sigma), S)$ is a bijection and for any $\beta : \pi_1(N \setminus \nu\Sigma) \to S$ we have
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$\mathrm{Im}(\pi_1(\Sigma) \to \pi_1(N \setminus \Sigma) \to S) = \mathrm{Im}(\pi_1(N \setminus \Sigma) \to S)$.
Put differently, we have to get from Alexander polynomials to information on the relation between a surface and its complement.

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Assume we have (N,ϕ) such that for any $\alpha:\pi_1(N)\to G$ we have that
(M) $\Delta^\alpha_{N,\phi}$ is monic (D) the degree of $\Delta^\alpha_{N,\phi}$ determined by $||\phi||_T$.
We claim that $\iota_\pm:\pi_1(\Sigma)\to\pi_1(N\setminus\nu\Sigma)$ induces an isomorphism of prosolvable completions,i.e. for any finite solvable group S the map $\iota_\pm^*:\mathrm{Hom}(\pi_1(N\setminus\nu\Sigma),S)\to\mathrm{Hom}(\pi_1(\Sigma),S)$ is a bijection and for any $\beta:\pi_1(N\setminus\nu\Sigma)\to S$ we have
$\mathrm{Im}(\pi_1(\Sigma)\to\pi_1(N\setminus\Sigma)\to S)=\mathrm{Im}(\pi_1(N\setminus\Sigma)\to S)$.
Put differently, we have to get from Alexander polynomials to information on the relation between a surface and its complement.

As for the classical Alexander polynomial we can express the twisted Alexander module in terms of a Meyer–Vietoris sequence relating it to the homology of Σ and $N\setminus\nu\Sigma$:

$$H_1(\Sigma;\mathbb{Z}G)\otimes\mathbb{Z}[t^{\pm1}]\xrightarrow{\iota_--t\iota_+}H_1(N\setminus\nu\Sigma;\mathbb{Z}G)\otimes\mathbb{Z}[t^{\pm1}]\to H_1(N;\mathbb{Z}[G][t^{\pm1}])$$

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Note that this is only information for homomorphisms from $\pi_1(N)$ to a finite group.

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Theorem A now follows from this data and lots of elementary group theory.

Proof of Theorem B.

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The following is a more precise version of Fact B:

Theorem B. For any 3–manifold there exists a finite cover such that the fundamental groups of all JSJ pieces are residually p.

Proof of Theorem C (1).

Let $\Sigma \subset N$. We have two inclusions $\iota_\pm : \Sigma \to N \setminus \nu\Sigma$.

Theorem C. If $\pi_1(N \setminus \nu\Sigma)$ is residually finite solvable and if

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Definition. A group π is called RFRS if π is residually finite solvable and 'the rank of finite index subgroups grows quickly with the index'.

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Remark.
(1) Free groups and surface groups are RFRS.
(2) Most 3–manifold groups are not RFRS.
(3) Are hyperbolic 3–manifold groups virtually RFRS?
(4) The $\pi_1(N \setminus \nu\Sigma)$ above is RFRS since 'solvably' $\pi_1(N \setminus \nu\Sigma)$ looks the same as a surface group.

Proof of Theorem C (2).

Agol's amazing theorem:

Theorem (Agol). Let W 3–manifold, $\pi_1(W)$ RFRS and $\phi \in H^1(W)$. Then there exists a finite solvable cover $p : \hat{W} \to W$ such that $p^*(\phi)$ lies on the closure of a fibered cone.

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