Twisted Alexander polynomials and symplectic manifolds

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Abstract. We show how twisted Alexander polynomials can be used to study the existence of symplectic structures on manifolds of the form $S^1 \times M^3$.

Symplectic structures

Let W be a 4–manifold. A 2–form ω is called a symplectic structure on W if $d\omega = 0$ and if ω is non–degenerate, i.e. $\omega \wedge \omega \neq 0$ everywhere.

Question 1. When does $S^1 \times M^3$ support a symplectic structure?

Theorem 1. (Thurston 1976) If M fibers over S^1, then $S^1 \times M$ is symplectic.

Proof. Let $p : M \to S^1$ be a fibration. Define $\theta = p^*(dt)$. This is a nowhere–vanishing 1–form. Give M a metric, 'small on the fibers'. Then we can find a harmonic, nowhere–vanishing 1–form θ_h on M with $[\theta_h] = [\theta] \in H^1(M;\mathbb{R})$.

Then

$$\omega = ds \wedge \theta_h + *\theta_h$$

is a symplectic form on $S^1 \times M$ since

$$\omega \wedge \omega = 2ds \wedge \theta_h \wedge *\theta_h.$$

Conjecture 1. $S^1 \times M$ is symplectic if and only if M fibers over S^1.

Let ω be symplectic structure on $S^1 \times M$. We can assume that $[\omega] \in H_2(S^1 \times M; \mathbb{Z})$. Write

$$[\omega] = dt \wedge \theta + \eta$$

in $H^2(S^1 \times M) = H^1(S^1) \otimes H^1(M) \oplus H^2(M)$. Write $\theta(\omega) = \theta$ for the Künneth component of ω in $H^1(M)$.

We get a more precise formulation of Conjecture 1.

Conjecture 2. If ω is a symplectic structure on $S^1 \times M$, then $(M, \theta(\omega))$ fibers over S^1, i.e. there exists a fibration $p : M \to S^1$ with $\theta(\omega) = p^*(1) \in H^1(M)$.

Alexander polynomials

Let $\phi \in H^1(M;\mathbb{Z})$. Consider the $\mathbb{Z}[t^{\pm 1}]$–module

$$H_1(M;\mathbb{Z}[t^{\pm 1}]).$$

Similarly to the Alexander polynomial of a knot we get an Alexander polynomial

$$\Delta_{M,\phi} \in \mathbb{Z}[t^{\pm 1}]$$

(defined as the order of $H_1(M;\mathbb{Z}[t^{\pm 1}])$).

Theorem 2. If (M,ϕ) fibers over S^1, then $\Delta_{M,\phi}$ is monic and

$$\deg(\Delta_{M,\phi}) = ||\phi||_{M,T} + 2\mathrm{div}(\phi).$$

Here $||\phi||_T$ denotes the Thurston norm (a generalization of knot genus) and div(ϕ) denotes the divisibility. We say Δ is monic if the top coefficient is ± 1.

Alexander polynomials and symplectic structures

Theorem 3. (Meng–Taubes, Kronheimer, Vidussi) If ω is a symplectic form on $S^1 \times M$, then $\Delta_{M,\phi(\omega)}$ is monic and

$$\deg(\Delta_{M,\phi(\omega)}) = ||\phi(\omega)||_{M,T} + 2\mathrm{div}(\phi(\omega)).$$

Proof.

1. Taubes' results on Seiberg–Witten invariants of symplectic manifolds.

2. Meng–Taubes showed that SW–invariants for $S^1 \times M$ correspond to the multivariable Alexander polynomial of M.

3. Get $\Delta_{M,\phi}$ from multivariable Alexander polynomial.

4. Result on the degree uses Donaldson's theorem on symplectic submanifolds dual to (multiples of) ω and the adjunction inequality.

Examples. Let $K \subset S^3$ a knot. Write M_K for zero–framed surgery along K. Note that we have a (up to sign) unique generator $\phi \in H^1(M_K)$. Then $\Delta_{M_K,\phi} = \Delta_K$, the Alexander polynomial of the knot K.

The Alexander polynomial detects most non–fibered knots. Hence Theorem 3 gives good evidence towards Conjecture 2.

Now consider the Pretzel knot $P = P(5, -3, 5)$.

P has genus one and $\Delta_P = t^2 - 3t + 1$, but P is not fibered. Kronheimer asked whether $S^1 \times M_P$ is symplectic. Here Theorem 3 is inconclusive.

Finite covers

Idea. Apply Theorem 3 to finite covers of $S^1 \times M$.

Let $\alpha : \pi_1(M) \to G$ be an epimorphism to a finite group G. Let

$$p : M_G \to M$$

be the induced cover. Then $p^*(\omega)$ is a symplectic form on $S^1 \times M_G$ with $\phi(p^*(\omega)) = p^*(\phi(\omega))$. By Gabai

$$||p^*(\phi(\omega))||_{M_G,T} = |G| \, ||\phi(\omega)||_{M,T}.$$

By the theorem $\Delta_{M_G,p^*(\phi(\omega))}$ is monic and

$$\deg(\Delta_{M_G,p^*(\phi(\omega))}) = |G| \, ||\phi(\omega)||_{M,T} + 2\mathrm{div}(p^*(\phi(\omega))).$$

We rewrite this using twisted Alexander polynomials.

Twisted Alexander polynomials

Let $\phi \in H^1(M)$ and let $\alpha : \pi_1(M) \twoheadrightarrow G$ be an epimorphism to a finite group G. We get a twisted homology $\mathbb{Z}[t^{\pm 1}]$–module

$$H_1(M; \mathbb{Z}[G][t^{\pm 1}]).$$

Let $\Delta^G_{M,\phi}$ be the corresponding twisted Alexander polynomial.

This polynomial can be computed using Fox calculus.

The following lemma shows that twisted Alexander polynomials are just Alexander polynomials of covers.

Lemma 1.

$$\Delta^G_{M,\phi} = \Delta_{M_G, p^*(\phi)}.$$

Main theorem

Theorem 4.(F–Vidussi) Let ω be a symplectic form on $S^1 \times M$. Let $\alpha : \pi_1(M) \twoheadrightarrow G$ be an epimorphism to a finite group. Then $\Delta^G_{M,\phi(\omega)}$ is monic and

$$\deg(\Delta^G_{M,\phi}) = |G| \, ||\phi(\omega)||_{M,T} + 2\mathrm{div}(p^*(\phi(\omega))).$$

1. $\mathrm{div}(p^*(\phi(\omega)))$ can be computed efficiently.

2. A similar statement holds for fibered manifolds.

3. 3–manifolds have 'many' epimorphisms to finite groups: if M is irreducible and $H^1(M) \neq 0$, then $\pi_1(M)$ is residually finite, i.e. for any $g \in \pi_1(M)$ there exists a finite group G and $\alpha : \pi_1(M) \to G$ such that $\alpha(g) \neq e \in G$.

Examples

Consider again the Pretzel knot $P = P(5, -3, 5)$. We found an epimorphism $\pi_1(M_P) \to S_5$ such that $\Delta^{S_5}_{M_P,\phi}$ is not monic. Therefore $S^1 \times M_P$ is not symplectic by Theorem 4.

There exist 13 knots K with up to 12 crossings which are not fibered, but such that Δ_K is monic and such that $\text{genus}(K) = \deg(\Delta_K(t))$. In all cases we can find $\alpha : \pi_1(M_P) \to G$ such that $\Delta^G_{M_P,\phi}$ is not monic.

Conjecture

We propose the following conjecture.

Conjecture 3. Let $\phi \in H^1(M;\mathbb{Z})$. Then (M,ϕ) fibers over S^1 if and only if for all $\pi_1(M) \to G$ we have $\Delta^G_{M,\phi(\omega)}$ is monic and

$$\deg(\Delta^G_{M,\phi}) = |G|\,||\phi(\omega)||_{M,T} + 2\mathrm{div}(p^*(\phi(\omega))).$$

Note that by Theorem 4 this conjecture implies in particular Conjecture 2.

Possible approach to Conjecture 3

Assume we have the conditions of Conjecture 3. If Thurston's geometrization holds then this implies that M is irreducible.

Now assume M irreducible. Let $S \subset M$ be a Thurston norm minimizing surface dual to ϕ. Since $\Delta_{M,\phi} \neq 0$ we can, by McMullen, assume S to be connected.

Let $N = M \setminus S \times (-1, 1)$. We have two embeddings

$$i_\pm : S \to \partial N.$$

By Stallings' theorem, S is a fiber if and only if $i_\pm : \pi_1(S) \to \pi_1(N)$ are isomorphisms.

Since S is Thurston norm minimizing we know that $i_\pm : S \to \partial M$ is injective.

Consider the long exact sequence

$$H_2(M;\mathbb{Z}[t^{\pm 1}])$$

$$\to H_1 S \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{i_- t - i_+} H_1 N \otimes \mathbb{Z}[t^{\pm 1}] \to H_1(M;\mathbb{Z}[t^{\pm 1}])$$

$$\to H_0 S \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{i_- t - i_+} H_0 N \otimes \mathbb{Z}[t^{\pm 1}] \to H_0(M;\mathbb{Z}[t^{\pm 1}])$$

(with $\mathbb{Z}[G]$–coefficients understood)

The condition on $\Delta^G_{M,\phi}$ ensures that for any $\alpha : \pi_1(M) \to G$ the maps

$$\begin{array}{rcl} i_\pm : H_1(S;\mathbb{Z}[G]) & \mapsto & H_1(N;\mathbb{Z}[G]) \\ i_\pm : H_0(S;\mathbb{Z}[G]) & \mapsto & H_0(N;\mathbb{Z}[G]) \end{array}$$

are isomorphisms.

Clearly the first isomorphism is interesting, but the second isomorphism also gives information on $\pi_1(S)$ and $\pi_1(N)$ since

$$\text{rank}(H_0(S;\mathbb{Z}[G])) = \mathbb{Z}[G/\alpha(\pi_1(S))].$$

Theorem(F–Vidussi) Assume ω is a symplectic structure on $S^1 \times M$, M irreducible.

1. If M has zero Thurston norm, then $(M, \phi(\omega))$ fibers over S^1.
2. All tori in the JSJ–decomposition fiber over S^1 w.r.t. $\phi(\omega)$.
3. All Seifert fibered pieces in the JSJ–decomposition fiber over S^1 w.r.t. $\phi(\omega)$.
4. If $\pi_1(M)$ is subgroup separable, then $(M, \phi(\omega))$ fibers over S^1.

Here a group π is called subgroup separable if for any subgroup $A \subset \pi$ and any $g \in \pi \setminus A$ there exists a homomorphism $\alpha : \pi \to G$, G a finite group, such that $\alpha(g) \notin \alpha(A)$. 3–manifolds groups are in general not subgroup separable, but it is an open question whether hyperbolic groups separate (surface) subgroups.

Surprisingly, we only need that $\Delta^G_{M,\phi(\omega)}$ is non–zero for every $\alpha : \pi_1(M) \to G$.