Complexity of surfaces in 4-manifolds with a free circle action

Stefan Friedl (joint with Stefano Vidussi)

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$$n\phi = p_* : \pi_1(N) \to \mathbb{Z} \text{ for some } n \in \mathbb{N}.$$

Surfaces in 4-manifolds

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Surfaces in 4-manifolds of the form $S^1 \times N$ I

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(3) Our theorem extends to S^1-bundles over such 3-manifolds.

Proof of the theorem I

The following is the 'great miracle of 3-manifold topology':

Theorem. (Agol, Przytycki-Wise, Wise - 2012) If N^3 is irreducible and not a graph manifold, then $\pi_1(N)$ is virtually a subgroup of a RAAG.

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An alternative proof of Agol's theorem is also given by F-Kitayama.

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