Examples of topologically slice knots

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Abstract. In the early 1980's Mike Freedman showed that all knots with trivial Alexander polynomial are topologically slice. In this talk we give the first new examples of topologically slice knots. We give a condition in terms of a non–commutative Blanchfield pairing for a knot to be topologically slice with fundamental group $\mathbb{Z} \ltimes \mathbb{Z}[1/2]$.

Slice knots

A knot $K \subset S^3$ is called (topologically) slice if it bounds a locally flat disk in D^4. We say K is *smoothly* slice if K bounds a smooth disk in D^4.

As an example consider the following class of knots.

Here C can be any knot, tied into the band. Such a knot bounds a 'ribbon', its only intersection in $\mathbb{R}^3$ are of the following type.

Pushing the interior of the horizontal ribbon into D^4 we get a smooth slice disk for K.

The high–dimensional case

Kervaire showed in 1965 that any even–dimensional knot is slice, more precisely, any knot $S^{2k} \cong K \subset S^{2k+2}, k \geq 1$ is slice.

In 1969 Levine classified the high–dimensional odd–dimensional knots. We give Kearton's version of the classification.

Theorem A knot $S^{2k+1} \cong K \subset S^{2k+3}, k \geq 1$ is slice if and only if the Blanchfield pairing

$$H_{k+1}(M_K, \mathbb{Z}[t^{\pm 1}]) \times H_{k+1}(M_K, \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

is metabolic.

1. M_K denotes the result of surgery along the knot K.

2. the Blanchfield pairing can be viewed as the odd–dimensional analogue of the intersection pairing on even–dimensional manifolds.

The classical case

In 1975 Casson and Gordon gave examples of knots $K \subset S^3$ such that the Blanchfield pairing is metabolic, but which are not slice.

This result proved in particular that the Whitney trick does not work in dimension 4.

In recent years Cochran, Orr and Teichner built an extensive sliceness obstruction theory.

On the positive side Freedman proved the following result:

Theorem If $\Delta_K(t) = 1$, then K is topologically slice.

Here $\Delta_K(t)$ denotes the Alexander polynomial of a knot K.

Note that Gompf gave examples of knots with $\Delta_K(t) = 1$ which are not *smoothly slice*.

The main theorem

Theorem(F – Teichner) Let $K \subset S^3$ be a knot and $G = \mathbb{Z}$ or $G = \mathbb{Z} \ltimes \mathbb{Z}[1/2]$.
Then there exists a (topological) slice disk D for K with $\pi_1(D^4 \setminus D) = G$
if and only if there exists an epimorphism $\varphi : \pi_1(M_K) \to G$ such that the Blanchfield form

$$H_1(M_K, \mathbb{Z}[G]) \times H_1(M_K, \mathbb{Z}[G]) \to K(G)/\mathbb{Z}[G]$$

vanishes.

Here $K(G)$ denotes the skew quotient field of G which exists since G is torsion–free and solvable.

M_K denotes the zero framed surgery along the knot K.

The main theorem in the case $G = \mathbb{Z}$

Theorem Let K be a knot.
Then there exists a (topological) slice disk D for K with $\pi_1(D^4 \setminus D) = \mathbb{Z}$
if and only if there exists an epimorphism $\varphi : \pi_1(M_K) \to \mathbb{Z}$ such that the Blanchfield form

$$H_1(M_K, \mathbb{Z}[\mathbb{Z}]) \times H_1(M_K, \mathbb{Z}[\mathbb{Z}]) \to K(\mathbb{Z})/\mathbb{Z}[\mathbb{Z}]$$

vanishes.

In the case $G = \mathbb{Z}$ we have $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$, $K(\mathbb{Z}) = \mathbb{Q}(t)$ and this theorem simplifies to Freedman's theorem since the Blanchfield form

$$H_1(M_K, \mathbb{Z}[t^{\pm 1}]) \times H_1(M_K, \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

vanishes if and only if $H_1(M_K, \mathbb{Z}[t^{\pm 1}]) = 0$ which in turn happens precisely when $\Delta_K(t) = 1$.

Example with $G = \mathbb{Z} \ltimes \mathbb{Z}[1/2]$

Consider again the following knot. Let D be the slice disk from above. Then $D^4 \setminus \nu D$ has the following description in Kirby calculus:

Then

$$\pi_1(D^4 \setminus \nu D) \cong \langle a, b \mid a^{-1}ba^{-1}bab^{-1} \rangle \cong \mathbb{Z} \ltimes \mathbb{Z}[1/2].$$

Note that $\partial(D^4 \setminus \nu D) = M_K$. Consider the map $\pi_1(M_K) \to \pi_1(D^4 \setminus \nu D) \cong \mathbb{Z} \ltimes \mathbb{Z}[1/2] =: G$. Then the Blanchfield form

$$H_1(M_K, \mathbb{Z}[G]) \times H_1(M_K, \mathbb{Z}[G]) \to K(G)/\mathbb{Z}[G]$$

vanishes on $\mathrm{Ker}(H_1(M_K, \mathbb{Z}[G]) \to H_1(D^4 \setminus \nu D, \mathbb{Z}[G]))$. But $H_1(D^4 \setminus \nu D, \mathbb{Z}[G]) = 0$, hence the Blanchfield form vanishes on all of $H_1(M_K, \mathbb{Z}[G])$.

We can show that $H_1(M_K, \mathbb{Z}[G]) \neq 0$, hence the condition that the Blanchfield form vanishes is weaker than the condition $H_1(M_K, \mathbb{Z}[G]) = 0$.

New examples

To get new examples of topologically slice knots we take the knot K as above and alter it using a satellite construction.

More precisely, we put K in a torus as follows.

Now wrap this torus around any other knot (e.g. the trefoil) and call the resulting knot S.

There exists a map $\pi_1(M_S) \to \pi_1(M_K)$ such that for $\pi_1(M_S) \to \pi_1(M_K) \to G = \mathbb{Z} \ltimes \mathbb{Z}[1/2]$ we have

$$\begin{aligned} & (H_1(M_S, \mathbb{Z}[G]), \text{Blanchfield form}) \\ \cong \ & (H_1(M_K, \mathbb{Z}[G]), \text{Blanchfield form}). \end{aligned}$$

In particular the Blanchfield of M_S vanishes as well, and S is topologically slice by our main theorem.

We conjecture that S is not smoothly slice.

Non–commutative invariants

Our main theorem gives a criterion in terms of a non–commutative Blanchfield form. This criterion can not be simplified to abelian invariants. In fact Livingston and Taehee Kim showed that for any Seifert matrix with non–trivial Alexander polynomial there exist non–slice knots which have this Seifert matrix.

High and low–dimensions

Theorem A knot $S^{2k+1} \cong K \subset S^{2k+3}, k \geq 1$ is slice if and only if the Blanchfield pairing

$$H_{k+1}(M_K, \mathbb{Z}[t^{\pm 1}]) \times H_{k+1}(M_K, \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

is metabolic.

Theorem Let $K \subset S^3$ be a knot and $G = \mathbb{Z}$ or $G = \mathbb{Z} \ltimes \mathbb{Z}[1/2]$. Then there exists a (topological) slice disk D for K with $\pi_1(D^4 \setminus D) = G$ if and only if there exists an epimorphism $\varphi : \pi_1(M_K) \to \mathbb{Z} \ltimes \mathbb{Z}[1/2]$ such that the Blanchfield form

$$H_1(M_K, \mathbb{Z}[G]) \times H_1(M_K, \mathbb{Z}[G]) \to K(G)/\mathbb{Z}[G]$$

vanishes.

In the case $K \subset S^3$ the condition is to find an epimorphism to an appropriate group such that a homological condition holds.

The obstruction no longer lies in a Witt group but in the existence of an appropriate epimorphism.

Outline of the proof: Reduction to surgery

Proposition A knot K is slice if and only if M_K bounds a 4-manifold W^4 with

1. $\pi_1(W)$ is normally generated by the image of a meridian for K,
2. $H_1(W) \cong \mathbb{Z}$,
3. $H_2(W) = 0$,

If K is slice, D a slice disk, then $W = D^4 \setminus \nu D$ has the required properties.

Conversely, consider W union a 2-handle along a meridian for K in M_K. By Freedman's solution of the topological Poincaré conjecture in dimension 4 this is homeomorphic to D^4.

The 2–handle in D^4 is a (thickened) slice disk for K.

Outline of the proof (2)

Assume from now on we have an epimorphism $\pi_1(M_K) \to G$, $G = \mathbb{Z}$ or $G = \mathbb{Z} \ltimes \mathbb{Z}[1/2]$. Let $X := K(G, 1)$, we get an induced map $M_K \to X$. Then

1. $\pi_1(X)$ is normally generated by the image of a meridian for K,

2. $H_1(X) \cong \mathbb{Z}$,

3. $H_2(X) = 0$,

Outline of the proof (3)

The main technical result is the following:

Proposition (X, M_K) is a Poincaré pair if and only if the Blanchfield form

$$H_1(M_K, \mathbb{Z}[G]) \times H_1(M_K, \mathbb{Z}[G]) \to K(G)/\mathbb{Z}[G]$$

vanishes.

In the proof we use that $K(G,1)$ in both cases is a 2–complex.

Furthermore there exists a degree one normal map from a (smooth) manifold pair

$$(N, M_K) \to (K(G,1), M_K).$$

Outline of the proof (4)

By Freedman–Teichner surgery works in dimension 4 for solvable groups. More precisely, the surgery sequence

$$\mathcal{S}^h_{TOP}(X,M) \Rightarrow \mathcal{N}_{TOP}(X,M) \Rightarrow L^h_4(\mathbb{Z}[G])$$

is exact for all Poincaré pairs (X,M) with $\pi_1(X) = G$.

We compute that in the case $G = \mathbb{Z}$ or $G = \mathbb{Z} \ltimes \mathbb{Z}[1/2]$ we have $L^h_4(\mathbb{Z}[G]) \cong \mathbb{Z}$, the isomorphism is given by the signature. Adding copies of Freedman's E_8–manifold to N we can assume that N represents the zero element in $L^h_4(\mathbb{Z}[G])$.

Therefore we can find a manifold W with the homotopy type of X, bounding M, hence by the above criterion K is topologically slice.

Summary of conditions on G

What did we really use in our assumption that $G = \mathbb{Z}$ or $G = \mathbb{Z} \ltimes \mathbb{Z}[1/2]$?

1. G is solvable, hence 'good', i.e. 4–dimensional surgery works.

2. $K(G, 1)$ is a 2–complex with $H_2(G) = 0$.

3. $L_4^h(\mathbb{Z}[G]) = \mathbb{Z}$, i.e. the surgery obstruction vanishes.

4. $\mathbb{Z}[G]$ has a quotient field $K(G)$ which we used to define the Blanchfield form.

Ribbon groups

A group G is called ribbon group if $H_1(G) = \mathbb{Z}$ and if it has a Wirtinger presentation of deficiency one, i.e. G has a presentation of the form

$$G = \langle g_1, \ldots, g_s | r_1, \ldots, r_{s-1} \rangle,$$

where $r_i = g_{h_i} g_{l_i}^{\epsilon_i} g_{k_i}^{-1} g_{l_i}^{-\epsilon_i}$ for some $h_i, k_i, l_i \in \{1, \ldots, s\}$ and $\epsilon_i \in \{-1, 1\}$.

Ribbon groups appear as the fundamental groups of ribbon disk complements, i.e. complements of a certain type of slice disks.

It is a wide open question whether all *smoothly* slice knots have such ribbon disks.

Conjectures

Let G be a ribbon group.

Whitehead conjecture:
$K(G,1)$ is a 2–complex with $H_2(G) = 0$.

Farrell–Jones conjecture:
$L_4(\mathbb{Z}[G]) = \mathbb{Z}$.

Atiyah–conjecture:
There exists a supring $U(G)$ of $\mathbb{Z}[G]$ such that the Blanchfield pairing

$$H_1(M_K, \mathbb{Z}[G]) \times H_1(M_K, \mathbb{Z}[G]) \to U(G)/\mathbb{Z}[G]$$

is defined.

The question whether surgery works for all groups G on the other hand is wide open.

Topologically sliceness conjecture

Conjecture If K is topologically slice then there exists an epimorphism $\varphi : \pi_1(M_K) \to G$ to a a ribbon group G such that the Blanchfield pairing

$$H_1(M_K, \mathbb{Z}[G]) \times H_1(M_K, \mathbb{Z}[G]) \to U(G)/\mathbb{Z}[G]$$

vanishes.

Conversely, if there exists a map $\varphi : \pi_1(M_K) \to G$ to a ribbon group G such that the Blanchfield pairing

$$H_1(M_K, \mathbb{Z}[G]) \times H_1(M_K, \mathbb{Z}[G]) \to U(G)/\mathbb{Z}[G]$$

vanishes, and such that surgery works for G, then K is topologically slice.