REIDEMEISTER TORSION, THE THURSTON NORM AND HARVEY'S INVARIANTS

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Abstract. Cochran introduced Alexander polynomials over non–commutative Laurent polynomial rings. Their degrees were studied by Cochran, Harvey and Turaev as they give lower bounds on the Thurston norm. We first extend Cochran's definition to twisted Alexander polynomials. We then show how Reidemeister torsion relates to these invariants and we give lower bounds on the Thurston norm in terms of the Reidemeister torsion. This gives in particular a concise formulation of the bounds of Cochran, Harvey and Turaev. The Reidemeister torsion approach also gives a natural approach to proving and extending certain monotonicity results of Cochran and Harvey.

1. Introduction

The following algebraic setup allows us to define twisted non–commutative Alexander polynomials. First let $\mathbb{K}$ be a (skew) field and $\gamma : \mathbb{K} \to \mathbb{K}$ a ring homomorphism. Then denote by $\mathbb{K}_\gamma[t^{\pm 1}]$ the skew Laurent polynomial ring over $\mathbb{K}$. More precisely the elements in $\mathbb{K}_\gamma[t^{\pm 1}]$ are formal sums $\sum_{i=m}^{n} a_i t^i$ $(m \le n \in \mathbb{Z})$ with $a_i \in \mathbb{K}$. Addition is given by addition of the coefficients, and multiplication is defined using the rule $t^i a = \gamma^i(a) t^i$ for any $a \in \mathbb{K}$.

Let X be a connected CW–complex with finitely many cells in dimension i. Given a representation $\pi_1(X) \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ we can consider the $\mathbb{K}_\gamma[t^{\pm 1}]$–modules $H_i^\alpha(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ and we define twisted non–commutative Alexander polynomials $\Delta_i^\alpha(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ (cf. Section 3.3 for details). Twisted Alexander polynomials over commutative Laurent polynomial rings were first introduced by Lin [Lin01], Alexander polynomials over skew Laurent polynomial rings were introduced by Cochran [Co04]. Our definition is a combination of the definitions in [KL99] and [Co04]. In Theorem 3.1 we describe the indeterminacy of these polynomials.

Denote by $\mathbb{K}_\gamma(t)$ the quotient field of $\mathbb{K}_\gamma[t^{\pm 1}]$. We denote the induced representation $\pi_1(X) \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d) \to \mathrm{GL}(\mathbb{K}_\gamma(t), d)$ by α as well. If the homology groups $H_i^\alpha(X; \mathbb{K}_\gamma(t)^d)$ vanish and if X is a finite connected CW–complex, then we can define the Reidemeister torsion $\tau(X, \alpha) \in K_1(\mathbb{K}_\gamma(t))/\pm\alpha(\pi_1(X))$. An important tool is the

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Dieudonné determinant which defines an isomorphism

$$\det : K_1(\mathbb{K}_\gamma(t)) \to \mathbb{K}_\gamma(t)^\times_{ab},$$

where $\mathbb{K}_\gamma(t)^\times_{ab}$ denotes the abelianization of the multiplicative group $\mathbb{K}_\gamma(t)^\times = \mathbb{K}_\gamma(t) \setminus \{0\}$. We can therefore study $\det(\tau(X,\alpha)) \in \mathbb{K}_\gamma(t)^\times_{ab}/\pm\det(\alpha(\pi_1(X)))$. We refer to Sections 2.3 and 3.1 for details. The following result generalizes well–known commutative results of Turaev [Tu86, Tu01] and Kirk–Livingston [KL99].

Theorem 1.1. *Let X be a finite connected CW complex of dimension n. Let $\alpha : \pi_1(X) \to GL(\mathbb{K}_\gamma[t^{\pm1}], d)$ be a representation such that $H^\alpha_*(X; \mathbb{K}_\gamma(t)^d) = 0$. Then $\Delta^\alpha_i(t) \neq 0$ for all i and*

$$\det(\tau(X,\alpha)) = \prod_{i=0}^{n-1} \Delta^\alpha_i(t)^{(-1)^{i+1}} \in \mathbb{K}_\gamma(t)^\times_{ab}/\{kt^e | k \in \mathbb{K} \setminus \{0\}, e \in \mathbb{Z}\}.$$

For $f(t) = \sum_{i=m}^n a_i t^i \in \mathbb{K}_\gamma[t^{\pm1}] \setminus \{0\}$ with $a_m \neq 0, a_n \neq 0$, we define its degree to be $\deg(f(t)) = n-m$. We can extend this to a degree function $\deg : \mathbb{K}_\gamma(t) \setminus \{0\} \to \mathbb{Z}$. We denote $\deg(\det(\tau(X,\alpha)))$ by $\deg(\tau(X,\alpha))$. Theorem 1.1 then implies that the degree of $\tau(X,\alpha)$ is the alternating sum of the degrees of the twisted Alexander polynomials (cf. Corollary 3.6).

We now turn to the study of 3–manifolds. Here and throughout the paper we will assume that all manifolds are compact, orientable and connected. Recall that given a 3–manifold M and $\phi \in H^1(M;\mathbb{Z})$ the *Thurston norm* ([Th86]) of ϕ is defined as

$$||\phi||_T = \min\{-\chi(\hat{S}) \,|\, S \subset M \text{ properly embedded surface dual to } \phi\}$$

where $\hat{S}$ denotes the result of discarding all connected components of S with positive Euler characteristic. As an example consider $X(K) = S^3 \setminus \nu K$, where $K \subset S^3$ is a knot and νK denotes an open tubular neighborhood of K in S^3. Let $\phi \in H^1(X(K);\mathbb{Z})$ be a generator, then $||\phi||_T = 2\,\text{genus}(K) - 1$.

Let X be a connected CW–complex and let $\phi \in H^1(X;\mathbb{Z})$. We identify henceforth $H^1(X;\mathbb{Z})$ with $\text{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z})$ and $\text{Hom}(\pi_1(X),\mathbb{Z})$. A representation $\alpha : \pi_1(X) \to \text{GL}(\mathbb{K}_\gamma[t^{\pm1}], d)$ is called *ϕ–compatible* if for any $g \in \pi_1(X)$ we have $\alpha(g) = At^{\phi(g)}$ for some $A \in \text{GL}(\mathbb{K}, d)$. This generalizes a notion of Turaev [Tu02b].

The following theorem gives lower bounds on the Thurston norm using Reidemeister torsion. It contains the lower bounds of McMullen [Mc02], Cochran [Co04], Harvey [Ha05], Turaev [Tu02b] and of the author together with Taehee Kim [FK05]. To our knowledge this theorem is the strongest of its kind. Not only does it contain these results, the formulation of the inequalities in [Mc02, Co04, Ha05, Tu02b] in terms of the degrees of Reidemeister torsion also gives a very concise reformulation of their results.

Theorem 1.2. *Let M be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(M;\mathbb{Z})$ and $\alpha : \pi_1(M) \to GL(\mathbb{K}_\gamma[t^{\pm1}], d)$ a ϕ–compatible representation. Then*

$\tau(M,\alpha)$ is defined if and only if $\Delta_1^\alpha(t)\ne 0$. Furthermore if $\tau(M,\alpha)$ is defined, then

$$||\phi||_T \ge \frac{1}{d}deg(\tau(M,\alpha)).$$

If (M,ϕ) fibers over S^1, then

$$||\phi||_T = \max\{0, \frac{1}{d}deg(\tau(M,\alpha))\}.$$

The most commonly used skew fields are the quotient fields $\mathbb{K}(G)$ of group rings $\mathbb{F}[G]$ ($\mathbb{F}$ a commutative field) for certain torsion–free groups G, we refer to Section 5.1 for details. The following theorem says roughly that 'larger groups give better bounds on the Thurston norm'. The main idea of the proof is to use the fact that Reidemeister torsion behaves well under ring homomorphisms, in contrast to Alexander polynomials. We refer to Section 6 or to [Ha06] for the definition of an admissible triple.

Theorem 1.3. *Let M be a 3–manifold with empty or toroidal boundary or let M be a 2–complex with $\chi(M)=0$. Let $\phi\in H^1(M;\mathbb{Z})$. Let $\alpha:\pi_1(M)\to GL(\mathbb{F},d)$, $\mathbb{F}$ a commutative field, be a representation and $(\varphi_G:\pi\to G, \varphi_H:\pi\to H,\phi)$ an admissible triple for $\pi_1(M)$, in particular we have epimorphisms $G\to H\to\mathbb{Z}$. Write $G'=Ker\{G\to\mathbb{Z}\}$ and $H'=Ker\{H\to\mathbb{Z}\}$.*

If $\tau(M,\varphi_H\otimes\alpha)\in K_1(\mathbb{K}(H')(t))$ is defined, then $\tau(M,\varphi_G\otimes\alpha)\in K_1(\mathbb{K}(G')(t))$ is defined. Furthermore in that case

$$deg(\tau(M,\varphi_G\otimes\alpha)) \ge deg(\tau(M,\varphi_H\otimes\alpha)).$$

A similar theorem holds for 2–complexes with Euler characteristic zero. As a special case consider the case that α is the trivial representation. Using Theorem 1.1 we can recover the monotonicity results of [Co04] and [Ha06]. We hope that our alternative proof using Reidemeister torsion will contribute to the understanding of their results.

The paper is organized as follows. In Section 2 we recall the definition of Reidemeister torsion. In Section 3 we introduce twisted non–commutative Alexander polynomials, we compute their indeterminacies in Theorem 3.1 and we prove Theorem 1.1. Beginning with Section 4 we concentrate on 3–manifolds. In particular in Section 4 we give the proof of Theorem 1.2. In Section 5 we give examples of ϕ–compatible representations. In Section 6 we prove Theorem 1.3 and in Section 7 we show that it implies Cochran's and Harvey's monotonicity results. We conclude with a few open questions in Section 8.

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2. Reidemeister torsion

2.1. **Definition of $K_1(R)$.** For the remainder of the paper we will only consider associative rings R with $1 \neq 0$ and with the property that if $r \neq s \in \mathbb{N}_0$, then R^r is not isomorphic to R^s as an R–module.

For such a ring R define $\mathrm{GL}(R) = \varinjlim \mathrm{GL}(R,d)$, where we have the following maps in the direct system: $\mathrm{GL}(R,d) \to \mathrm{GL}(R,d+1)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Then $K_1(R)$ is defined as $\mathrm{GL}(R)/[\mathrm{GL}(R),\mathrm{GL}(R)]$. Note that $K_1(R)$ is an abelian group. For details we refer to [Mi66] or [Tu01]. There exists a canonical map $\mathrm{GL}(R,d) \to K_1(R)$ for every d, in particular there exists a homomorphism from the units of R into $K_1(R)$. By abuse of notation we denote the image of $A \in \mathrm{GL}(R,d)$ in $K_1(R)$ by A as well. We denote by $-A$ the product of $A \in K_1(R)$ by the image of (-1) under the map $\mathrm{GL}(R,1) \to K_1(R)$.

We will often make use of the observation (cf. [Ro94, p. 61]) that for $A \in \mathrm{GL}(R,d_1), B \in \mathrm{GL}(R,d_2)$ the product $AB \in K_1(R)$ is given by

$$AB = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in K_1(R).$$

2.2. **Definition of Reidemeister torsion.** Let C_* be a finite free chain complex of R–modules. By this we mean a chain complex of free finite right R–modules such that $C_i = 0$ for all but finitely many $i \in \mathbb{Z}$. Let $\mathcal{C}_i \subset C_i$ be a basis for all i with $C_i \neq 0$. Assume that $B_i = \mathrm{Im}(C_{i+1}) \subset C_i$ is free, pick a basis $\mathcal{B}_i$ of B_i and a lift $\tilde{\mathcal{B}}_i$ of $\mathcal{B}_i$ to C_{i+1}. We write $\mathcal{B}_i\tilde{\mathcal{B}}_{i-1}$ for the collection of elements given by $\mathcal{B}_i$ and $\tilde{\mathcal{B}}_{i-1}$. Since C_* is acyclic this is indeed a basis for C_i. Then we define the *Reidemeister torsion* of the based acyclic complex $(C_*, \{\mathcal{C}_i\})$ to be

$$\tau(C_*, \{\mathcal{C}_i\}) = \prod [\mathcal{B}_i\tilde{\mathcal{B}}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}} \in K_1(R),$$

where $[d/e]$ denotes the matrix of a basis change, i.e. $[d/e] = (a_{ij})$ where $d_i = \sum_j a_{ji}e_j$. Note that in contrast to [Tu01] we view vectors as column vectors. This means that our matrix is the transpose of the matrix in [Tu01, p. 1].

It is easy to see that $\tau(C_*, \{\mathcal{C}_i\})$ is independent of the choice of $\{\mathcal{B}_i\}$ and of the choice of the lifts $\tilde{\mathcal{B}}_i$. If the R–modules B_i are not free, then one can show that they are stably free and a stable basis will then make the definition work again. We refer to [Mi66, p. 369] or [Tu01, p. 13] for the full details.

2.3. **Reidemeister torsion of a CW–complex.** Let X be a connected CW–complex. Denote the universal cover of X by $\tilde{X}$. We view $C_*(\tilde{X})$, the chain complex of the universal cover, as a chain complex of right $\mathbb{Z}[\pi_1(X)]$–modules, where the $\mathbb{Z}[\pi_1(X)]$–module structure is given via deck transformations.

Let R be a ring. Let $\alpha : \pi_1(X) \to \mathrm{GL}(R,d)$ be a representation. This equips R^d with a left $\mathbb{Z}[\pi_1(X)]$–module structure. We can therefore consider the chain complex

$C_*^\alpha(X;R^d) = C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R^d$. Note that this is a finite free chain complex of (right) R–modules. We denote its homology by $H_*^\alpha(X;R^d)$, we drop α from the notation if it is clear from the context.

Now assume that X is a finite connected CW–complex. If $H_i^\alpha(X;R^d) = H_i(C_*^\alpha(X;R^d)) \neq 0$ for some i then we write $\tau(X,\alpha) = 0$. Otherwise denote the i–cells of X by $\sigma_i^1, \dots, \sigma_i^{r_i}$ and denote by $e_1, \dots, e_d$ the standard basis of R^d. Pick an orientation for each cell σ_i^j, and also pick a lift $\tilde{\sigma}_i^j$ for each cell σ_i^j to the universal cover $\tilde{X}$. We get a basis

$$\mathcal{C}_i = \{\tilde{\sigma}_i^1 \otimes e_1, \dots, \tilde{\sigma}_i^1 \otimes e_d, \dots, \tilde{\sigma}_i^{r_i} \otimes e_1, \dots, \tilde{\sigma}_i^{r_i} \otimes e_d\}$$

for $C_i^\alpha(X;R^d)$. Then we can define

$$\tau(C_*^\alpha(X;R^d), \{\mathcal{C}_i\}) \in K_1(R).$$

This element depends only on the ordering and orientation of the cells and on the choice of lifts of the cells to the universal cover. Therefore

$$\tau(X,\alpha) = \tau(C_*^\alpha(X,R^d), \{\mathcal{C}_i\}) \in K_1(R)/\pm\alpha(\pi_1(X))$$

is a well–defined invariant of the CW–complex X.

Now let M be a compact PL–manifold. Pick any finite CW–structure for M to define $\tau(M,\alpha) \in K_1(R)/\pm\alpha(\pi_1(M))$. By Chapman's theorem [Ch74] this is a well–defined invariant of the manifold, i.e. independent of the choice of the CW–structure.

2.4. **Computation of Reidemeister torsion.** We explain an algorithm for computing Reidemeister torsion which was formulated by Turaev [Tu01, Section 2.1] in the commutative case.

In the following assume that we have a finite free chain complex of R–modules

$$0 \to C_m \xrightarrow{\partial_m} C_{m-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0.$$

Let $A_i = (a_{jk}^i)$ be the matrix representing ∂_i corresponding to the given bases. Note that in contrast to [Tu01, Section 2.1] we view the elements in $R^{rank(C_i)}$ as column vectors.

Following Turaev [Tu01, p. 8] we define a *matrix chain* for C to be a collection of sets $\xi = (\xi_0, \xi_1, \dots, \xi_m)$ where $\xi_i \subset \{1, 2, \dots, \mathrm{rank}(C_i)\}$ so that $\xi_0 = \emptyset$. Given a matrix chain ξ we define $A_i(\xi)$, $i = 1, \dots, m$ to be the matrix formed by the entries a_{jk}^i with $j \notin \xi_{i-1}$ and $k \in \xi_i$. Put differently the matrix $(a_{jk}^i)_{jk}$ is given by considering only the ξ_i–columns of A_i and with the ξ_{i-1}–rows removed.

We say that a matrix chain ξ is a τ–chain if $A_1(\xi), \dots, A_m(\xi)$ are square matrices. The following is the generalization of [Tu01, Theorem 2.2] to the non–commutative setting. Turaev's proof can easily be generalized to this more general setting.

Theorem 2.1. *Let ξ be a τ–chain such that $A_i(\xi)$ is invertible for all odd i. Then $A_i(\xi)$ is invertible for all even i if and only if $H_*(C) = 0$. Furthermore if $H_*(C) = 0$,* *then*

$$\tau(C) = \epsilon \prod_{i=1}^{m} A_i(\xi)^{(-1)^i} \in K_1(R)$$

for some $\epsilon \in \{\pm 1\}$.

This proposition is the reason why Reidemeister torsion behaves in general well under ring homomorphisms.

3. Reidemeister torsion and Alexander polynomials

3.1. **Laurent polynomial rings and the Dieudonné determinant.** For the remainder of this paper let $\mathbb{K}$ be a (skew) field and let $\mathbb{K}_\gamma[t^{\pm 1}]$ be a skew Laurent polynomial ring. By [DLMSY03, Corollary 6.3] the ring $\mathbb{K}_\gamma[t^{\pm 1}]$ has a classical quotient field $\mathbb{K}_\gamma(t)$ which is flat over $\mathbb{K}_\gamma[t^{\pm 1}]$ (cf. [Ra98, p. 99]). In particular we can view $\mathbb{K}_\gamma[t^{\pm 1}]$ as a subring of $\mathbb{K}_\gamma(t)$ and any element in $\mathbb{K}_\gamma(t)$ is of the form $f(t)g(t)^{-1}$ for some $f(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ and $g(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$. We refer to Theorem 5.1 for a related result. Recall that we write $\mathbb{K}_\gamma(t)^\times = \mathbb{K}_\gamma(t) \setminus \{0\}$.

In the following we mean by an elementary column (row) operation the addition of a right multiple (left multiple) of one column (row) to a different column (row). Let A be an invertible $k \times k$–matrix over the skew field $\mathbb{K}_\gamma(t)$. After elementary row operations we can turn A into a diagonal matrix $D = (d_{ij})$. Then the Dieudonné determinant $\det(A) \in \mathbb{K}_\gamma(t)^\times_{ab} = \mathbb{K}_\gamma(t)^\times/[\mathbb{K}_\gamma(t)^\times, \mathbb{K}_\gamma(t)^\times]$ is defined to be $\prod_{i=1}^{k} d_{ii}$. This is a well–defined map. Note that the Dieudonné determinant is invariant under elementary row and column operations. The Dieudonné determinant induces an isomorphism $\det : K_1(\mathbb{K}_\gamma(t)) \to \mathbb{K}_\gamma(t)^\times_{ab}$. Using the last observation in Section 2.1 it is easy to see that $A = \det(A) \in K_1(\mathbb{K}_\gamma(t))$. We will often make use of this equality. We refer to [Ro94, Theorem 2.2.5 and Corollary 2.2.6] for more details.

In the introduction we defined $\deg : \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\} \to \mathbb{N}$. This can be extended to a homomorphism $\deg : \mathbb{K}_\gamma(t)^\times \to \mathbb{Z}$ via $\deg(f(t)g(t)^{-1}) = \deg(f(t)) - \deg(g(t))$ for $f(t), g(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$. Clearly the degree map vanishes on $[\mathbb{K}_\gamma(t)^\times, \mathbb{K}_\gamma(t)^\times]$ and we get an induced homomorphism $K_1(\mathbb{K}_\gamma(t)) \to \mathbb{K}_\gamma(t)^\times_{ab} \to \mathbb{Z}$ which we also denote by deg.

3.2. **Orders of $\mathbb{K}_\gamma[t^{\pm 1}]$–modules.** Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$–module. The ring $\mathbb{K}_\gamma[t^{\pm 1}]$ is a principal ideal domain (PID) since $\mathbb{K}$ is a skew field. We can therefore find an isomorphism

$$H \cong \bigoplus_{i=1}^{l} \mathbb{K}_\gamma[t^{\pm 1}]/p_i(t)\mathbb{K}_\gamma[t^{\pm 1}]$$

for $p_i(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ for $i = 1, \dots, l$. Following [Co04] we define $\operatorname{ord}(H) = \prod_{i=1}^{l} p_i(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$. This is called the *order of H*.

Note that $\text{ord}(H) \in \mathbb{K}_\gamma[t^{\pm 1}]$ has a high degree of indeterminacy. For example writing the $p_i(t)$ in a different order will change $\text{ord}(H)$. Furthermore we can change $p_i(t)$ by multiplication by any element of the form kt^e where $k \in \mathbb{K}^\times = \mathbb{K} \setminus \{0\}$ and $e \in \mathbb{Z}$. The following theorem can be viewed as saying that these are all possible indeterminacies.

Theorem 3.1. *Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$–module. Then $ord(H) = 0$ if and only if H is not $\mathbb{K}_\gamma[t^{\pm 1}]$–torsion. If $ord(H) \neq 0$, then $ord(H) \in \mathbb{K}_\gamma[t^{\pm 1}]$ is well–defined considered as an element in $\mathbb{K}_\gamma(t)^\times_{ab}$ up to multiplication by an element of the form $kt^e, k \in \mathbb{K}^\times$ and $e \in \mathbb{Z}$.*

The first statement is clear. We postpone the proof of the second statement of the theorem to Section 3.4. We refer to [Co04, p. 367] for an alternative discussion of the indeterminacy of $\text{ord}(H)$. Note that the idea of considering $\text{ord}(H)$ as an element in $\mathbb{K}_\gamma(t)^\times_{ab}$ is already present in [Co04, p. 367].

It follows from Theorem 3.1 that $\deg(\text{ord}(H))$ is well–defined. In fact we have the following interpretation of $\text{ord}(H)$.

Lemma 3.2. [Co04, p. 368] *Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$–torsion module. Then*

$$deg(ord(H)) = \dim_{\mathbb{K}}(H).$$

Here we used that by [St75, Proposition I.2.3] and [Coh85, p. 48] every right $\mathbb{K}$–module V is free and has a well-defined dimension $\dim_{\mathbb{K}}(V)$.

Proof. It is easy to see that for $f(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$ we have

$$\deg(f(t)) = \dim_{\mathbb{K}}(\mathbb{K}_\gamma[t^{\pm 1}]/f(t)\mathbb{K}_\gamma[t^{\pm 1}]).$$

The lemma is now immediate. □

3.3. **Alexander polynomials.** Let X be a connected CW–complex with finitely many cells in dimension i. Let $\alpha : \pi_1(X) \to \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a representation. The right $\mathbb{K}_\gamma[t^{\pm 1}]$–module $H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ is called twisted (non–commutative) Alexander module. Similar modules were studied in [Co04], [Ha05] and [Tu02b]. Note that $H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ is a finitely generated $\mathbb{K}_\gamma[t^{\pm 1}]$–module since we assumed that X has only finitely many cells in dimension i and since $\mathbb{K}_\gamma[t^{\pm 1}]$ is a PID. We now define $\Delta_i^\alpha(t) = \text{ord}(H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)) \in \mathbb{K}_\gamma[t^{\pm 1}]$, this is called the (twisted) *i–th Alexander polynomial* of (X, α).

The degrees of these polynomials (corresponding to one–dimensional representations) have been studied recently in various contexts (cf. [Co04, Ha05, Ha06, Tu02b, LM05, FK05b, FH06]). We hope that by determining the indeterminacy of the Alexander polynomials (Theorem 3.1) more information can be extracted from the Alexander polynomials than just the degrees.

3.4. **Proof of Theorem 3.1.** We first point out that $\mathbb{K}_\gamma[t^{\pm 1}]$ is a Euclidean ring with respect to the degree function. This means that given $f(t), g(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$ we can find $a(t), r(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ such that $f(t) = g(t)a(t) + r(t)$ and such that either $r(t) = 0$ or $\deg(r(t)) < \deg(g(t))$.

Let A be an $r \times s$–matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ of rank r. Here and in the following the rank of a matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ will be understood as the rank of the matrix considered as a matrix over the skew field $\mathbb{K}_\gamma(t)$. Note that $\operatorname{rank}(A) = r$ implies that in particular $s \geq r$. Since $\mathbb{K}_\gamma[t^{\pm 1}]$ is a Euclidean ring we can perform a sequence of elementary row and column operations to turn A into a matrix of the form $\begin{pmatrix} D & 0_{r\times(s-r)} \end{pmatrix}$ where D is an $r \times r$–matrix and $0_{r\times(s-r)}$ stands for the $r \times (s-r)$–matrix consisting only of zeros. Since A is of rank r it follows that D has rank r as well, in particular D is a square matrix which is invertible over $\mathbb{K}_\gamma(t)$ and we can consider its Dieudonné determinant $\det(D)$. We define $\det(A) = \det(D) \in \mathbb{K}_\gamma(t)^\times_{ab}$.

Lemma 3.3. *Let A be a (square) matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ which is invertible over $\mathbb{K}_\gamma(t)$.*

1. *The Dieudonné determinant $\det(A) \in \mathbb{K}_\gamma(t)^\times_{ab}$ can be represented by an element in $\mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$.*
2. *If $A \in GL(\mathbb{K}_\gamma[t^{\pm 1}], d)$, then $\det(A) \in \mathbb{K}_\gamma(t)^\times_{ab}$ can be represented by an element of the form $kt^e, k \in \mathbb{K}^\times, e \in \mathbb{Z}$.*
3. *The Dieudonné determinant induces a homomorphism*
$$\det : K_1(\mathbb{K}_\gamma(t)) \to \mathbb{K}_\gamma(t)^\times_{ab}$$
which sends $K_1(\mathbb{K}_\gamma[t^{\pm 1}])$ to $\{kt^e | k \in \mathbb{K}^\times, e \in \mathbb{Z}\}/[\mathbb{K}_\gamma(t)^\times, \mathbb{K}_\gamma(t)^\times] \subset \mathbb{K}_\gamma(t)^\times_{ab}$.

Proof. The first statement follows from the discussion preceding the lemma. Now let $A \in \operatorname{GL}(\mathbb{K}_\gamma[t^{\pm 1}], r)$. It follows from Lemma 3.2 applied to $H = \mathbb{K}_\gamma[t^{\pm 1}]^r / A\mathbb{K}_\gamma[t^{\pm 1}]^r$ that $\deg(\det(A)) = 0$. This proves the second statement. The last statement follows from the second statement and the fact that the Dieudonné determinant induces a homomorphism $\det : K_1(\mathbb{K}_\gamma(t)) \to \mathbb{K}_\gamma(t)^\times_{ab}$. □

Proposition 3.4. *Let A be an $r \times s$–matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ of rank r. Then $\det(A) \in \mathbb{K}_\gamma(t)^\times_{ab}$ is well–defined up to multiplication by an element of the form $kt^e, k \in \mathbb{K}^\times, e \in \mathbb{Z}$. Furthermore $\det(A)$ is invariant under elementary row and column operations.*

Proof. First note that the effect of an elementary row operation on A over $\mathbb{K}_\gamma[t^{\pm 1}]$ can be described by left multiplication by a matrix $P \in \operatorname{GL}(\mathbb{K}_\gamma[t^{\pm 1}], r)$. Similarly an elementary column operation on A over $\mathbb{K}_\gamma[t^{\pm 1}]$ can be described by right multiplication by an $s \times s$–matrix $Q \in \operatorname{GL}(\mathbb{K}_\gamma[t^{\pm 1}], s)$.

Now assume we have $P_1, P_2 \in \operatorname{GL}(\mathbb{K}_\gamma[t^{\pm 1}], r)$ and $Q_1, Q_2 \in \operatorname{GL}(\mathbb{K}_\gamma[t^{\pm 1}], s)$ such that $P_iAQ_i = \begin{pmatrix} D_i & 0_{r\times(s-r)} \end{pmatrix}, i = 1, 2$ where D_i is an $r \times r$–matrix. We are done once we show that $\det(D_1) = kt^e \det(D_2) \in \mathbb{K}_\gamma(t)^\times_{ab}$ for some $k \in \mathbb{K}^\times, e \in \mathbb{Z}$. Let $E_i = P_i^{-1} D_i$. Then by Lemma 3.3 we only have to show that $E_1 = E_2 \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}])$.

We have $\begin{pmatrix} E_1 & 0 \end{pmatrix} Q_1^{-1} = \begin{pmatrix} E_2 & 0 \end{pmatrix} Q_2^{-1}$. Let $Q := Q_2^{-1}Q_1 \in \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], s)$, we therefore get the equality $\begin{pmatrix} E_1 & 0 \end{pmatrix} = \begin{pmatrix} E_2 & 0 \end{pmatrix} Q$. Now write $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$ where Q_{ij} is a $n_i \times n_j$ matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ with $n_1 = r$ and $n_2 = s - r$. We get the equality

$$\begin{pmatrix} E_1 & 0 \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} E_2 & 0 \\ 0 & \mathrm{id}_{s-r} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

It follows in particular that Q_{22} is invertible over $\mathbb{K}_\gamma(t)$. Furthermore we have

$$E_1 \cdot Q_{22} = E_2 \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}]).$$

Note that $\deg : K_1(\mathbb{K}_\gamma(t)) \to \mathbb{Z}$ vanishes on $K_1(\mathbb{K}_\gamma[t^{\pm 1}])$ by Lemma 3.3. We therefore get $\deg(\det(E_1)) + \deg(\det(Q_{22})) = \deg(\det(E_2))$, in particular $\deg(\det(E_1)) \le \deg(\det(E_2))$. But by symmetry we have $\deg(\det(E_2)) \le \deg(\det(E_1))$. In particular $\deg(\det(Q_{22})) = 0$. The proposition now follows immediately from Lemma 3.3 since $\deg(f(t)) = 0$ for $f(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$ if and only if $f(t) = kt^e$ for some $k \in \mathbb{K}^\times, e \in \mathbb{Z}$.

The last statement is immediate. □

Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$–module. We say that an $r \times s$–matrix A is a presentation matrix for H if the following sequence is exact:

$$\mathbb{K}_\gamma[t^{\pm 1}]^s \xrightarrow{A} \mathbb{K}_\gamma[t^{\pm 1}]^r \to H \to 0.$$

We say that A has full rank if the rank of A equals r. Note that A has full rank if and only if $H \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t) = 0$.

The following lemma clearly implies Theorem 3.1.

Lemma 3.5. *Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$–module and let A_1, A_2 be presentation matrices for H. Then A_1 has full rank if and only if A_2 has full rank. Furthermore if A_i has full rank, then*

$$\det(A_1) = \det(A_2) \in \mathbb{K}_\gamma(t)^\times_{ab}/\{kt^e | k \in \mathbb{K} \setminus \{0\}, e \in \mathbb{Z}\}.$$

Proof. It is well–known that any two presentation matrices for H differ by a sequence of matrix moves of the following forms and their inverses:

(1) Permutation of rows or columns.
(2) Replacement of the matrix A by $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.
(3) Addition of an extra column of zeros to the matrix A.
(4) Addition of a right scalar multiple of a column to another column.
(5) Addition of a left scalar multiple of a row to another row.

This result is proven [Li97, Theorem 6.1] in the commutative case, but the proof carries through in the case of the base ring $\mathbb{K}_\gamma[t^{\pm 1}]$ as well (cf. also [Ha05, Lemma 9.2]).

Clearly none of the moves changes the status of being of full rank, and if a representation is of full rank, then it is easy to see that none of the moves changes the determinant. □

3.5. **Proof of Theorem 1.1.** Now let X be a finite connected CW complex of dimension n. Let $\alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a representation such that $H_*^\alpha(X; \mathbb{K}_\gamma(t)^d) = 0$. (Recall that we denote the induced representation $\pi_1(X) \to \mathrm{GL}(\mathbb{K}_\gamma(t), d)$ by α as well). Furthermore recall that $\mathbb{K}_\gamma(t)$ is flat over $\mathbb{K}_\gamma[t^{\pm 1}]$, in particular $H_i(X; \mathbb{K}_\gamma(t)^d) = H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)$. It follows that $H_i(X; \mathbb{K}_\gamma(t)^d) = 0$ if and only if $H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ is $\mathbb{K}_\gamma[t^{\pm 1}]$–torsion, which is equivalent to $\Delta_i^\alpha(t) \neq 0$. This proves the first statement of Theorem 1.1. To conclude the proof of Theorem 1.1 it remains to prove the following claim.

Claim. If $H_*^\alpha(X; \mathbb{K}_\gamma(t)^d) = 0$, then

$$\det(\tau(X, \alpha)) = \prod_{i=0}^{n-1} \Delta_i^\alpha(t)^{(-1)^{i+1}} \in \mathbb{K}_\gamma(t)^\times_{ab} / \{kt^e | k \in \mathbb{K} \setminus \{0\}, e \in \mathbb{Z}\}.$$

Proof. Let $C_* = C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{K}_\gamma[t^{\pm 1}]^d$. Note that any $\mathbb{K}_\gamma[t^{\pm 1}]$–basis for C_* also gives a basis for $C_* \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)$, which we will always denote by the same symbol.

Denote by $\mathcal{C}_*$ the $\mathbb{K}_\gamma[t^{\pm 1}]$–basis of C_* as in Section 2.3. Let $r_i := \dim_{\mathbb{K}(t)}(C_i \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t))$ and let $s_i := \dim_{\mathbb{K}(t)}(\mathrm{Ker}\{C_i \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t) \xrightarrow{\partial_i} C_{i-1} \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)\})$. Note that $s_i + s_{i-1} = r_i$ since $H_*(X; \mathbb{K}_\gamma(t)^d) = 0$. Furthermore note that $\mathrm{Ker}\{C_i \xrightarrow{\partial_i} C_{i-1}\} \subset C_i$ is a free direct summand of C_i of rank s_i since $\mathbb{K}_\gamma[t^{\pm 1}]$ is a PID. We can therefore pick $\mathbb{K}_\gamma[t^{\pm 1}]$–bases $\mathcal{C}_i' = \{v_1, \dots, v_{r_i}\}$ for C_i such that $\{v_1, \dots, v_{s_i}\}$ is a basis for $\mathrm{Ker}\{C_i \xrightarrow{\partial_i} C_{i-1}\}$. Note that the base changes from $\mathcal{C}_i$ to $\mathcal{C}_i'$ are given by matrices which are invertible over $\mathbb{K}_\gamma[t^{\pm 1}]$. It follows that

$$\tau(C_* \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t), \{\mathcal{C}_i\}) = \tau(C_* \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t), \{\mathcal{C}_i'\}) \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}]).$$

Let A_i be the $r_{i-1} \times r_i$–matrix representing $\partial_i : C_i \to C_{i-1}$ with respect to the bases $\mathcal{C}_i'$ and $\mathcal{C}_{i-1}'$. Let $\xi_i := \{s_i + 1, \dots, r_i\}, i = 1, \dots, n$ and $\xi_0 := \emptyset$. Let $A_i(\xi)$ as in Theorem 2.1. Note that $A_i(\xi)$ is an $s_{i-1} \times s_{i-1}$–matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$. In particular $\xi := (\xi_0, \dots, \xi_n)$ is a τ–chain. It is easy to see that

$$A_i = \begin{pmatrix} 0_{s_{i-1} \times s_i} & A_i(\xi) \\ 0_{s_{i-2} \times s_i} & 0_{s_{i-2} \times s_{i-1}} \end{pmatrix}.$$

Since A_i has rank s_{i-1} it follows that $A_i(\xi)$ is invertible over $\mathbb{K}_\gamma(t)$. It follows from Theorem 2.1 that

$$\tau(C_* \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t), \{\mathcal{C}_i'\}) = \prod_{i=1}^{n} A_i(\xi)^{(-1)^i} \in K_1(\mathbb{K}_\gamma(t)).$$

We also have short exact sequences

$$0 \to \mathbb{K}_\gamma[t^{\pm 1}]^{s_{i-1}} \xrightarrow{A_i(\xi)} \mathbb{K}_\gamma[t^{\pm 1}]^{s_{i-1}} \to H_{i-1}(C_*) = H_{i-1}(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) \to 0.$$

In particular $(A_i(\xi))$ is a presentation matrix for $H_{i-1}(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$. It therefore follows from Lemma 3.5 that $\det(A_i(\xi)) = \Delta_{i-1}^\alpha(t)$. ∎

The following corollary now follows immediately from the fact that $\deg : \mathbb{K}_\gamma(t)^\times \to \mathbb{Z}$ is a homomorphism and from Lemma 3.2.

Corollary 3.6. *Let X be a finite connected CW complex of dimension n. Let $\alpha : \pi_1(X) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a representation such that $H_*^\alpha(X;\mathbb{K}_\gamma(t)^d) = 0$. Then*

$$deg(\tau(X,\alpha)) = \sum_{i=0}^{n-1} (-1)^{i+1} deg(\Delta_i^\alpha(t)) = \sum_{i=0}^{n-1} (-1)^{i+1} \dim_{\mathbb{K}}(H_i(X;\mathbb{K}_\gamma[t^{\pm 1}]^d).$$

Remark. In the case that $H_*(X;\mathbb{K}_\gamma(t)^d) \neq 0$ we can pick $\mathbb{K}_\gamma[t^{\pm 1}]$–bases $\mathcal{H}_i$ for the $\mathbb{K}_\gamma[t^{\pm 1}]$–free parts of $H_i(X;\mathbb{K}_\gamma[t^{\pm 1}]^d)$. These give bases for $H_i(X;\mathbb{K}_\gamma(t)^d) = H_i(X;\mathbb{K}_\gamma[t^{\pm 1}]^d) \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)$ and we can consider $\tau(X,\alpha,\{\mathcal{H}_i\}) = \tau(C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{K}_\gamma(t)^d, \{\mathcal{H}_i\}) \in K_1(\mathbb{K}_\gamma(t))$ (cf. [Mi66] for details). If we consider $\tau(X,\alpha,\{\mathcal{H}_i\}) \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}])$ then this is independent of the choice of $\{\mathcal{H}_i\}$. The proof of Theorem 1.1 can be generalized to show that it is the alternating product of the orders of the $\mathbb{K}_\gamma[t^{\pm 1}]$–torsion submodules of $H_*(X;\mathbb{K}_\gamma[t^{\pm 1}]^d)$ (cf. also [KL99] in the commutative case).

4. 3–manifolds and 2–complexes

We now restrict ourselves to ϕ–compatible representations since these have a closer connection to the topology of a space.

Lemma 4.1. *Let X be a connected CW–complex with finitely many cells in dimensions zero and one. Let $\phi \in H^1(X;\mathbb{Z})$ non–trivial and let $\alpha : \pi_1(X) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a ϕ–compatible representation. Then $\Delta_0^\alpha(t) \neq 0$. If X is in fact an k–manifold, then $\Delta_k^\alpha(t) = 1$.*

We need the following notation. If $A = (a_{ij})$ is an $r \times s$–matrix over $\mathbb{Z}[\pi_1(X)]$ and $\alpha : \pi_1(X) \to \mathrm{GL}(R,d)$ a representation. Then we denote by $\alpha(A)$ the $rd \times sd$–matrix over R obtained by replacing each entry $a_{ij} \in \mathbb{Z}[\pi_1(X)]$ of A by the $d \times d$–matrix $\alpha(a_{ij})$.

Proof. First equip X with a CW–structure with one 0–cell and n 1–cells $g_1, \dots, g_n$. We denote the corresponding elements in $\pi_1(X)$ by $g_1, \dots, g_n$ as well. Since ϕ is non–trivial there exists at least one i such that $\phi(g_i) \neq 0$. Write $C_* = C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{K}_\gamma[t^{\pm 1}]^d$. The boundary map $\partial_1 : C_1 \to C_0$ is represented by the matrix

$$(\alpha(1-g_1), \dots, \alpha(1-g_n)) = (\mathrm{id} - \alpha(g_1), \dots, \mathrm{id} - \alpha(g_n)).$$

Since α is ϕ–compatible it follows that $\alpha(1-g_i) = \mathrm{id} - At^{\phi(g_i)}$ for some matrix $A \in \mathrm{GL}(\mathbb{K}, d)$. The first statement of the lemma now follows from Lemma 4.2.

If X is a closed k–manifold then equip X with a CW–structure with one k–cell. Since ϕ is primitive and ϕ–compatible an argument as above shows that $\partial_k : C_k \to C_{k-1}$ has full rank, i.e. $H_k(X;\mathbb{K}_\gamma[t^{\pm 1}]^d) = 0$. Hence $\Delta_k^\alpha(t) = 1$. If X is a k–manifold with boundary, then it is homotopy equivalent to a $k-1$–complex, and hence $H_k(X;\mathbb{K}_\gamma[t^{\pm 1}]^d) = 0$. $\square$

Lemma 4.2. *Let $\mathbb{K}_\gamma[t^{\pm 1}]$ be a skew Laurent polynomial ring and let A, B be invertible $d \times d$–matrices over $\mathbb{K}$ and $r \neq 0$. Then deg$(\det(A+Bt^r)) = dr$. In particular $A+Bt^r$ is invertible over $\mathbb{K}_\gamma(t)$.*

We point out that Harvey [Ha05, Proposition 9.1] proves a related result.

Proof. We can clearly assume that $r > 0$. Let $\{e_1, \dots, e_d\}$ be a basis for $\mathbb{K}^d$. Consider the projection map $p : \mathbb{K}_\gamma[t^{\pm 1}]^d \to P = \mathbb{K}_\gamma[t^{\pm 1}]^d/(A + Bt^r)\mathbb{K}_\gamma[t^{\pm 1}]^d$. Note that by Lemma 3.2 we are done once we show that $p(e_i t^j), i \in \{1, \dots, d\}, j \in \{0, \dots, r-1\}$ form a basis for P as a right $\mathbb{K}$–vector space.

It follows easily from the fact that A, B are invertible that this is indeed a generating set. Let $v \in \mathbb{K}_\gamma[t^{\pm 1}]^d \setminus \{0\}$. We can write $v = \sum_{i=n}^m v_i t^i, v_i \in \mathbb{K}^d$ with $v_n \neq 0, v_m \neq 0$. Since A, B are invertible it follows that $(A + Bt^r)v$ has terms with t–exponent n and terms with t–exponent $m + r$. This observation can be used to show that the above vectors are linearly independent in P. $\square$

We can now give the proof of Theorem 1.2.

Proof of Theorem 1.2. Now let M be a 3–manifold whose boundary is empty or consists of tori. Note that a standard duality argument shows that $2\chi(M) = \chi(\partial M) = 0$. Let $\phi \in H^1(M; \mathbb{Z})$ be non–trivial, and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ a ϕ–compatible representation.

We first show that $H_*(M; \mathbb{K}_\gamma(t)^d) = 0$ if and only if $\Delta_1^\alpha(t) \neq 0$. Recall that in Section we showed that $H_i(M; \mathbb{K}_\gamma(t)^d) = 0$ if and only if $\Delta_i^\alpha(t) \neq 0$. It now follows from Lemma 4.1 that $H_i(M; \mathbb{K}_\gamma(t)^d) = 0$ for $i = 0, 3$. If $\Delta_1^\alpha(t) \neq 0$, then $H_1(M; \mathbb{K}_\gamma(t)^d) = 0$. Since $\chi(H_i(M; \mathbb{K}_\gamma(t)^d)) = d\chi(M) = 0$ it follows that $H_2(M; \mathbb{K}_\gamma(t)^d) = 0$.

Claim.

$$||\phi||_T \geq \frac{1}{d}\left(-\dim_{\mathbb{K}}\left(H_0^\alpha(M; \mathbb{K}_\gamma[t^{\pm 1}]^d)\right) + \dim_{\mathbb{K}}\left(H_1^\alpha(M; \mathbb{K}_\gamma[t^{\pm 1}]^d)\right) - \dim_{\mathbb{K}}\left(H_2^\alpha(M; \mathbb{K}_\gamma[t^{\pm 1}]^d)\right)\right).$$

Furthermore this inequality becomes an equality if (M, ϕ) fibers over S^1 and if $M \neq S^1 \times D^2, M \neq S^1 \times S^2$.

First note that if ϕ vanishes on $X \subset M$ then α restricted to $\pi_1(X)$ lies in $\mathrm{GL}(\mathbb{K}, d) \subset \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ since α is ϕ–compatible. Therefore $H_i^\alpha(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) \cong H_i^\alpha(X; \mathbb{K}^d) \otimes_{\mathbb{K}} \mathbb{K}_\gamma[t^{\pm 1}]$. The proofs of [FK05, Theorem 3.1] and [FK05, Theorem 6.1] can now easily be translated to this non–commutative setting. This proves the claim.
Combining the results of the claim with Lemma 3.2 and Corollary 3.6 we immediately get a proof for Theorem 1.2. $\square$

In order to relate Theorem 1.2 to the results of [Co04, Ha05, Tu02b] we need the following computations for one–dimensional ϕ–compatible representations. Recall that $\phi \in H^1(X; \mathbb{Z})$ is called primitive if the corresponding map $\phi : H_1(X; \mathbb{Z}) \to \mathbb{Z}$ is surjective.

Lemma 4.3. *Let X be a connected CW–complex with finitely many cells in dimensions zero and one. Let $\phi \in H^1(X;\mathbb{Z})$ primitive. Let $\alpha : \pi_1(X) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], 1)$ be a ϕ–compatible one–dimensional representation. If $\alpha(\pi_1(X)) \subset GL(\mathbb{K}_\gamma[t^{\pm 1}], 1)$ is cyclic, then $\Delta_0^\alpha(t) = at - 1$ for some $a \in \mathbb{K} \setminus \{0\}$. Otherwise $\Delta_0^\alpha(t) = 1$.*

Proof. Equip X with a CW–structure with one 0–cell and then consider the chain complex for X as in Lemma 4.1. The lemma now follows easily from the observation that in $\mathbb{K}_\gamma[t^{\pm 1}]$ we have $\gcd(1 - at, 1 - bt) = 1$ if $a \neq b \in \mathbb{K}$. □

Lemma 4.4. *Let X be a 3–manifold with empty or toroidal boundary or let X be a 2–complex with $\chi(X) = 0$. Let $\phi \in H^1(M;\mathbb{Z})$ non–trivial. Let $\alpha : \pi_1(M) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], 1)$ be a ϕ–compatible one–dimensional representation. Assume that $\Delta_1^\alpha(t) \neq 0$. If X is a closed 3–manifold, then $deg(\Delta_2^\alpha(t)) = deg(\Delta_0^\alpha(t))$, otherwise $\Delta_2^\alpha(t) = 1$.*

Proof. First assume that X is a 3–manifold. Then the lemma follows from combining [Tu02b, Sections 4.3 and 4.4] with [FK05, Lemmas 4.7 and 4.9]. Note that the results of [FK05] also hold in the non–commutative setting. If X is a 2–complex then the argument in the proof of Theorem 1.2 shows that $H_2(X;\mathbb{K}_\gamma(t)^d) = 0$. But since X is a 2–complex we have $H_2(X;\mathbb{K}_\gamma[t^{\pm 1}]^d) \subset H_2(X;\mathbb{K}_\gamma(t)^d)$, hence $H_2(X;\mathbb{K}_\gamma[t^{\pm 1}]^d) = 0$ and $\Delta_2^\alpha(t) = 1$. □

Remark. Note that we can not apply the duality results of [FK05, Lemma 4.12 and Proposition 4.13] since the natural involution on $\mathbb{Z}[G]$ does not necessarily extend to an involution on $\mathbb{K}_\gamma[t^{\pm 1}]$, i.e. the representation $\mathbb{Z}[G] \to \mathbb{K}_\gamma[t^{\pm 1}]$ is not necessarily unitary.

It now follows immediately from Lemma 4.3 and 4.4 and the discussion in Section 5 that Theorem 1.2 contains the results of [Mc02], [Co04], [Ha05], [Tu02b] and [FK05].

Remark. Given a 2–complex X Turaev [Tu02a] defined a norm $||-||_X : H^1(X;\mathbb{R}) \to \mathbb{R}$, modelled on the definition of the Thurston norm of a 3–manifold. In [Tu02a] and [Tu02b] Turaev gives lower bounds for the Turaev norm which have the same form as certain lower bounds for the Thurston norm. Going through the proofs in [FK05] it is not hard to see that the obvious version of Theorem 1.2 for 2–complexes also holds.

If M is a 3–manifold with boundary, then it is homotopy equivalent to a 2–complex X. It is not known whether the Thurston norm of M agrees with the Turaev norm on X. But the fact that Theorem 1.2 holds in both cases, and the observation that $\deg(\tau(X, \alpha))$ is a homotopy invariant by Theorem 1.1 suggests that they do in fact agree.

5. Examples for skew fields and ϕ–compatible representations

5.1. **Skew fields of group rings.** A group G is called locally indicable if for every finitely generated subgroup $U \subset G$ there exists a non–trivial homomorphism $U \to \mathbb{Z}$.

Theorem 5.1. *Let G be a locally indicable and amenable group and let $\mathbb{F}$ be a commutative field. Then the following hold.*

(1) $\mathbb{F}[G]$ *is an Ore domain, in particular it embeds in its classical right ring of quotients* $\mathbb{K}(G)$.
(2) $\mathbb{K}(G)$ *is flat over* $\mathbb{F}[G]$.

It follows from [Hi40] that $\mathbb{F}[G]$ has no zero divisors. The first part now follows from [Ta57] or [DLMSY03, Corollary 6.3]. The second part is a well–known property of Ore localizations (cf. e.g. [Ra98, p. 99]). We call $\mathbb{K}(G)$ the *Ore localization* of $\mathbb{F}[G]$.

A group G is called poly–torsion–free–abelian (PTFA) if there exists a filtration

$$1 = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G$$

such that G_i/G_{i-1} is torsion free abelian. It is well–known that PTFA groups are amenable and locally indicable (cf. [St74]). The group rings of PTFA groups played an important role in [COT03], [Co04] and [Ha05].

5.2. **Examples for ϕ–compatible representations.** Let X be a connected CW–complex and $\phi \in H^1(X;\mathbb{Z})$. We give examples of ϕ–compatible representations.

Let $\mathbb{F}$ be a commutative field. Note that $\phi \in H^1(X;\mathbb{Z}) \cong \operatorname{Hom}(H_1(X;\mathbb{Z}), \langle t\rangle)$ induces a ϕ–compatible representation $\phi : \mathbb{Z}[\pi_1(X)] \to \mathbb{F}[t^{\pm 1}]$. Furthermore if $\alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{F}, d)$ is a representation, then $\pi_1(X)$ acts via $\alpha \otimes \phi$ on the $\mathbb{F}[t^{\pm 1}]$–module $\mathbb{F}^d \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \cong \mathbb{F}[t^{\pm 1}]^d$. We therefore get a representation $\alpha \otimes \phi : \pi_1(X) \to \mathrm{GL}(\mathbb{F}[t^{\pm 1}], d)$, which is clearly ϕ–compatible. In this particular case Theorem 1.2 was proved in [FK05].

To describe the ϕ–compatible representations of Cochran [Co04] and Harvey [Ha05, Ha06] we need the following definition.

Definition. Let π be a group, $\phi : \pi \to \mathbb{Z}$ an epimorphism and $\varphi : \pi \to G$ an epimorphism to a locally indicable and amenable group G such that there exists a map $\phi_G : G \to \mathbb{Z}$ (which is necessarily unique) such that

$$\begin{tikzcd} \pi \arrow[r, "\varphi"] \arrow[dr, "\phi"'] & G \arrow[d, "\phi_G"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes. Following [Ha06, Definition 1.4] we call (φ, ϕ) an *admissible pair*. If ϕ_G is an isomorphism, then (φ, ϕ) is called *initial*.

Now let $(\varphi : \pi_1(X) \to G, \phi)$ be an admissible pair for $\pi_1(X)$. In the following we denote $\operatorname{Ker}\{\phi : G \to \mathbb{Z}\}$ by $G'(\phi)$. When the homomorphism ϕ is understood we will write G' for $G'(\phi)$. Clearly G' is still a locally indicable and amenable group. Let $\mathbb{F}$ be any commutative field and $\mathbb{K}(G')$ the Ore localization of $\mathbb{F}[G]$. Pick an element $\mu \in G$ such that $\phi(\mu) = 1$. Let $\gamma : \mathbb{K}(G') \to \mathbb{K}(G')$ be the homomorphism given by

$\gamma(a) = \mu a \mu^{-1}$. Then we get a representation

$$\begin{array}{rcl} G & \to & \mathrm{GL}(\mathbb{K}(G')_\gamma[t^{\pm 1}],1) \\ g & \mapsto & (g\mu^{-\phi(g)}t^{\phi(g)}). \end{array}$$

It is clear that $\alpha : \pi_1(X) \to G \to \mathrm{GL}(\mathbb{K}(G')_\gamma[t^{\pm 1}],1)$ is ϕ–compatible. Note that the ring $\mathbb{K}(G')_\gamma[t^{\pm 1}]$ and hence the above representation depends on the choice of μ. We will nonetheless suppress γ in the notation since different choices of splittings give isomorphic rings. We often make use of the fact that $f(t)g(t)^{-1} \to f(\mu)g(\mu)^{-1}$ defines an isomorphism $\mathbb{K}(G')(t) \to \mathbb{K}(G)$ (cf. [Ha05, Proposition 4.5]). Similarly $\mathbb{Z}[G'][t^{\pm 1}] \xrightarrow{\cong} \mathbb{Z}[G]$.

An important example of admissible pairs is provided by Harvey's rational derived series of a group G (cf. [Ha05, Section 3]). Let $G_r^{(0)} = G$ and define inductively

$$G_r^{(n)} = \big\{ g \in G_r^{(n-1)} \,|\, g^k \in \big[G_r^{(n-1)}, G_r^{(n-1)}\big] \text{ for some } k \in \mathbb{Z}\setminus\{0\}\big\}.$$

Note that $G_r^{(n-1)}/G_r^{(n)} \cong \big(G_r^{(n-1)}/\big[G_r^{(n-1)}, G_r^{(n-1)}\big]\big)/\mathbb{Z}$–torsion. By [Ha05, Corollary 3.6] the quotients $G/G_r^{(n)}$ are PTFA groups for any G and any n. If $\phi : G \to \mathbb{Z}$ is an epimorphism, then $(G \to G/G_r^{(n)}, \phi)$ is an admissible pair for (G, ϕ) for any $n > 0$.

For example if K is a knot, $G = \pi_1(S^3 \setminus K)$, then it follows from [St74] that $G_r^{(n)} = G^{(n)}$, i.e. the rational derived series equals the ordinary derived series (cf. also [Co04] and [Ha05]).

Remark. Recall that for a knot K the knot exterior $S^3 \setminus \nu K$ is denoted by $X(K)$. Let $\pi = \pi_1(X(K))$ and let $\phi \in H^1(X(K);\mathbb{Z})$ primitive. Then

$$\overline{\delta}_n(K) = \dim_{\mathbb{K}(\pi'/(\pi')_r^{(n)})}(H_1(X(K), \mathbb{K}(\pi'/(\pi')_r^{(n)})[t^{\pm 1}])$$

is a knot invariant for $n > 0$. Cochran [Co04, p. 395, Question 5] asked whether $K \mapsto \overline{\delta}_n(K)$ is of finite type.

Eisermann [Ei00, Lemma 7] shows that the genus is not a finite type knot invariant. Cochran [Co04] showed that $\overline{\delta}_n(K) \le 2\,\mathrm{genus}(K)$ (cf. also Theorem 1.2 together with Corollary 3.6 and Lemmas 4.3 and 4.4). Eisermann's argument can now be used to show that $K \mapsto \overline{\delta}_n(K)$ is not of finite type either.

Let X be again be a connected CW–complex and $\phi \in H^1(X;\mathbb{Z})$. The two types of ϕ–compatible representations given above can be combined as follows. Let $\alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{F}, d)$ be a representation and let $\varphi : \pi_1(X) \to G$ be a homomorphism such that (φ, ϕ) is an admissible pair. Denote the Ore localization of $\mathbb{F}[G']$ by $\mathbb{K}(G')$. Then $\pi_1(X)$ acts via $\varphi \otimes \alpha$ on $\mathbb{K}(G')[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{F}^d \cong \mathbb{K}(G')[t^{\pm 1}]^d$. We therefore get a ϕ–compatible representation $\varphi \otimes \alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{K}(G')[t^{\pm 1}], d)$.

6. Comparing different ϕ–compatible maps

We now recall a definition from [Ha06].

Definition. Let π be a group and $\phi : \pi \to \mathbb{Z}$ an epimorphism. Furthermore let $\varphi_1 : \pi \to G_1$ and $\varphi_2 : \pi \to G_2$ be epimorphisms to locally indicable and amenable groups G_1 and G_2. We call $(\varphi_1, \varphi_2, \phi)$ an *admissible triple* for π if there exist epimorphisms $\varphi_2^1 : G_1 \to G_2$ (which is not an isomorphism) and $\phi_2 : G_2 \to \mathbb{Z}$ such that $\varphi_2 = \varphi_2^1 \circ \varphi_1$, and $\phi = \phi_2 \circ \varphi_2$.

The situation can be summarized in the following diagram

$$\begin{tikzcd} & G_1 \arrow[d, "\varphi_2^1"] \\ \pi \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"] \arrow[dr, "\phi"'] & G_2 \arrow[d, "\phi_2"] \\ & \mathbb{Z}. \end{tikzcd}$$

Note that in particular $(\varphi_i, \phi), i = 1, 2$ are admissible pairs for π. Given an admissible triple we can pick splittings $\mathbb{Z} \to G_i$ of $\varphi_i, i = 1, 2$ which make the following diagram commute:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[dr] & G_1 \arrow[d, "\varphi_2^1"] \\ & G_2. \end{tikzcd}$$

We therefore get an induced commutative diagram of ring homomorphisms

$$\begin{tikzcd} \mathbb{Z}[\pi] \arrow[r] \arrow[dr] & \mathbb{Z}[G_1'][t^{\pm 1}] \arrow[d, "\varphi_2^1"] \\ & \mathbb{Z}[G_2'][t^{\pm 1}]. \end{tikzcd}$$

Note that we are suppressing the notation for the twisting in the skew Laurent polynomial rings. Denote the ϕ–compatible maps $\mathbb{Z}[\pi] \to \mathbb{K}(G_i')[t^{\pm 1}], i = 1, 2$ by φ_i as well. For convenience we recall Theorem 1.3.

Theorem 1.3. *Let M be a 3–manifold whose boundary is a (possibly empty) collection of tori or let M be a 2–complex with $\chi(M) = 0$. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, d)$ be a representation and $(\varphi_1, \varphi_2, \phi)$ an admissible triple for $\pi_1(M)$. If $\tau(M, \varphi_2 \otimes \alpha) \neq 0$, then $\tau(M, \varphi_1 \otimes \alpha) \neq 0$. Furthermore in that case*

$$deg(\tau(M, \varphi_1 \otimes \alpha)) \geq deg(\tau(M, \varphi_2 \otimes \alpha)).$$

6.1. **Proof of Theorem 1.3 for closed 3–manifolds.** In this section let M be a closed 3–manifold. Choose a triangulation of M. Let T be a maximal tree in the 1-skeleton of the triangulation and let T' be a maximal tree in the dual 1-skeleton.

Following [Mc02, Section 5] we collapse T to form a single 0-cell and join the 3-simplices along T' to form a single 3-cell. Since $\chi(M)=0$ the number n of 1–cells equals the number of 2–cells. Consider the chain complex of the universal cover $\tilde{M}$:

$$0 \to C_3(\tilde{M})^1 \xrightarrow{\partial_3} C_2(\tilde{M})^n \xrightarrow{\partial_2} C_1(\tilde{M})^n \xrightarrow{\partial_1} C_0(\tilde{M})^1 \to 0,$$

where the supscript indicates the rank over $\mathbb{Z}[\pi_1(M)]$. Picking appropriate lifts of the (oriented) cells of M to cells of $\tilde{M}$ we get bases $\tilde{\sigma}_i = \{\tilde{\sigma}_i^1, \dots, \tilde{\sigma}_i^{r_i}\}$ for the $\mathbb{Z}[\pi_1(M)]$–modules $C_i(\tilde{M})$, such that if A_i denotes the matrix corresponding to ∂_i, then A_1 and A_3 are of the form

$$\begin{array}{rcll} A_3 &=& (1-g_1,\dots,1-g_n)^t, & g_i \in \pi_1(M) \\ A_1 &=& (1-h_1,\dots,1-h_n), & h_i \in \pi_1(M). \end{array}$$

Clearly $\{h_1,\dots,h_n\}$ is a generating set for $\pi_1(M)$. Since M is a closed 3–manifold $\{g_1,\dots,g_n\}$ is a generating set for $\pi_1(M)$ as well. In particular we can find $k,l \in \{1,\dots,n\}$ such that $\phi(g_k)\neq 0, \phi(h_l)\neq 0$.

In the following we write $\alpha_i = \varphi_i \otimes \alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{K}(G_i')[t^{\pm 1}] \otimes \mathbb{F}^d) \to \mathrm{GL}(\mathbb{K}(G_i')[t^{\pm 1}], d)$, $i=1,2$ and we write $\varphi = \varphi_2^1$.

Lemma 6.1.

$$\begin{array}{rcccl} deg(\alpha_1(1-h_l)) &=& deg(\alpha_2(1-h_l)) &=& d|\phi(h_l)| \\ deg(\alpha_1(1-g_k)) &=& deg(\alpha_2(1-g_k)) &=& d|\phi(g_k)|. \end{array}$$

In particular the matrices $\alpha_i(1-h_l), \alpha_i(1-g_k)$ are invertible over $\mathbb{K}(G_i')(t)$ for $i=1,2$.

Proof. Note that $\alpha_i(1-h_l) = \mathrm{id} - \alpha_i(h_l)$, $\alpha_i(1-g_k) = \mathrm{id} - \alpha_i(g_k)$ and that $\phi(h_l)\neq 0$, $\phi(g_k)\neq 0$. The lemma now follows from Lemma 4.2 since α_1 and α_2 are ϕ–compatible. □

Denote by B the result of deleting the k–th column and the l–row of A_2.

Lemma 6.2. *$\tau(M,\alpha_i)\neq 0$ if and only if $\alpha_i(B)$ is invertible. Furthermore if $\tau(M,\alpha_i)\neq 0$, then*

$$\tau(M,\alpha_i) = \alpha_i(1-g_k)^{-1}\alpha_i(B)\alpha_i(1-h_l)^{-1} \in K_1(\mathbb{K}(G_i')(t))/\pm\alpha_i(\pi_1(M)).$$

Proof. Denote the standard basis of $\mathbb{K}(G_i')(t)^d$ by $e_1,\dots,e_d$. We equip $C_j = C_j^{\alpha_i}(M;\mathbb{K}(G_i')(t)^d)$ with the ordered bases

$$\mathcal{C}_j = \{\tilde{\sigma}_j^1\otimes e_1,\dots,\tilde{\sigma}_j^1\otimes e_d,\dots,\tilde{\sigma}_i^{r_j}\otimes e_1,\dots,\tilde{\sigma}_i^{r_j}\otimes e_d\}.$$

Now let

$$\begin{array}{rcl} \xi_0 &=& \emptyset, \\ \xi_1 &=& \{ld+1,\dots,l(d+1)\}, \\ \xi_2 &=& \{1,\dots,nd\}\setminus\{kd+1,\dots,(k+1)d\}, \\ \xi_3 &=& \{1,\dots,d\}. \end{array}$$

Then $\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$ is a τ–chain for for C_*. Note that $A_1(\xi) = \alpha_i(1-h_l), A_2(\xi) = \alpha_i(B)$ and $A_3(\xi) = \alpha_i(1-g_k)$. Clearly $A_1(\xi)$ and $A_3(\xi)$ are invertible by Lemma 6.1. The proposition now follows immediately from Theorem 2.1. $\square$

Now assume that $\tau(M, \alpha_2) \neq 0$. Then $\alpha_2(B)$ is invertible over $\mathbb{K}(G_2')(t)$ by Lemma 6.2. Note that $\alpha_i(B)$ is defined over $\mathbb{Z}[G_i'][t^{\pm 1}] \subset \mathbb{K}(G_i')(t)$. In particular $\alpha_2(B) = \varphi(\alpha_1(B))$. It follows from the following lemma that $\alpha_1(B)$ is invertible as well.

Lemma 6.3. *Let P be an $r \times s$–matrix over $\mathbb{Z}[G_1'][t^{\pm 1}]$. If*

$$\begin{array}{rcl} \mathbb{Z}[G_2]^s & \to & \mathbb{Z}[G_2]^r \\ v & \mapsto & \varphi(P)v \end{array}$$

is invertible (injective) over $\mathbb{K}(G_2')(t)$, then P is invertible (injective) over $\mathbb{K}(G_1')(t)$.

Proof. Assume that multiplication by $\varphi(P)$ is injective over $\mathbb{K}(G_2')(t)$. Since $\mathbb{Z}[G_2] \to \mathbb{K}(G_2')(t) = \mathbb{K}(G_2)$ is injective it follows that $\varphi(P) : \mathbb{Z}[G_2]^s \to \mathbb{Z}[G_2]^r$ is injective. By Proposition 6.4 the map $P : \mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ is injective as well. Since $\mathbb{K}(G_1')(t) = \mathbb{K}(G_1)$ is flat over $\mathbb{Z}[G_1]$ it follows that $P : \mathbb{K}(G_1')(t)^s \to \mathbb{K}(G_1')(t)^r$ is injective.

If $\varphi(P)$ is invertible over the skew field $\mathbb{K}(G_2')(t)$, then $r = s$. But an injective homomorphism between vector spaces of the same dimension over a skew field is in fact an isomorphism. This shows that P is invertible over $\mathbb{K}(G_1')(t)$. $\square$

Proposition 6.4. *If G_1 is locally indicable, and if $\mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ is a map such that $\mathbb{Z}[G_1]^s \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2] \to \mathbb{Z}[G_1]^r \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2]$ is injective, then $\mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ is injective as well.*

Proof. Let $K = \operatorname{Ker}\{\varphi : G_1 \to G_2\}$. Clearly K is again locally indicable. Note that $\mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ can also be viewed as a map between free $\mathbb{Z}[K]$–modules. Pick any right inverse $\lambda : G_2 \to G_1$ of φ. It is easy to see that $g \otimes h \mapsto g\lambda(h) \otimes 1, g \in G_1, h \in G_2$ induces an isomorphism

$$\mathbb{Z}[G_1] \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2] \to \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[K]} \mathbb{Z}.$$

By assumption $\mathbb{Z}[G_1]^s \otimes_{\mathbb{Z}[K]} \mathbb{Z} \to \mathbb{Z}[G_2]^r \otimes_{\mathbb{Z}[K]} \mathbb{Z}$ is injective. Since K is locally indicable it follows immediately from [Ge83] or [HS83] (cf. also [St74] for the case of PTFA groups) that $\mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ is injective. $\square$

By Lemma 6.2 we now showed that if $\tau(M, \alpha_2) \neq 0$, then $\tau(M, \alpha_1) \neq 0$. Furthermore

$$\deg(\tau(M, \alpha_i)) = \deg(\alpha_i(B)) - \deg(\alpha_i(1-g_k)) - \deg(\alpha_i(1-h_l)), i = 1, 2.$$

Theorem 1.3 now follows immediately from Lemma 6.1 and from the following proposition.

Proposition 6.5. *Let P be an $r \times r$–matrix over $\mathbb{Z}[G_1'][t^{\pm 1}]$. If $\varphi(P)$ is invertible then*

$$deg(P) \geq deg(\varphi(P)).$$

Remark. (1) If $\varphi : R \to S$ is a homomorphism of *commutative* rings, and if P is a matrix over $R[t^{\pm 1}]$, then clearly

$$\begin{array}{rcl} \deg(P) = \deg(\det(P)) & \geq & \deg(\varphi(\det(P))) = \deg(\det(\varphi(P))) \\ & = & \deg(\varphi(P)). \end{array}$$

Similarly, several other results in this paper, e.g. Theorem 3.1 and Lemma 6.1 are clear in the commutative world, but require more effort in our non-commutative setting.

(2) If

$$(\mathbb{Z}[G_1'], \{f \in \mathbb{Z}[G_1'] | \varphi(f) \neq 0 \in \mathbb{Z}[G_2']\})$$

has the Ore property, then one can give an elementary proof of the proposition by first diagonalizing over $\mathbb{K}(G_2')$ and then over $\mathbb{K}(G_1')$. Since this is not known to be the case, we have to give a more indirect proof.

The following proof is based on arguments in [Co04] and [Ha06].

Proof of Proposition 6.5. Let $s = \deg(\varphi(P))$. Pick a map $f : \mathbb{Z}[G_1']^s \to \mathbb{Z}[G_1'][t^{\pm 1}]^r$ such that the induced map

$$\mathbb{K}(G_2')^s \to \mathbb{K}(G_2')[t^{\pm 1}]^r \to \mathbb{K}(G_2')[t^{\pm 1}]^r / \varphi(P)\mathbb{K}(G_2')[t^{\pm 1}]^r$$

is an isomorphism. Denote by $0 \to C_1 \xrightarrow{P} C_0 \to 0$ the complex

$$0 \to \mathbb{Z}[G_1'][t^{\pm 1}]^r \xrightarrow{P} \mathbb{Z}[G_1'][t^{\pm 1}]^r \to 0,$$

and denote by $0 \to D_0 \to 0$ the complex with $D_0 = \mathbb{Z}[G_1']^s$. We have a chain map $D_* \to C_*$ given by $f : D_0 \to C_0$. Denote by $\mathrm{Cyl}(D_* \xrightarrow{f} C_*)$ the mapping cylinder of the complexes. We then get a short exact sequence of complexes

$$0 \to D_* \to \mathrm{Cyl}(D_* \xrightarrow{f} C_*) \to \mathrm{Cyl}(D_* \xrightarrow{f} C_*)/D_* \to 0.$$

More explicitly we get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & C_1 \oplus D_0 \arrow[r, "\begin{pmatrix} \mathrm{id} & \mathrm{id} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} P & -f \\ 0 & \mathrm{id} \end{pmatrix}"] & C_1 \oplus D_0 \arrow[r] \arrow[d, "(P\ -f)"] & 0 \arrow[d] \\
0 \arrow[r] & D_0 \arrow[r, "\begin{pmatrix} 0 & \mathrm{id} \end{pmatrix}"] & C_0 \oplus D_0 \arrow[r] & C_0 \arrow[r] & 0.
\end{tikzcd}$$

Recall that $\mathrm{Cyl}(D_* \xrightarrow{f} C_*)$ and C_* are chain homotopic. Using the definition of f we therefore see that

$$f : H_0(D_*; \mathbb{K}(G_2')) \to H_0(\mathrm{Cyl}(D_* \xrightarrow{f} C_*), \mathbb{K}(G_2'))$$

is an isomorphism. Since P is invertible over $\mathbb{K}(G_2')(t)$ it follows that $H_1(\mathrm{Cyl}(D_* \xrightarrow{f} C_*); \mathbb{K}(G_2')) = 0$. It follows from the long exact homology sequence corresponding to the above short exact sequence of chain complexes that $H_1(\mathrm{Cyl}(D_* \xrightarrow{f} C_*)/D_*; \mathbb{K}(G_2')) =$

0, i.e. the matrix $\begin{pmatrix} P & -f \end{pmatrix}$ is injective over $\mathbb{K}(G_2')$. It follows from Lemma 6.3 that $H_1(\mathrm{Cyl}(D_* \xrightarrow{f} C_*)/D_*; \mathbb{K}(G_1')) = 0$ as well. Again looking at the long exact homology sequence we get that

$$f : H_0(D_*; \mathbb{K}(G_1')) \to H_0(\mathrm{Cyl}(D_* \xrightarrow{f} C_*); \mathbb{K}(G_1')) = H_0(C_*; \mathbb{K}(G_1'))$$

is an injection. Hence

$$\begin{array}{rcl} \deg(\varphi(P)) = s & = & \dim_{\mathbb{K}(G_2')}(H_0(D_*; \mathbb{K}(G_2'))) \\ & = & \dim_{\mathbb{K}(G_1')}(H_0(D_*; \mathbb{K}(G_1'))) \\ & \leq & \dim_{\mathbb{K}(G_1')}(H_0(C_*; \mathbb{K}(G_1'))) \\ & = & \deg(P). \end{array}$$

□

6.2. **Proof of Theorem 1.3 for 3–manifolds with boundary and for 2–complexes.** First let X be a finite connected 2–complex with $\chi(X) = 0$. We can give X a CW–structure with one 0–cell. If n denotes the number n of 1–cells, then $n-1$ equals the number of 2–cells. Now consider the chain complex of the universal cover $\tilde{X}$:

$$0 \to C_2(\tilde{X})^{n-1} \xrightarrow{\partial_2} C_1(\tilde{X})^n \xrightarrow{\partial_1} C_0(\tilde{X})^1 \to 0.$$

As in Section 6.1 we pick lifts of the cells of X to cells of $\tilde{X}$ to get bases such that if A_i denotes the matrix corresponding to ∂_i, then

$$A_1 = (1 - h_1, \dots, 1 - h_n).$$

Note that $\{h_1, \dots, h_n\}$ is a generating set for $\pi_1(X)$. Let $l \in \{1, \dots, n\}$ such that $\phi(l) \neq 0$. The proof of Lemma 6.2 can easily be modified to prove the following.

Lemma 6.6. *Denote by B the result of deleting the l–row of A_2. Then $\tau(X, \alpha) \neq 0$ if and only if $\alpha(B)$ is invertible. Furthermore if $\tau(X, \alpha) \neq 0$, then*

$$\tau(X, \alpha) = \alpha(B)\alpha(1 - h_l)^{-1} \in K_1(\mathbb{K}(G_i')(t))/ \pm \alpha_i(\pi_1(X)).$$

The proof of Theorem 1.3 for closed manifolds can now easily be modified to cover the case of 2–complexes X with $\chi(X) = 0$.

Now let M be again a 3–manifold whose boundary consists of a non–empty set of tori. A duality argument shows that $\chi(M) = \frac{1}{2}\chi(\partial(M)) = 0$. Clearly M is homotopy equivalent to a 2–complex. Reidemeister torsion is not a homotopy invariant but the following lemma still allows us to reduce the case of a 3–manifold with boundary to the case of a 2–complex.

Lemma 6.7. [Tu01, p. 56 and Theorem 9.1] *Let M be a 3–manifold with boundary. Then there exists a 2–complex X and a* simple *homotopy equivalence $M \to X$. In particular, if $\alpha : \pi_1(X) \cong \pi_1(M) \to GL(R, d)$ is a representation such that $H_*(X, R^d) = 0$, then*

$$\tau(M, \alpha) = \tau(X, \alpha) \in K_1(R)/ \pm \alpha(\pi_1(M)).$$

Theorem 1.3 for 3–manifolds with boundary now follows from Theorem 1.3 for 2–complexes X with $\chi(X) = 0$.

7. Harvey's monotonicity theorem for groups

Let π be a finitely presented group and let $(\varphi : \pi \to G, \phi : \pi \to \mathbb{Z})$ be an admissible pair for π. Consider $G' = G'(\phi_G)$ and pick a splitting $\mathbb{Z} \to G$ of ϕ_G. As in Section 5.2 we can consider the skew Laurent polynomial ring $\mathbb{K}(G')[t^{\pm 1}]$ together with the ϕ–compatible representation $\pi \to \mathrm{GL}(\mathbb{K}(G')[t^{\pm 1}], 1)$.

Following [Ha06, Definition 1.6] we define $\overline{\delta}_G(\phi)$ to be zero if $H_1(\pi, \mathbb{K}(G')[t^{\pm 1}])$ is not $\mathbb{K}(G')[t^{\pm 1}]$–torsion and

$$\overline{\delta}_G(\phi) = \dim_{\mathbb{K}(G')}(H_1(\pi, \mathbb{K}(G')[t^{\pm 1}]))$$

otherwise. We give an alternative proof for the following result of Harvey [Ha06, Theorem 2.9].

Theorem 7.1. *If $\pi = \pi_1(M)$, M a closed 3–manifold, and if $(\varphi_1 : \pi \to G_1, \varphi_2 : \pi \to G_2, \phi)$ is an admissible triple for π, then*

$$\begin{array}{rcll} \overline{\delta}_{G_1}(\phi) & \geq & \overline{\delta}_{G_2}(\phi), & \textit{if } (\varphi_2, \phi) \textit{ is not initial,} \\ \overline{\delta}_{G_1}(\phi) & \geq & \overline{\delta}_{G_2}(\phi) - 2, & \textit{otherwise.} \end{array}$$

Proof. We clearly only have to consider the case that $\overline{\delta}_{G_2}(\phi) > 0$. We can build $K(\pi, 1)$ by adding i–handles to M with $i \geq 3$. It therefore follows that for the admissible pairs $(\varphi_i : \pi \to G_i, \phi)$ we have

$$\overline{\delta}_{G_i}(\phi) = \dim_{\mathbb{K}(G_i')}(H_1(K_1(\pi, 1); \mathbb{K}(G_i')[t^{\pm 1}])) = \dim_{\mathbb{K}(G_i')}(H_1(M; \mathbb{K}(G_i')[t^{\pm 1}])).$$

We combine this equality with Theorem 1.3, Corollary 3.6 and Lemmas 3.2, 4.3, 4.4. The theorem follows now immediately from the observation that $\mathrm{Im}\{\pi_1(M) \to G_i \to \mathrm{GL}(\mathbb{K}(G_i')[t^{\pm 1}], 1)\}$ is cyclic if and only if $\phi : G_i \to \mathbb{Z}$ is an isomorphism. □

This monotonicity result gives in particular an obstruction for a group π to be the fundamental group of a closed 3–manifold. For example Harvey [Ha06, Example 3.2] shows that as an immediate consequence we get the (well–known) fact that $\mathbb{Z}^m, m \geq 4$ is not a 3–manifold group.

Remark. In [FK05b] the author and Taehee Kim consider the case $\pi = \pi_1(M)$, where M is a closed 3–manifold. Given an admissible pair $(\varphi : \pi \to G, \phi)$ we show (under a mild assumption) that $\overline{\delta}_G(\phi)$ is even, generalizing a result of Turaev ([Tu86, p. 141]). Furthermore in [FH06] the author and Shelly Harvey will show that given $\pi \to G$, G locally indicable and amenable, the map

$$\begin{array}{rcl} \mathrm{Hom}(G, \mathbb{Z}) & \to & \mathbb{Z} \\ \phi & \mapsto & \overline{\delta}_G(\phi) \end{array}$$

defines a seminorm on $\mathrm{Hom}(G, \mathbb{Z})$.

Let π be a finitely presented group of deficiency at least one, for example $\pi = \pi_1(M)$ where M is a 3–manifold with boundary. Using a presentation of deficiency one we can build a 2–complex X with $\chi(X) = 0$ and $\pi_1(X) = \pi$. The same proof as the proof of Theorem 7.1 now gives the following theorem of Harvey [Ha06, Theorem 2.2]. In the case that $\pi = \pi_1(S^3 \setminus K)$, K a knot, this was first proved by Cochran [Co04].

Theorem 7.2. *If π is a finitely presented group of deficiency one and if $(\varphi_1, \varphi_2, \phi)$ is an admissible triple for π, then*

$$\begin{array}{rcll} \overline{\delta}_{G_1}(\phi) & \geq & \overline{\delta}_{G_2}(\phi), & \text{if } (\varphi_2,\phi) \text{ is not initial,} \\ \overline{\delta}_{G_1}(\phi) & \geq & \overline{\delta}_{G_2}(\phi) - 1, & \text{otherwise.} \end{array}$$

8. Open questions and problems

Let M be a 3–manifold and $\phi \in H^1(M;\mathbb{Z})$. We propose the following three problems for further study.

(1) If $(\varphi : \pi_1(M) \to G, \phi)$ is an admissible pair for $\pi_1(M)$ and if $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, d)$ factors through φ, does it follow that

$$\frac{1}{d}\deg(\tau(M,\alpha)) \leq \deg(\tau(M, \mathbb{Z}[\pi_1(M)] \to \mathbb{K}(G')(t)))?$$

Put differently, are the Thurston norm bounds of Cochran and Harvey optimal, i.e. at least as good as the Thurston norm bounds of [FK05] for any representation factoring through G.

(2) It is well–known that in many cases $\deg(\tau(M, \mathbb{Z}[\pi_1(M)] \to \mathbb{K}(G')(t))) < ||\phi||_T$ for any admissible pair $(\varphi : \pi_1(M) \to G, \phi)$. For example this is the case if K is a knot with $\Delta_K(t) = 1$ and $M = X(K)$. It is an interesting question whether invariants can be defined for any map $\pi_1(M) \to G$, G a (locally indicable) torsion–free group. For example it might be possible to work with $\mathcal{U}(G)$ the algebra of affiliated operators (cf. e.g. [Re98]) instead of $\mathbb{K}(G)$. If such an extension is possible, then it is a natural question whether the Thurston norm is determined by such more general bounds. This might be too optimistic in the general case, but it could be true in the case of a knot complement.

(3) If (M, ϕ) fibers over S^1, then it is well–known that the corresponding Alexander polynomial defined over $\mathbb{Z}[t^{\pm 1}]$ is monic, i.e. the top coefficient is ± 1. Because of the high–degree of indeterminacy of Alexander polynomials over skew Laurent polynomial rings a corresponding statement is meaningless. Since Reidemeister torsion has a much smaller indeterminacy it is potentially possible to use it to extend the above fiberedness obstruction as in [GKM05].

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