

# THE VIRTUAL FIBERING THEOREM FOR 3-MANIFOLDS

STEFAN FRIEDL AND TAKAHIRO KITAYAMA

ABSTRACT. In 2007 Agol showed that if  $N$  is an aspherical compact 3-manifold with empty or toroidal boundary such that  $\pi_1(N)$  is virtually RFRS, then  $N$  is virtually fibered. We give a largely self-contained proof of Agol's theorem using complexities of sutured manifolds.

## 1. INTRODUCTION

In 1982 Thurston [Th82, Question 18] asked whether every hyperbolic 3-manifold is virtually fibered, i.e. whether every hyperbolic 3-manifold admits a finite cover which fibers over  $S^1$ .

Evidence towards an affirmative answer was given by many authors, including Agol–Boyer–Zhang [ABZ08], Aitchison–Rubinstein [AR99], Button [Bu05], DeBlois [DeB10], Gabai [Ga86], Guo–Zhang [GZ09], Leininger [Lei02], Reid [Re95] and Walsh [Wa05].

The first general virtual fibering theorem was proved by Agol in 2007. In order to state the theorem we need two more definitions:

- (1) a group  $\pi$  is *residually finite rationally solvable* or *RFRS* if  $\pi$  admits a filtration  $\pi = \pi_0 \supset \pi_1 \supset \pi_2 \dots$  such that the following hold:
  - (a)  $\bigcap_k \pi_k = \{1\}$ ,
  - (b) for any  $k$  the group  $\pi_k$  is a normal, finite index subgroup of  $\pi$ ,
  - (c) for any  $k$  the map  $\pi_k \rightarrow \pi_k/\pi_{k+1}$  factors through  $\pi_k \rightarrow H_1(\pi_k; \mathbb{Z})/\text{torsion}$ .
- (2) Given a 3-manifold  $N$ , we say  $\phi \in H^1(N; \mathbb{Q}) = \text{Hom}(\pi_1(N), \mathbb{Q})$  is *fibered* if there exists an  $n \in \mathbb{N}$  and a fibration  $p: N \rightarrow S^1$  such that  $\phi = \frac{1}{n} \cdot p_*: \pi_1(N) \rightarrow \mathbb{Q}$ .

We can now state Agol's [Ag08, Theorem 5.1] theorem.

**Theorem 1.1. (Agol)** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary such that  $\pi_1(N)$  is virtually RFRS. Let  $\phi \in H^1(N; \mathbb{Q})$  be non-trivial. Then there exists a finite cover  $q: \tilde{N} \rightarrow N$  such that  $q^*\phi$  is the limit of a sequence of fibered classes in  $H^1(\tilde{N}; \mathbb{Q})$ .*

The key idea in the proof of the theorem is that the RFRS condition ensures that given a Thurston norm minimizing surface one can find ‘enough’ surfaces in finite covers to ‘reduce the complexity of the guts’ by perturbing the initial surface

appropriately. Agol uses the theory of ‘least-weight taut normal surfaces’ introduced and developed by Oertel [Oe86] and Tollefson and Wang [TW96] to carry through this program.

In the introduction to [Ag08] Agol writes that ‘the natural setting [...] lies in sutured manifold hierarchies’. We pick up this suggestion and provide a proof of Theorem 1.1 using sutured manifolds and their hierarchies. In our proof we only use standard results about the Thurston norm and sutured manifold decompositions (see [Th86, Ga83]) and a complexity for sutured manifolds defined by Gabai [Ga83]. At the core our argument also follows the above ‘key idea’, but for the most part the treatment of the argument is somewhat different from Agol’s original proof.

In a stunning turn of events it has been shown over the last few years that most 3-manifold groups are in fact virtually RFRS. More precisely, the following theorem was proved by Agol [Ag12] and Wise [Wi12] in the hyperbolic case and by Przytycki-Wise [PW12] in the case of a 3-manifold with a non-trivial JSJ decomposition.

**Theorem 1.2. (Agol, Przytycki-Wise, Wise)** *If  $N$  is an irreducible 3-manifold with empty or toroidal boundary which is not a graph manifold, then  $\pi_1(N)$  is virtually RFRS.*

Furthermore it follows from work of Liu [Li11] and Przytycki-Wise [PW11] that the fundamental group of a graph manifold with boundary is also virtually RFRS. Finally Liu [Li11] showed that the fundamental group of a closed aspherical graph manifold is virtually RFRS if and only if  $N$  is non-positively curved, i.e. if it admits a Riemannian metric of non-positive curvature. Combining these results with Theorem 1.1 we thus obtain the following result.

**Theorem 1.3.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary and let  $\phi \in H^1(N; \mathbb{Q})$  be non-trivial. Suppose one of the following two conditions hold:*

- (1)  $N$  is not a closed graph manifold, or
- (2)  $N$  is a closed graph manifold which is non-positively curved,

*then there exists a finite cover  $q: \tilde{N} \rightarrow N$  such that  $q^*\phi$  is the limit of a sequence of fibered classes in  $H^1(\tilde{N}; \mathbb{Q})$ .*

*Remark.* (1) If  $\pi_1(N)$  is infinite and virtually RFRS, then there exists a finite cover with positive first Betti number. It therefore follows from Theorem 1.3 that if  $N$  is an irreducible 3-manifold with empty or toroidal boundary which is not a graph manifold, then  $N$  is virtually fibered. In particular Theorem 1.3 gives an affirmative answer to Thurston’s question.

- (2) The work of Agol [Ag12], Przytycki-Wise [PW12] and Wise [Wi12] resolves not only Thurston’s Virtual Fibered Conjecture but also has a long list of other consequences. We refer to the survey paper [AFW12] for a comprehensive discussion.

- (3) Let  $N$  be an aspherical 3-manifold. If  $N$  is not a closed graph manifold, then it follows from work of Leeb [Leb95] (see also [Br99, Br01]) that  $N$  is non-positively curved. Combining this with the above results we see that an aspherical 3-manifold  $N$  is non-positively curved if and only if  $\pi_1(N)$  is virtually RFRS.
- (4) There are graph manifolds which are virtually fibered but whose fundamental groups are not virtually RFRS. One class of such graph manifolds is given by Sol-manifolds (see e.g. [Ag12, AFW12]).
- (5) It follows from classical arguments that the conclusion of Theorem 1.3 in fact holds for any virtually fibered graph manifold.
- (6) The conclusion that any cohomology class can be approximated by fibered classes in a suitable finite cover has been crucial in the applications to twisted Alexander polynomials and to the study of surfaces of minimal complexity in 4-manifolds with a free  $S^1$ -action (see [FV12a, FV12b]).

For completeness' sake we also mention Agol's result on finite covers of taut sutured manifolds, even though it plays no role in the later part of the paper. Using the argument in the proof of Theorem 1.1 and using a 'doubling' argument Agol proves that given any taut sutured manifold with virtually RFRS fundamental group there exists a finite cover which admits a depth one taut oriented foliation. (We will not define these notions and we refer instead to [Ga83, Ag08, CC03] for background information and precise definitions.) More precisely, Agol [Ag08, Theorem 6.1] proved the following result.

**Theorem 1.4. (Agol)** *Let  $(N, \gamma)$  be a taut sutured manifold such that  $\pi_1(N)$  is virtually RFRS. Then there exists a finite covering  $p: (\tilde{N}, \tilde{\gamma}) \rightarrow (N, \gamma)$  such that  $(\tilde{N}, \tilde{\gamma})$  admits a depth one taut oriented foliation.*

In the above discussion we already saw that the fundamental group of any irreducible 3-manifold with non-trivial toroidal boundary is virtually RFRS. A straightforward doubling argument (see e.g. [AFW12, Section 5.3]) shows that in fact the fundamental group of any irreducible 3-manifold with non-empty incompressible boundary is virtually RFRS. Combining this observation with Theorem 1.4 we obtain the following theorem.

**Theorem 1.5.** *Let  $(N, \gamma)$  be a taut-sutured manifold. Then there exists a finite covering  $p: (\tilde{N}, \tilde{\gamma}) \rightarrow (N, \gamma)$  such that  $(\tilde{N}, \tilde{\gamma})$  admits a depth one taut-oriented foliation.*

The paper is organized as follows. In Sections 2 and 3 we recall some standard facts about the Thurston norm and sutured manifolds. Along the way we will also make some preparations for the proof of Proposition 4.1. This proposition allows us to carefully arrange surfaces to 'cut the guts' of a given surface. This result is the technical heart of the paper and we give a very detailed proof of it. In Proposition 4.2 we then summarize the effect of 'cutting by a surface' on the complexities of the

guts of a given surface. Finally in the last section we present our proof of Theorem 1.1.

**Convention.** All manifolds are assumed to be compact and oriented. We do not assume that spaces are connected, nonetheless, if we talk about the fundamental group of a space without specifying a base point, then we implicitly assume that the space is connected. All surfaces in a 3-manifold are assumed to be properly embedded, unless we say explicitly otherwise. If  $N$  is a 3-manifold and  $R \subset N$  a properly embedded surface and  $a > 0$ , then we denote by  $R \times [-a, a]$  a neighborhood of  $R$  such that  $(R \times [-a, a]) \cap \partial N = \partial R \times [-a, a]$ . Finally, given a submanifold  $S \subset M$  we denote by  $\nu S$  an open tubular neighborhood around  $S$ .

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## 2. THE THURSTON NORM

**2.1. The Thurston norm and fibered classes.** Let  $S$  be a surface with connected components  $S_1 \cup \dots \cup S_k$ . We then refer to

$$\chi_-(S) := \sum_{i=1}^k \max\{-\chi(S_i), 0\}$$

as the *complexity* of  $S$ . Now let  $N$  be a 3-manifold with empty or toroidal boundary and let  $\phi \in H^1(N; \mathbb{Z})$ . It is well-known that any class in  $H^1(N; \mathbb{Z})$  is dual to a properly embedded surface. The *Thurston norm of  $\phi$*  is defined as

$$x_N(\phi) := \min\{\chi_-(S) \mid S \subset N \text{ properly embedded and dual to } \phi\}.$$

We will drop the subscript ‘ $N$ ’, when the manifold  $N$  is understood.

Thurston [Th86] showed that  $x$  is a seminorm on  $H^1(N; \mathbb{Z})$ , which implies that  $x$  can be extended to a seminorm on  $H^1(N; \mathbb{Q})$ . We denote the seminorm on  $H^1(N; \mathbb{Q})$  also by  $x$ . Throughout the paper we will freely go back and forth between  $H^1(N; \mathbb{Q})$  and  $H_2(N, \partial N; \mathbb{Q})$ . In particular we will consider the Thurston norm also for classes in  $H_2(N, \partial N; \mathbb{Q})$ .

Thurston furthermore proved that the Thurston norm ball

$$B(N) := \{\phi \in H^1(N; \mathbb{Q}) \mid x(\phi) \leq 1\}$$

is a (possibly non-compact) finite convex polytope. A *Thurston cone of  $N$*  is defined to be either an open cone  $\{rf \mid r > 0, f \in F\}$  on a face  $F$  of  $B(N)$  or a maximal connected subset of  $H^1(N; \mathbb{Q}) \setminus \{0\}$  on which  $x$  vanishes. The Thurston cones have the following properties:

- (1) if  $\phi, \psi$  lie in a Thurston cone  $C$ , then  $\phi + \psi \in C$  and given any  $r > 0$  we have  $r\phi \in C$ ,
- (2) the Thurston cones are disjoint and their union equals  $H^1(N; \mathbb{Q}) \setminus \{0\}$ ,

- (3) the Thurston norm is additive precisely on the closures of Thurston cones, i.e. given  $\phi, \psi \in H^1(N; \mathbb{Q})$  we have

$$x(\phi + \psi) = x(\phi) + x(\psi) \Leftrightarrow \text{there exists a Thurston cone } C \text{ with } \phi, \psi \in \overline{C}.$$

In the following we say that an integral class  $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$  is *fibred* if there exists a fibration  $p: N \rightarrow S^1$  such that  $\phi = p_*: \pi_1(N) \rightarrow \mathbb{Z}$ . We say  $\phi \in H^1(N; \mathbb{Q})$  is *fibred* if a non-trivial integral multiple of  $\phi$  is fibred. Thurston [Th86] showed that the set of fibred classes equals the union of top-dimensional Thurston cones. These cones are referred to as the *fibred cones* of  $N$ .

**2.2. Subordination.** Given two non-zero cohomology classes  $\phi, \psi \in H^1(N; \mathbb{Q})$  we say  $\phi$  is *subordinate to*  $\psi$  if  $\phi \in \overline{C}$  where  $C$  is the unique Thurston cone which contains  $\psi$ . We collect several properties of subordination in a lemma:

- Lemma 2.1.** (1) *Subordination is transitive, i.e. if  $\phi$  is subordinate to  $\psi$  and  $\psi$  is subordinate to  $\varphi$ , then  $\phi$  is subordinate to  $\varphi$ .*  
 (2) *Given any two non-zero cohomology classes  $\phi, \psi \in H^1(N; \mathbb{Q})$  there exists an  $m \in \mathbb{N}$  such that  $\phi$  is subordinate to  $m\phi + \psi$ .*  
 (3) *If  $\phi$  is subordinate to  $m\phi + \psi$  for some  $m$ , then  $\phi$  is also subordinate to  $k\phi + \psi$  for any  $k \geq m$ .*  
 (4) *Let  $p: \tilde{N} \rightarrow N$  be a finite cover and let  $\phi, \psi \in H^1(N; \mathbb{Q})$  be two non-zero cohomology classes. Then  $\phi$  is subordinate to  $\psi$  if and only if  $p^*\phi$  is subordinate to  $p^*\psi$ .*

The first three statements are straightforward to verify. The last statement is an immediate consequence of the fact that given any cover  $p: \tilde{N} \rightarrow N$  of degree  $k$  and any  $\phi \in H^1(N; \mathbb{Q})$  we have  $x(p^*\phi) = k \cdot x(\phi)$  (see [Ga83, Corollary 6.18]). Put differently,  $p^*: H^1(N; \mathbb{Q}) \rightarrow H^1(\tilde{N}; \mathbb{Q})$  is up to the scaling factor  $k$  an isometry of vector spaces.

### 3. COMPLEXITIES FOR SUTURED MANIFOLDS

**3.1. Sutured manifolds.** A sutured manifold  $(M, R_-, R_+, \gamma)$  consists of a 3-manifold  $M$  together with a decomposition of its boundary

$$\partial M = -R_- \cup \gamma \cup R_+$$

into oriented submanifolds where the following conditions hold:

- (1)  $\gamma$  is a disjoint union of annuli.
- (2)  $R_-$  and  $R_+$  are disjoint.
- (3) If  $A$  is a component of  $\gamma$ , then  $R_- \cap A$  is a boundary component of  $A$  and of  $R_-$ , and similarly for  $R_+ \cap A$ . Furthermore,  $[R_+ \cap A] = [R_- \cap A] \in H_1(A; \mathbb{Z})$  where we endow  $R_{\pm} \cap A$  with the orientation coming from the boundary of  $R_{\pm}$ .

(Here we give  $\partial M$  the orientation such that  $R_+$  are precisely those components of  $\partial M \setminus \gamma$  whose normal vectors point out of  $M$ .)

We sometimes just write  $(M, \gamma)$  instead of  $(M, R_-, R_+, \gamma)$ , but it is important to remember that  $R_-$  and  $R_+$  are part of the structure of a sutured manifold.

Finally a simple example of a sutured manifold is given as follows: Let  $R$  be a surface, then

$$(R \times [-1, 1], R \times -1, R \times 1, \partial R \times [-1, 1])$$

is a sutured manifold. We refer to such a sutured manifold as a *product sutured manifold*.

**3.2. Taut sutured manifolds and Thurston norm minimizing surfaces.** A sutured manifold  $(M, R_-, R_+, \gamma)$  is called *taut* if  $M$  is irreducible and if  $R_-$  and  $R_+$  have minimal complexity among all surfaces representing  $[R_-] = [R_+] \in H_2(M, \gamma; \mathbb{Z})$ .

Let  $R$  be a surface in a closed 3-manifold  $N$ . We say that  $R$  is *good* if  $R$  has no spherical components and no component which bounds a solid torus. Furthermore we say  $R$  is *Thurston norm minimizing* if  $R$  has minimal complexity in its homology class  $[R] \in H_2(N, \partial N; \mathbb{Z})$ . It is clear that any homology class can be represented by a good Thurston norm minimizing surface.

Note that if  $R$  is a good Thurston norm minimizing surface in an irreducible 3-manifold, then a standard argument using the Loop Theorem (see [He76, Chapter 4]) shows that  $R$  is also  $\pi_1$ -injective.

To a surface  $R$  in a closed 3-manifold  $N$  we now associate the sutured manifold

$$N(R) = (N \setminus R \times (-1, 1), R \times 1, R \times -1, \emptyset).$$

We conclude this section with the following two observations regarding  $N(R)$ :

- (1) If  $N$  is irreducible and if  $R$  is a Thurston norm minimizing surface without spherical components, then  $N(R)$  is a taut sutured manifold.
- (2) The surface  $R$  is a fiber of a fibration  $N \rightarrow S^1$  if and only if  $N(R)$  is a product sutured manifold.

**3.3. Sutured manifold decompositions.** We now recall the definition of a sutured manifold decomposition which also goes back to Gabai [Ga83]. Let  $(M, R_-, R_+, \gamma)$  be a sutured manifold. We say that a properly embedded surface  $S$  is a *decomposition surface* if the following condition holds: for any component  $A$  of  $\gamma$  every component of  $A \cap S$  is either a non-separating arc in  $A$ , or it is a closed curve which is homologous to  $[A \cap R_-] = [A \cap R_+] \in H_1(A; \mathbb{Z})$ .

Let  $S$  be a decomposition surface of  $(M, R_-, R_+, \gamma)$ . Gabai [Ga83, Definition 3.1] defines the sutured manifold decomposition

$$(M, R_-, R_+, \gamma) \xrightarrow{S} (M', R'_-, R'_+, \gamma')$$

where

$$\begin{aligned} M' &= M \setminus S \times (-1, 1), \\ \gamma' &= (\gamma \cap M') \cup \overline{\nu(S'_+ \cap R_-)} \cup \overline{\nu(S'_- \cap R_+)}, \\ R'_+ &= ((R_+ \cap M') \cup S'_+) \setminus \text{int } \gamma' \\ R'_- &= ((R_- \cap M') \cup S'_-) \setminus \text{int } \gamma'. \end{aligned}$$

Here  $S'_+$  (respectively  $S'_-$ ) is the union of the components of  $(S \times -1 \cup S \times 1) \cap M'$  whose normal vector points out of (respectively into)  $M'$ . We say that a decomposition surface  $S$  is *taut* if all the components of the sutured manifold decomposition along  $S$  are taut.

We make the following observations:

- (1) If  $\gamma = \emptyset$ , then any surface in  $M$  is a decomposition surface for  $(M, \gamma)$ .
- (2) If each component of  $S$  is a  $\pi_1$ -injective surface, then for any component of  $M'$  the inclusion into  $M$  induces a monomorphism of fundamental groups.
- (3) If  $N$  is a closed 3-manifold and if  $R \subset N$  is a closed surface, then  $R$  is a decomposition surface for the sutured manifold  $(N, \emptyset, \emptyset, \emptyset)$ , and  $N(R)$  is precisely the result of the decomposition along  $R$ .
- (4) If  $(M, \gamma)$  is a sutured manifold and if  $S \subset M$  is a decomposition surface which is boundary parallel, then the resulting sutured manifolds  $(M', \gamma')$  is a union of product sutured manifolds and a sutured manifold  $(M'_0, \gamma'_0)$  which is canonically diffeomorphic to  $(M, \gamma)$ .

We conclude this section with the following elementary but very useful lemma (see also [Ga83, Lemma 3.12]).

**Lemma 3.1.** *Let  $(M, R_-, R_+, \gamma)$  be a taut sutured manifold. Suppose  $S$  is a decomposition surface in  $M$  such that one of the following holds:*

- (1)  *$S$  is an annulus such that one boundary component of  $A$  lies on  $R_-$  and the other one lies on  $R_+$ , or*
- (2)  *$S$  is a disk such that  $S \cap R_-$  and  $S \cap R_+$  consist of an interval each,*

*then  $S$  is a taut decomposition surface.*

**3.4. The guts of a sutured manifold.** A sutured submanifold of a sutured manifold  $(M, R_-, R_+, \gamma)$  is a submanifold  $N$  of  $M$  such that

$$(N, N \cap R_-, N \cap R_+, \overline{\partial N \setminus (N \cap R_- \cup N \cap R_+)})$$

is a sutured manifold. An isotopy between two sutured submanifolds  $N_0$  and  $N_1$  is a map  $f: N \times [0, 1] \rightarrow M$  with  $N_i = f(N, i)$ ,  $i = 0, 1$  and such that for any  $t \in [0, 1]$  the image  $f(N, t)$  is a sutured submanifold. In the following, when no confusion can occur, we will refer to a sutured submanifold just as a submanifold.

A *product submanifold* of  $(M, R_-, R_+, \gamma)$  is a submanifold  $P$  such that

$$(P, P \cap R_-, P \cap R_+, \overline{\partial P \setminus (P \cap R_- \cup P \cap R_+)})$$

is a product sutured manifold. We say a product submanifold is *essential* if no component of  $P \cap R_-$  is a disk.

The following theorem is now a well-known consequence of the Characteristic Pair Theorem of Jaco-Shalen [JS79] and Johannson [Jo79]. (See also the exposition in [MS88, Section IV.4].)

**Theorem 3.2.** *Let  $(M, R_-, R_+, \gamma)$  be an irreducible sutured manifold. Then there exists an essential product submanifold  $P \subset M$  with the following properties:*

- (1) *No component of  $P$  can be isotoped as a sutured submanifold into another component of  $P$ .*
- (2) *If  $Q \subset M$  is a connected essential product submanifold, then it can be isotoped into a component of  $P$ .*

*Furthermore such a product submanifold is unique up to isotopy.*

A *window* of  $(M, R_-, R_+, \gamma)$  is a component of the essential product submanifold  $P \subset M$ , furthermore a *gut* of  $(M, R_-, R_+, \gamma)$  is a component of  $\overline{M \setminus P}$ . We collect several basic facts about windows and guts in the following lemma:

**Lemma 3.3.** *Let  $(M, R_-, R_+, \gamma)$  be a sutured manifold such that  $M$  is irreducible. Then the following hold:*

- (1) *If  $A_1, \dots, A_k$  denote the annuli and disks separating windows from guts, then the windows and guts are obtained from  $M$  by sutured manifold decomposition along  $A_1 \cup \dots \cup A_k$ .*
- (2) *The guts and windows are  $\pi_1$ -injective submanifolds of  $M$ .*
- (3) *The fundamental group of a gut is non-trivial.*
- (4) *If  $(M, R_-, R_+, \gamma)$  is taut, then the windows and guts are also taut.*
- (5) *If  $p: (\widetilde{M}, \widetilde{\gamma}) \rightarrow (M, \gamma)$  is a finite cover of sutured manifolds, then the windows and guts of  $(\widetilde{M}, \widetilde{\gamma})$  are precisely the preimages of the windows and guts of  $(M, \gamma)$ .*

Here the first statement is elementary. The second statement follows from the observation that the  $A_1, \dots, A_k$  are  $\pi_1$ -injective if  $M$  is irreducible, the third statement is a consequence of the irreducibility of  $M$  (or alternatively of the Poincaré conjecture) and the fourth statement is a consequence of Lemma 3.1. Finally the last statement is an immediate corollary to standard facts about characteristic pairs.

**3.5. The double-curve sum of surfaces.** Let  $N$  be a closed 3-manifold and let  $R$  and  $F$  be two embedded surfaces which are in general position. Note that by the standard ‘cut and paste’ technique applied to the intersection curves of  $R$  and  $F$  we can turn the immersed surface  $R \cup F$  into an embedded surface  $R \uplus F$ . The surface  $R \uplus F$  is sometimes called the double-curve sum of  $R$  and  $F$ . Note that  $R \uplus F$  represents the same homology class as  $R \cup F$  and that furthermore  $R \uplus F$  has the same complexity as  $R \cup F$ .

Now let  $R$  and  $F$  be two properly embedded surfaces in  $N$  in general position.



- (1) A *filling ball* for  $(R, F)$  is an embedded ball  $B \subset N$  such that  $\partial B \subset R \cup F$  as oriented surfaces.
- (2) A *filling solid torus* for  $(R, F)$  is an embedded solid torus  $X \subset N$  such that  $\partial X \subset R \cup F$  as oriented surfaces.

(Here we view  $B$  and  $X$  as oriented manifolds where the orientation does not necessarily have to agree with the orientation of  $N$ .) We then say that  $R$  and  $F$  form a *good pair* if there are no filling balls and no filling solid tori for  $(R, F)$ .

We will later on make use of the following elementary lemma:

**Lemma 3.4.** *Let  $N$  be a closed irreducible 3-manifold and let  $R$  and  $F$  be a good pair of embedded surfaces in  $N$ . Then the following hold:*

- (1)  $R$  and  $F$  are good,
- (2)  $R \uplus F$  is good,
- (3)  $F \cap N(R)$  is a decomposition surface for  $N(R)$ ,
- (4) there exist decomposition annuli  $C_1, \dots, C_k$  which are in one-to-one correspondence with the components of  $R \cap F$  such that the following diagram commutes:

$$\begin{array}{ccc}
 N & \overset{R}{\rightsquigarrow} & N(R) \\
 \downarrow R \uplus F & & \downarrow F \cap N(R) \\
 N(R \uplus F) & \overset{C_1 \cup \dots \cup C_k}{\rightsquigarrow} & (M, \gamma)
 \end{array}$$

A schematic illustration for  $R \uplus F$  and the decomposition annuli  $C_i$  is given in Figure 1.

**3.6. Complexity of sutured manifolds.** Gabai [Ga83, Definition 4.11] associates to each connected sutured manifold  $(M, R_-, R_+, \gamma)$  an invariant  $c(M, R_-, R_+, \gamma) \in \mathcal{O}$  which we refer to as the *complexity* of  $(M, R_-, R_+, \gamma)$ . Here  $\mathcal{O}$  is a totally ordered set with the property that any strictly descending chain in  $\mathcal{O}$  starting at a given element is finite. We denote the minimal element of  $\mathcal{O}$  by 0. We refer to [Ga83, Definition 4.3] for details.<sup>1</sup>

Gabai [Ga83, Section 4] proved the following theorem.

**Theorem 3.5.** *Let  $(M, \gamma)$  be a connected sutured manifold and let  $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$  be a sutured manifold decomposition along a connected decomposition surface  $S$ . Suppose that  $(M, \gamma)$  and  $(M', \gamma')$  are taut. Let  $(M'_0, \gamma'_0)$  be a component of  $(M', \gamma')$ . Then*

$$c(M'_0, \gamma'_0) \leq c(M, \gamma).$$

*Furthermore, if  $S$  is not boundary parallel, e.g. if  $[S]$  is non-trivial in  $H_2(M, \partial M; \mathbb{Z})$ , then*

$$c(M'_0, \gamma'_0) < c(M, \gamma).$$

<sup>1</sup>Note that our notation and name differ from [Ga83]: Gabai denotes this invariant  $\bar{C}(M, R_-, R_+, \gamma)$  and calls it the ‘reduced complexity’.

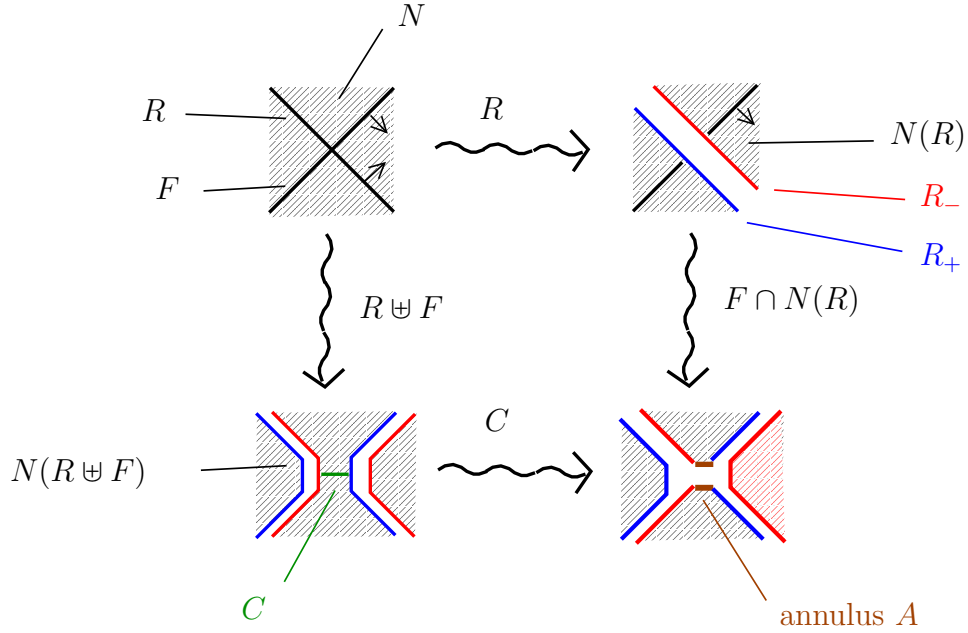


FIGURE 1. Schematic picture for decomposing along  $R$  and  $F$ .

*Remark.* (1) We could also use the complexity  $\hat{C}(M, R_-, R_+, \gamma)$  introduced by Scharlemann [Sc89, Definition 4.12]. It follows from [Sc89, Definition 4.12, Remark 4.13 (b) and Theorem 4.17] that the conclusion of Theorem 3.5 holds in an analogous way for Scharlemann's complexity.

- (2) Juhász [Ju06, Ju08] defines and studies in detail the 'sutured Floer homology  $SFH(M, \gamma)$ ' for 'balanced' sutured manifolds. The total rank of  $SFH(M, \gamma)$  is a very useful complexity for balanced sutured manifolds and it has properties similar to Theorem 3.5. It would be interesting to give another proof of Theorem 5.1 using  $SFH(M, \gamma)$ . This though would require some adjustments since not all sutured manifolds which occur in our proof are balanced (e.g. if  $(M, R_-, R_+, \gamma)$  is balanced, then  $R_-$  and  $R_+$  have no closed components).

#### 4. PERTURBATIONS OF HOMOLOGY CLASSES

The key to proving the Virtual Fiber Theorem is to show that given a good Thurston norm minimizing surface  $R$  and a homology class  $\psi \in H_2(N; \mathbb{Z})$  one can find a surface  $F$  such that given any gut or window  $X$  of  $N(R)$  the intersection  $F \cap X$  is a taut decomposition surface for  $X$  which represents the same class as the restriction of  $\psi$  to  $H_2(X, \partial X; \mathbb{Z})$ .

We start out with the following proposition.

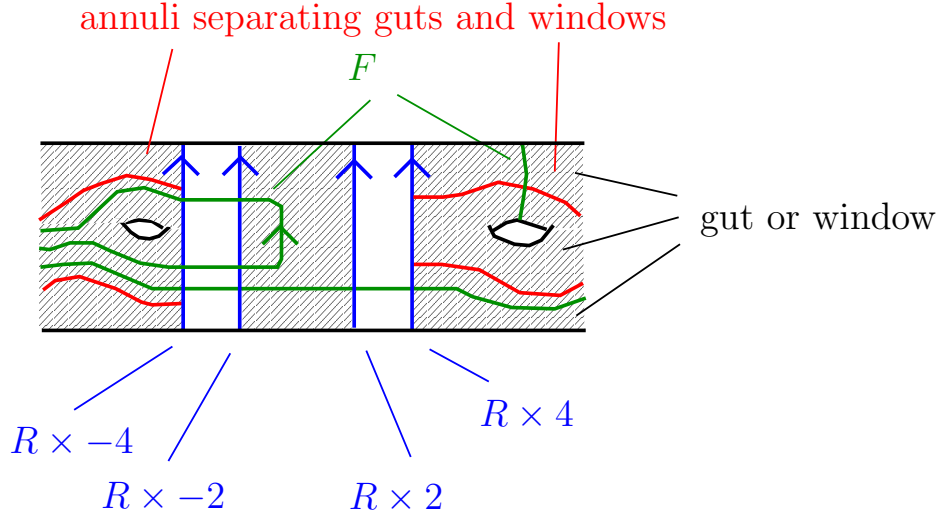


FIGURE 2. Schematic picture for Proposition 4.1.

**Proposition 4.1.** *Let  $N$  be a closed irreducible connected 3-manifold and let  $R$  be a good Thurston norm minimizing surface. Then given any  $\psi \in H_2(N; \mathbb{Z})$  there exists an  $m \in \mathbb{N}$  and a surface  $F$  with the following properties:*

- (W1)  $[R]$  is subordinate to  $m[R] + \psi$  and  $F$  represents  $m[R] + \psi$ ,
- (W2)  $F \uplus (R \times -3 \cup R \times 3)$  is Thurston norm minimizing,
- (W3) the intersections  $F \cap R \times [-4, -2]$  and  $F \cap R \times [2, 4]$  are product surfaces,
- (W4) if  $X$  is a gut or a window of  $N \setminus R \times (-4, 4)$ , then  $F \cap X$  is a decomposition surface,
- (W5)  $F$  and  $R \times -3 \cup R \times 3$  are a good pair.

This proposition is the technical heart of our proof of the Virtual Fiberings Theorem and we therefore give a detailed proof of the proposition. A very schematic picture for Proposition 4.1 is given in Figure 2.

*Proof.* Let  $N$  be a closed irreducible 3-manifold and let  $R$  be a good Thurston norm minimizing surface. We pick a tubular neighborhood  $R \times [-5, 5]$  for  $R$ . We write  $M = N \setminus R \times (-1, 1)$  which we view as a sutured manifold  $(M, \gamma)$  in the usual way.

We furthermore denote by  $(M_i, R_{i+}, R_{i-}, \gamma_i)$ ,  $i = 1, \dots, r$  the guts and windows of the sutured manifold  $M$ . Finally we denote by  $A_1, \dots, A_s$  the collection of all the components of the  $\gamma_i$ . Note that we can and will assume that for each  $A_i$  the intersection with  $R \times [-5, -1]$  and  $R \times [1, 5]$  consists of a union of product annuli.

Before we state the first claim of the proof we need to introduce one more definition. Let  $S$  be a surface and let  $c$  be a component of  $S \cap A_i$  which is a closed curve. We pick a  $j$  such that  $A_i$  is a component of  $\gamma_j$ , i.e. such that  $A_i$  lies on  $\partial M_j$ . Note that  $c$  is a component of  $\partial(S \cap M_j)$  and it thus inherits an orientation. We now say that  $c$  is *positive* if  $[c] = [R_{j\pm} \cap A_i] \in H_1(A_i; \mathbb{Z})$  and we say  $c$  is *negative* if  $[c] = -[R_{j\pm} \cap A_i]$ .

It is straightforward to see that if we chose the other  $M_k$  for which  $A_i$  is a component of  $\gamma_k$ , then the orientation of  $c$  flips and  $[R_{k\pm} \cap A_i] = -[R_{j\pm} \cap A_i]$ , which implies that we would get the same sign.

We can now formulate our first claim.

*Claim.* Let  $\psi \in H_2(N; \mathbb{Z})$ . There exists an  $l \in \mathbb{N}$  and a surface  $D$  with the following properties:

- (D1)  $[R]$  is subordinate to  $l[R] + \psi$  and  $D$  represents  $l[R] + \psi$ ,
- (D2) for any set of real numbers  $-5 < x_1 < x_2 < \dots \leq x_t < 5$  the surface  $D \natural \bigcup_i R \times x_i$  is Thurston norm minimizing,
- (D3) the intersection  $D \cap R \times [-4, 4]$  is a product surface,
- (D4) given any  $i \in \{1, \dots, s\}$  the surface  $D$  intersects  $A_i$  transversely and any component of  $D \cap A_i$  is either an arc or it is a closed curve which is positive,
- (D5)  $D$  is a good surface.

We first note that by Lemma 2.1 there exists a  $k \in \mathbb{N}$  such that  $[R]$  is subordinate to  $l[R] + \psi$  for any  $l \geq k$ . By a general position argument we can find a Thurston norm minimizing surface  $C$  in  $N$  which represents  $k[R] + \psi$ , which intersects all the annuli  $A_i$  transversely and such that  $C \cap R \times [-5, 5]$  is a product surface.

Since  $[R]$  is subordinate to  $[C]$  it follows that  $[R]$  and  $[C]$  lie on the closure of a Thurston cone, which in turn implies that for any  $-5 < x_1 < x_2 < \dots \leq x_t < 5$  we have

$$x(t[R] + [C]) = x(t[R]) + x([C]) = t\chi_-(R) + \chi_-(C) = \chi_-(C \natural \bigcup_{i=1}^t R \times x_i).$$

This shows that  $C \natural \bigcup_i R \times x_i$  is Thurston norm minimizing. In particular  $C$  satisfies (D1) to (D3).

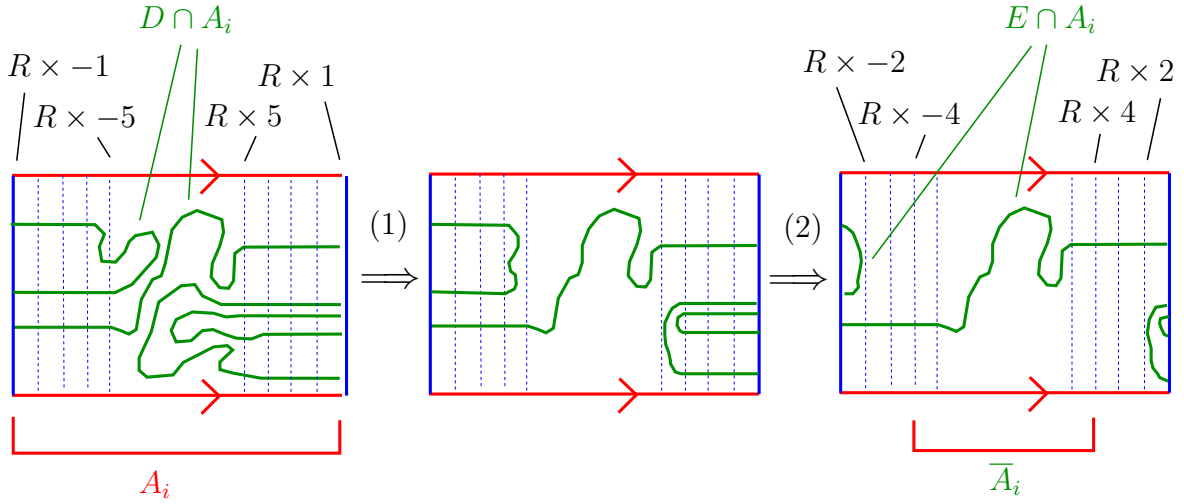
We now let

$$d := \text{maximal number of negative components of any } C \cap A_i$$

and we consider

$$D := C \natural \bigcup_{i=1}^d R \times (4 + \frac{i}{d}).$$

It follows easily from  $R_{i+} = (R \times 1) \cap M_i$ ,  $i = 1, \dots, r$  that for any  $A_i$  there are now at least as many positive components of  $D \cap A_i$  as there are negative components. Using the standard ‘cut and paste’ method we can arrange that given any  $A_i$  the intersection  $D \cap A_i$  contains no null-homologous closed loops and no anti-parallel closed loops. Note that if we remove a pair of anti-parallel closed loops then we lower the number of positive and negative components each by one. It now follows that any component of  $D \cap A_i$  is either an arc, or it is a closed curve which is positive. We thus arranged that  $D$  satisfies (D4). Since all of the above operations can be performed outside of  $R \times [-4, 4]$  it is clear that  $D$  also has Properties (D1) to (D3).


 FIGURE 3. Modification of  $D$  in a neighborhood of  $A_i$ .

We finally turn  $D$  into a good surface by removing all components of  $D$  which are spheres or which bound a compressible torus. This concludes the proof of the claim.

For each  $A_i$  we now perform successively two isotopies of  $D$  in a small neighborhood of  $A_i$ , i.e. in a neighborhood which does not intersect any of the other  $A_j$ :

- (1) We first apply an isotopy outside of  $R \times [-4, 4]$  which pulls the separating arcs of  $D \cap A_i$  either into  $A_i \cap (R \times (-5, -1])$  or into  $A_i \cap (R \times [1, 5))$  and which leaves all the other intersections of  $D$  with  $A_i$  untouched.
- (2) We then apply an isotopy in  $R \times [-5, -1] \cup R \times [1, 5]$  which pulls the separating arcs into  $A_i \cap (R \times (-2, -1])$  or into  $A_i \cap (R \times [1, 2))$  and which again leaves all the other intersections of  $D$  with  $A_i$  untouched.

Note that such isotopies exist since  $D \cap A_i$  contains no null-homologous closed loops. Also note that we can perform the isotopies in such a way that the intersection of the resulting surface  $E$  with  $R \times [-4, -2] \cup R \times [2, 4]$  is still a product surface. We illustrate the two isotopies in Figure 3.

It is now time to pause for a minute and see what we have achieved so far.

*Claim.* The surface  $E$  has Properties (W1) to (W4).

We consider the sutured manifold  $\overline{M} := N \setminus R \times (-4, 4)$ . Note that the guts and the windows of  $\overline{M}$  are precisely the intersection of the guts and the windows of  $M = N \setminus R \times (-1, 1)$  with  $\overline{M}$ . In the following we write  $\overline{M}_i = \overline{M} \cap M_i$ ,  $\overline{\gamma}_i = \overline{M} \cap \gamma_i$  and  $\overline{A}_i = \overline{M} \cap A_i$  for all  $i$ .

We first point out that Properties (W1) and (W2) are preserved under isotopy, so they are clearly satisfied by  $E$ . As we discussed above, the surface  $E$  has Property (W3).

Finally let  $\overline{A}_i$  be any of the annuli. It follows from (D4) and the type of isotopy we applied that any component of  $E \cap \overline{A}_i = (E \cap A_i) \cap (N \setminus R \times (-4, 4))$  is either a *non-separating arc* or a closed curve which is positive. This is equivalent to saying that  $E$  satisfies (W4). This concludes the proof of the claim.

So it now remains to modify  $E$  to arrange (W5).<sup>2</sup> We will do so over the next two claims.

*Claim.* There exists a good surface  $E$  which has Properties (W1) to (W4) and which satisfies

(W5') There exists no filling ball and no filling solid tori for  $(E, R \times -3 \cup R \times 3)$  which lies in  $N \setminus R \times (-3, 3)$ .

We will prove the claim using the complexity  $b_0(E \cap (R \times -3 \cup R \times 3))$ . It suffices to show that if  $E$  is a good surface with Properties (W1) to (W4) which does not satisfy (W5'), then there exists a good surface with Properties (W1) to (W4) with lower complexity.

So let  $E$  be a good surface with Properties (W1) to (W4) which admits a filling solid torus  $X$  for  $(E, R \times -3 \cup R \times 3)$  which lies in  $N \setminus R \times (-3, 3)$ . Since  $E$  and  $R$  are good it follows that  $X$  touches  $E$  and  $R \times -3 \cup R \times 3$ . (It is in fact straightforward to see that  $\partial X \cap (R \times -3 \cup R \times 3)$  lies either completely in  $R \times -3$  or in  $R \times 3$ .) We now replace  $E$  by

$$(E \setminus (X \cap E)) \cup (X \cap (R \times -3 \cup R \times 3))$$

and push the components of  $X \cap (R \times -3 \cup R \times 3)$  into  $R \times (-2, 2)$ . These two steps are illustrated in Figure 4. Finally we delete all components of the new surface which are spheres or which bound solid tori.

Note that the fact that  $X$  is a filling solid torus implies that the resulting surface is homologous to  $E$  and in particular oriented. Also note that any component of the intersection of the new surface with any of the  $A_j$  is a component of the intersection of  $E$  with  $A_j$ . It is now straightforward to see that the resulting surface is a good surface that still has Properties (W1) to (W4). Furthermore it is clear that the number of components of the intersection with  $R \times -3 \cup R \times 3$  went down. We thus lowered the complexity.

We now suppose that  $(E, R \times -3 \cup R \times 3)$  admits a filling ball  $B$  which lies in  $N \setminus R \times (-3, 3)$ . Then exactly the same argument as above, with  $X$  replaced by  $B$ , shows that we can find a new surface of lower complexity. This concludes the proof of the claim.

We now turn to the last claim of the proof of the proposition.

*Claim.* There exists a good surface  $F$  which has Properties (W1) to (W5).

<sup>2</sup>Note that we could of course have picked  $C$  initially such that  $C$  and  $R \times -3 \cup R \times 3$  are a good pair, but this property can get lost in the step from the surface  $D$  to the surface  $E$ .

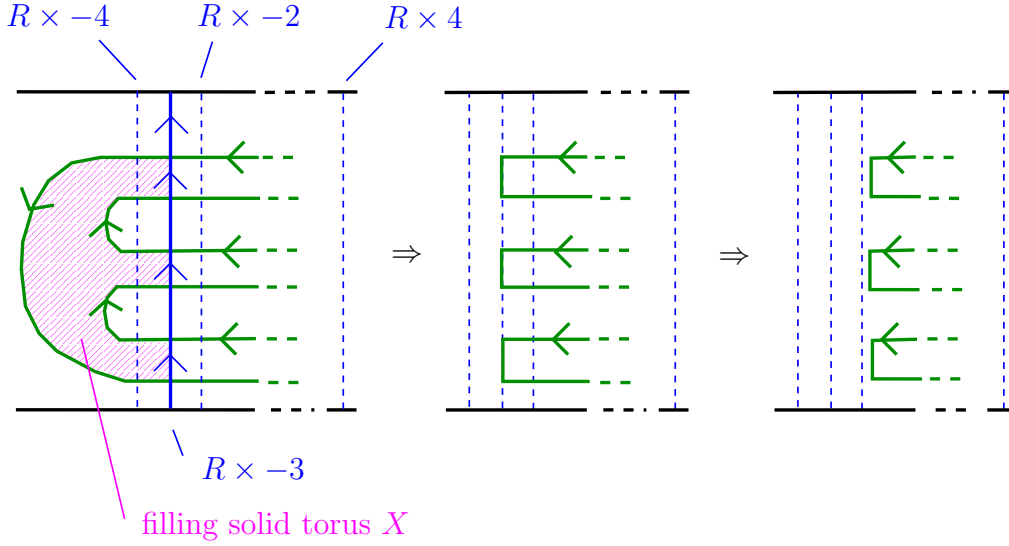


FIGURE 4. Replacing  $X \cap E$  by  $X \cap R \times -3$  and pushing into  $R \times [-2, 2]$ .

Let  $E$  be a good surface which has Properties (W1) to (W4) and (W5'). We denote by  $C_-, C_+ \subset R$  the collection of curves such that  $E \cap R \times \pm 2 = C_{\pm} \times \pm 2$ . We can and will assume that  $C_-$  and  $C_+$  are in general position. We also write  $C = C_- \cup C_+$ .

We denote by  $c$  the number of components of  $R \setminus C$ . Note that the closures of the components of  $R \setminus C$  (equipped with the orientation coming from  $R$ ) give naturally rise to a basis for  $H_2(R, C; \mathbb{Z})$ . We denote the corresponding isomorphism  $H_2(R, C; \mathbb{Z}) \rightarrow \mathbb{Z}^c$  by  $\Phi$  and we denote by  $p: R \times [-2, 2] \rightarrow R$  the canonical projection map.

If  $S \subset R \times [-2, 2]$  is a surface with  $\partial S \subset C_- \times -2 \cup C_+ \times 2$ , then we refer to  $\Phi(p_*([S, \partial S])) \in \mathbb{Z}^c$  as the *coordinates of  $S$* .

If  $S \subset R \times [-2, 2]$  is a surface with  $\partial S \subset C_- \times -2 \cup C_+ \times 2$ , then we say that  $S$  is *negative* if  $\Phi(p_*([S, \partial S]))$  has non-positive coordinates and at least one coordinate is negative. Similarly we define what it means for  $S$  to be *positive*. Note that if  $S \subset R \times [-2, 2]$  is a surface with  $\partial S \subset C_- \times -2$ , then  $S$  is isotopic rel boundary to a surface in  $R \times -2$ , it follows that  $S$  is either negative or positive. The same conclusion holds for surfaces  $S \subset R \times [-2, 2]$  with  $\partial S \subset C_+ \times 2$ .

Finally, given a surface  $E \subset N$  with  $E \cap (R \times -2 \cup R \times 2) = C_- \times -2 \cup C_+ \times 2$  we consider the complexity

$$- \sum_{\substack{S \text{ component} \\ \text{of } E \cap R \times [-2, 2]}} \text{sum of the negative coordinates of } \Phi(p_*([S, \partial S])).$$

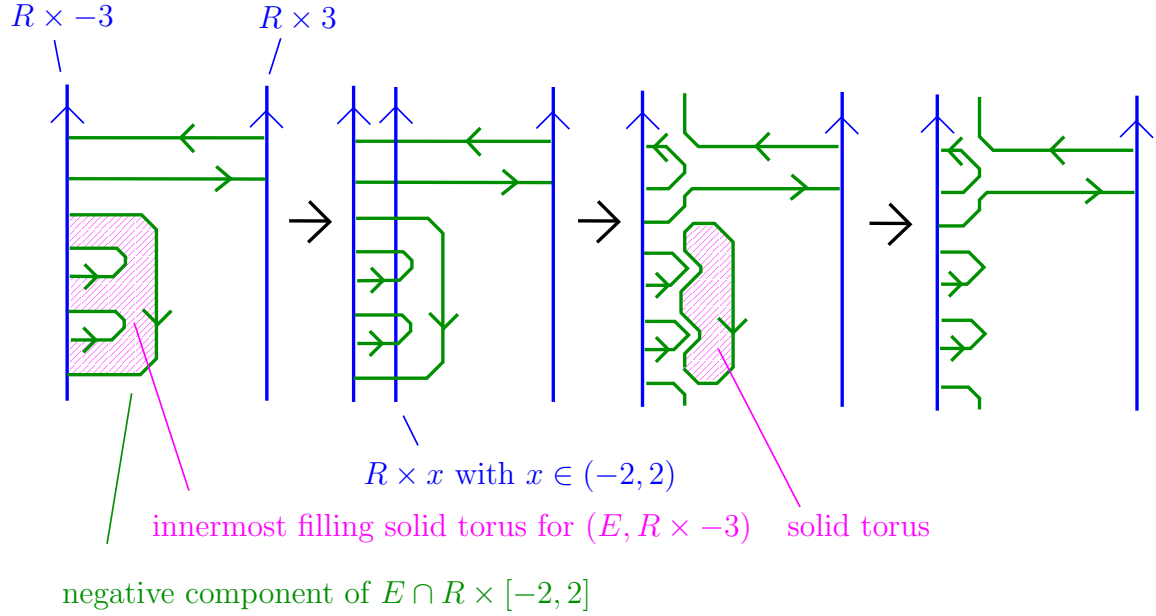


FIGURE 5. Replacing  $E$  by  $E \natural (R \times x)$  and deleting any components bounding balls and solid tori.

In order to prove the claim it suffices to show that if  $E$  is a good surface with Properties (W1) to (W4) and (W5') which does not satisfy (W5), then there exists a good surface with Properties (W1) to (W4) and (W5') with lower complexity.

So let  $E$  be a good surface with Properties (W1) to (W4) and (W5') with  $E \cap (R \times -2 \cup R \times 2) = C_- \times -2 \cup C_+ \times 2$  which admits a filling solid torus  $X$  for  $(E, R \times -3 \cup R \times 3)$ . Note that the intersection of  $X$  with  $N \setminus R \times (-3, 3)$  is either empty, or a filling ball or a filling solid torus. By (W5') the last two cases can not occur, we thus conclude that the filling solid torus  $X$  has to lie in  $R \times [-3, 3]$ .

Note that the oriented surface  $X \cap (R \times -3 \cup R \times 3)$  has non-negative coordinates and at least one coordinate is positive. Since  $X \cap (R \times -3 \cup R \times 3)$  is homologous to  $-X \cap E$  it follows that the surface  $X \cap E$  has non-positive coordinates and at least one component of  $X \cap E$  has a negative coordinate. Finally note that  $X$  intersects either  $R \times -3$  or  $R \times 3$ , without loss of generality we can assume that the former is the case. By the above this implies that  $X \cap E$  contains a negative component.

We pick an  $x \in (-2, 2)$  such that  $\partial X \cap R \times x$  is isotopic in  $X$  to  $\partial X \cap R \times -2$ . We now consider the surface  $E \natural (R \times x)$ . Note that the coordinates of  $R \times x$  are  $(1, \dots, 1)$ . Since  $R \times x$  intersects a negative component of  $E$  it is now straightforward to verify (see e.g. Figure 5 for an illustration) that the surface  $E \natural (R \times x)$  has lower complexity than  $E$ . We now delete all components of  $E \natural (R \times x)$  which bound balls or solid tori. It is easy to see, using (D2), that the resulting surface still has Properties (W1) to (W4) and by the above it has lower complexity than  $E$ .



Finally, if  $(E, R \times -3 \cup R \times 3)$  admits a filling ball  $B$ , then exactly the same argument as above shows that we can again find a surface which satisfies (W1) to (W4) and (W5') and which has lower complexity.

This concludes the proof of the claim.  $\square$

We will now study how the guts are affected by decomposing along the surface which is given to us by Proposition 4.1. Before we state the next result we introduce one more definition. Let  $N$  be a closed irreducible 3-manifold. We say that a subset  $G \subset N$  is *homologically visible in  $N$*  if the map  $H_1(G; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z})/\text{torsion}$  is non-trivial. Otherwise we say that  $G$  is *invisible*.

We can now formulate the following proposition.

**Proposition 4.2.** *Let  $N$  be a closed irreducible connected 3-manifold and let  $R \subset N$  be a good Thurston norm minimizing surface. Suppose that  $m \in \mathbb{N}$  and  $F \subset N$  have Properties (W1) to (W5). We put  $S = (R \times -3 \cup R \times 3) \uplus F$ . Then to each gut  $G$  of  $N \setminus R \times (-4, 4)$  we can associate a collection  $\Phi(G)$  of guts of  $N(S)$  with the following properties:*

- (1) *The guts of  $N(S)$  are the disjoint union of all the  $\Phi(G)$ .*
- (2) *Any gut in  $\Phi(G)$  is a subset of  $G$ .*
- (3) *If  $G$  is invisible, then any gut in  $\Phi(G)$  is also invisible.*
- (4) *If  $G$  is a gut of  $N(R)$ , then one of the following two statement holds:*
  - (a) *either any element in  $\Phi(G)$  has lower complexity than  $G$ , or*
  - (b)  *$\Phi(G)$  consists of one element  $G'$  and there exists an isotopy of  $N$  which restricts to a diffeomorphism  $G \rightarrow G'$  as sutured manifolds.*
- (5) *If  $G$  is a gut such that  $[F \cap G] \neq 0 \in H_2(G, \partial G; \mathbb{Z})$ , then any element in  $\Phi(G)$  has lower complexity than  $G$ .*

*Proof.* We consider the sutured manifold  $M = N \setminus R \times (-4, 4)$  and we denote by  $G_1, \dots, G_k$  the guts and by  $P_1, \dots, P_l$  the windows of  $M$ . We furthermore denote by  $A$  the union of the annuli separating the windows and the guts. Recall that we can and will view the  $G_i$  and the  $P_j$  as the result of cutting  $M$  along  $A$ . By (W4) we can decompose  $G_1, \dots, G_k$  and  $P_1, \dots, P_l$  along  $F$  and we obtain new sutured manifolds  $G_1^F, \dots, G_k^F$  and  $P_1^F, \dots, P_l^F$ .

We also consider the product sutured manifold  $Q := R \times [-2, 2]$ . We can decompose  $Q$  along  $F \cap Q$  and we obtain a sutured manifold  $Q^F$ . Note that we can and will identify  $N(R \times -3 \cup R \times 3)$  with  $Q \cup M$ .

Finally we put  $S = (R \times -3 \cup R \times 3) \uplus F$  and we consider the following diagram

$$\begin{array}{ccccc}
 N & \xrightarrow{R \times -3 \cup R \times 3} & Q \cup M & \xrightarrow{A} & Q \cup \bigcup_i G_i \cup \bigcup_i P_i \\
 \downarrow S & & \downarrow F & & \downarrow F \\
 N(S) & \xrightarrow{C} & X & \xrightarrow{A \cap X} & Q^F \cup \bigcup_i G_i^F \cup \bigcup_i P_i^F.
 \end{array}$$

We now make several explanations and observations:

- (1) The decompositions along  $F$  are understood to be along the intersection of  $F$  with the given submanifold of  $N$ .
- (2) It follows from (W2) and (W5) and Lemma 3.4 that  $N(S)$  is taut.
- (3) By  $C$  we denote the union of the decomposition annuli from Lemma 3.4 which correspond to the components of  $F \cap (R \times -3 \cup R \times 3)$ .
- (4) It follows from Lemma 3.4 that the first square of the diagram is commutative. It is straightforward to verify that the second square is also commutative.
- (5) The neighborhoods of  $C$  are product submanifolds of  $N(S)$ , it thus follows that the guts of  $N(S)$  are precisely the guts of  $X$ .
- (6) It follows from (W4) that the components of  $C$  and  $A \cap X$  are annuli and disks as in Lemma 3.1. Since  $N(S)$  is taut it now follows that  $Q^F$  and that the  $G_i^F$  and  $P_i^F$  are also taut.
- (7) It is well-known that if we decompose a product sutured manifold along a taut decomposition surface, then the result is also a product sutured manifold. We thus see that the guts of  $N(S)$  are precisely the disjoint union of the guts of the  $G_i^F$ .

To each gut  $G_i$  of  $N(R)$  we now associate

$$\Phi(G_i) := \text{guts of } G_i^F.$$

By the above the guts of  $N(S)$  are the disjoint union of  $\{\Phi(G_i)\}_{i=1, \dots, k}$ . By construction any  $J \in \Phi(G_i)$  is a subset of  $G_i$ . In particular the map  $H_1(J; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z})$  factors through  $H_1(G_i; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z})$ . It follows that if  $G_i$  is invisible, then any component of  $G_i^F$  is invisible as well. It furthermore follows immediately from Theorem 3.5, applied iteratively to the components of  $F \cap G_i$ , that the fourth and the fifth statement also hold.  $\square$

## 5. THE PROOF OF THE VIRTUAL FIBERING THEOREM

For the reader's convenience we recall Agol's theorem.

**Theorem 5.1. (Agol)** *Let  $N$  be an irreducible connected 3-manifold with empty or toroidal boundary such that  $\pi_1(N)$  is virtually RFRS. Let  $\phi \in H^1(N; \mathbb{Q})$  be non-trivial. Then there exists a finite cover  $q: \tilde{N} \rightarrow N$  such that  $q^*\phi$  is subordinate to a fibered class.*

In Section 5.1 we will provide the proof of Theorem 5.1 in the case of closed 3-manifolds. In Section 5.2 we will then deduce the case of non-trivial boundary from the closed case by a ‘doubling’ argument.

**5.1. The Virtual Fibered Theorem for closed 3-manifolds.** In this section we will give a proof of Theorem 5.1 in the case that  $N$  is a closed irreducible connected 3-manifold with virtually RFRS fundamental group. Let  $\phi \in H^1(N; \mathbb{Q})$  be non-trivial. In light of Lemma 2.1 (4) we can without loss of generality assume that  $\pi = \pi_1(N)$

is already RFRS. We can therefore find a filtration  $\pi = \pi_0 \supset \pi_1 \supset \pi_2 \dots$  such that the following hold:

- (1)  $\bigcap_k \pi_k = \{1\}$ ,
- (2) for any  $k$  the group  $\pi_k$  is a normal, finite index subgroup of  $\pi$ ,
- (3) for any  $k$  the map  $\pi_k \rightarrow \pi_k/\pi_{k+1}$  factors through  $\pi_k \rightarrow H_1(\pi_k; \mathbb{Z})/\text{torsion}$ .

Given a non-trivial subgroup  $\Gamma \subset \pi_k$  we define its *invisibility*  $i(\Gamma \subset \pi_k)$  as follows:

$$i(\Gamma \subset \pi_k) := \min\{l \in \mathbb{N} \mid \Gamma \subset \pi_{k+l} \text{ and } H_1(\Gamma; \mathbb{Z}) \rightarrow H_1(\pi_{k+l}; \mathbb{Z})/\text{torsion} \text{ is non-trivial}\}.$$

It follows from Properties (1) and (3) of a RFRS group that the invisibility of any non-trivial subgroup is defined.

In the following, given  $k \in \mathbb{N}$ , we denote by  $N_k$  the cover of  $N$  corresponding to  $\pi_k$  and for  $j \geq k$  we denote the covers  $N_j \rightarrow N_k$  by  $q$ . Now let  $R \subset N_k$  be a good Thurston norm minimizing surface. We say that two guts  $G$  and  $G'$  of  $N_k(R)$  are *equivalent* if there exists a deck transformation  $\Phi$  of the covering  $N_k \rightarrow N$  and an isotopy  $\Psi$  of  $N_k$  such that  $\Psi \circ \Phi$  restricts to a diffeomorphism  $G \rightarrow G'$  of sutured manifolds. Note that equivalent guts have in particular the same complexity.

We can now introduce the following invariants:

- $m_c(N_k, R) :=$  maximal complexity of a gut of  $N_k(R)$ ,
- $n_c(N_k, R) :=$  number of equivalence classes of guts of  $N_k(R)$  with maximal complexity,
- $i(N_k, R) :=$  maximal invisibility among all guts of  $N_k(R)$  of maximal complexity,
- $m_v(N_k, R) :=$  maximal complexity of a visible gut of  $N_k(R)$ ,
- $n_v(N_k, R) :=$  number of equivalence classes of visible guts of  $N_k(R)$  with maximal complexity.

If  $N_k(R)$  has no guts, then all these invariants are understood to be 0.

We finally define the complexity  $f(N_k, R)$  to be the lexicographically ordered quintuple

$$f(N_k, R) := (m_c(N_k, R), n_c(N_k, R), i(N_k, R), m_v(N_k, R), n_v(N_k, R)).$$

Note that  $f(N_k, R)$  is the zero vector if and only if  $N_k(R)$  has no guts, i.e. if  $R$  is a fiber of a fibration.

We now want to prove the following lemma, which by the above implies the theorem.

**Lemma 5.2.** *Let  $R$  be a good Thurston norm minimizing surface in  $N$ . Then there exists a  $j$  and a good Thurston norm minimizing surface  $R_j$  in  $N_j$  such that the following two conditions hold:*

- (1)  $q^*([R]) \in H_2(N_j; \mathbb{Z})$  is subordinate to  $[R_j]$ , and
- (2)  $f(N_j, R_j)$  is the zero vector.

This lemma in turn follows from the following lemma:

**Lemma 5.3.** *Let  $R_k$  be a good Thurston norm minimizing surface in  $N_k$  such that  $f(N_k, R_k)$  is not the zero vector. Then there exists a  $j \geq k$  and a good Thurston norm minimizing surface  $\tilde{R}_j$  in  $N_j$  such that*

- (1)  $q^*([R_k]) \in H_2(N_j; \mathbb{Z})$  is subordinate to  $[\tilde{R}_j]$ , and  
(2)  $f(N_j, \tilde{R}_j) < f(N_k, R_k)$ .

In our proof of Lemma 5.3 we first suppose that every gut of  $N_k(R_k)$  is invisible. We then consider the covering  $q: N_{k+1} \rightarrow N_k$  and we write  $R_{k+1} = q^{-1}(R_k)$ . It follows from Lemma 3.3 that the guts of  $N_{k+1}(R_{k+1})$  are precisely the preimages under  $q$  of the guts of  $N_k(R_k)$ . Now note that if  $G$  is a gut of  $N_k(R_k)$ , then the assumption that  $G$  is invisible implies that the map

$$\pi_1(G) \rightarrow \pi_1(N_k) \rightarrow H_1(N_k; \mathbb{Z})/\text{torsion} \rightarrow \pi_k/\pi_{k+1}$$

is trivial. This implies that the components of  $q^{-1}(G)$  are all diffeomorphic to  $G$ . It follows that  $m_c(N_{k+1}, R_{k+1}) = m_c(N_k, R_k)$ .

Note that all the components of  $q^{-1}(G)$  are furthermore equivalent. Since  $N_{k+1} \rightarrow N$  is a regular cover it now follows easily that two guts of  $N_{k+1}(R_{k+1})$  are equivalent if and only if their projections to  $N_k(R_k)$  are equivalent. We thus see that  $n_c(N_{k+1}, R_{k+1}) = n_c(N_k, R_k)$ . On the other hand we clearly have  $i(N_{k+1}, R_{k+1}) = i(N_k, R_k) - 1$ . We thus showed that  $f(N_{k+1}, R_{k+1}) < f(N_k, R_k)$ .

We now turn to the case that there exists a gut of  $N_k(R_k)$  which is visible. Among all visible guts of  $N_k(R_k)$  we take a gut  $G$  of maximal complexity. We denote by  $G_1 = G, G_2, \dots, G_l$  the guts which are equivalent to  $G$ . Note that all these guts are also visible. There exists therefore a homomorphism  $H_1(N; \mathbb{Z}) \rightarrow \mathbb{Z}$  which is non-trivial when restricted to each  $G_j$ . Put differently, there exists a  $\psi \in H_2(N_k; \mathbb{Z}) = H^1(N_k; \mathbb{Z})$  such that the restriction to each  $G_j$  is non-zero.

By Proposition 4.1 there exists an  $m \in \mathbb{N}$  such that  $[R_k]$  is subordinate to  $m[R_k] + \psi$  and a surface  $F$  in  $N_k$  which represents  $m[R_k] + \psi$  and which has Properties (W2) to (W5). We set  $S = (R \times -3 \cup R \times 3) \uplus F$ . It now suffices to show the following claim:

*Claim.*

$$f(N_k, S) < f(N_k, R_k).$$

First note that it follows immediately from Proposition 4.2 (1) and (4) that (up to isotopy)

$$\{\text{guts of } N_k(S) \text{ of complexity } m_c(N_k, R_k)\} \subset \{\text{guts of } N_k(R_k) \text{ of complexity } m_c(N_k, R_k)\},$$

and that furthermore no gut of  $N_k(S)$  has complexity larger than  $m_c(N_k, R_k)$ . It follows that

$$(1) \quad (m_c(N_k, S), n_c(N_k, S), i(N_k, S)) \leq (m_c(N_k, R_k), n_c(N_k, R_k), i(N_k, R_k)).$$

Furthermore it follows from Proposition 4.2 (1), (3) and (4) that  $N_k(S)$  contains no visible gut of complexity larger than  $m_v(N_k, R_k)$  and that

$$\begin{aligned} &\{\text{visible guts of } N_k(S) \text{ of complexity } m_v(N_k, R_k)\} \\ &\subset \{\text{visible guts of } N_k(R_k) \text{ of complexity } m_v(N_k, R_k)\}. \end{aligned}$$

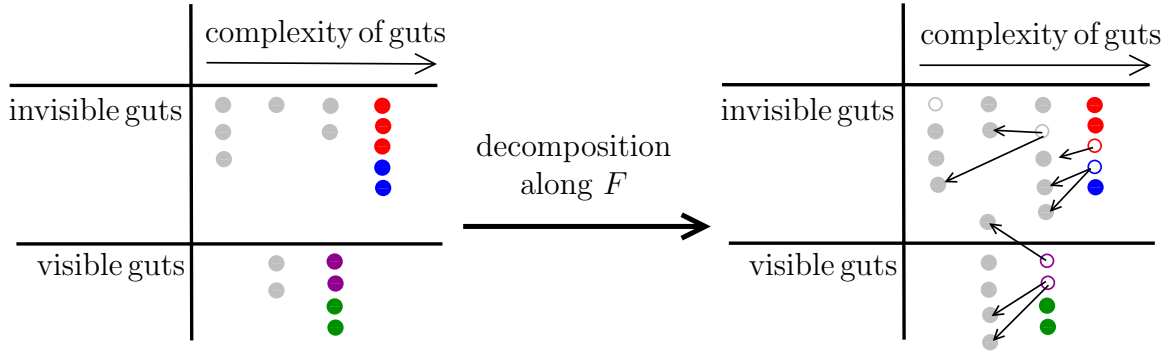


FIGURE 6. Schematic picture of the decomposition of guts along  $F$ : guts get cut into pieces of smaller complexity and invisible guts stay invisible. The colors indicate equivalence classes of guts.

Now note that for any  $G_j$  we have

$$[S]|_{G_j} = (m[R_k] + \psi)|_{G_j} = \psi|_{G_j} \neq 0 \in H_2(G_j, \partial G_j; \mathbb{Z}).$$

It therefore follows from Proposition 4.2 (4) and (5) that

$$\begin{aligned} & \#\{\text{equivalence classes of visible guts of } N_k(S) \text{ of complexity } m_v(N_k, R_k)\} \\ & \leq \#\{\text{equivalence classes of visible guts of } N_k(R_k) \text{ of complexity } m_v(N_k, R_k)\} - 1. \end{aligned}$$

Putting these observations together we see that

$$(2) \quad (m_v(N_k, S), n_v(N_k, S)) < (m_v(N_k, R_k), n_v(N_k, R_k)).$$

Combining the inequalities (1) and (2) we see that  $f(N_k, S) < f(N_k, R_k)$ . This concludes the proof of the claim and thus of Theorem 5.1.

For the purpose of the next section we also state the following lemma which we implicitly proved in the above:

**Lemma 5.4.** *Let  $N$  be a closed irreducible 3-manifold and let  $R \subset N$  be a good Thurston norm minimizing surface. Suppose there exists a filtration  $\pi = \pi_0 \supset \pi_1 \supset \pi_2 \dots$  such that the following hold:*

- (1) *for any gut  $G$  of  $N(R)$  we have  $\cap_k(\pi_1(G) \cap \pi_k) = \{1\}$ ,*
- (2) *for any  $k$  the group  $\pi_k$  is a normal, finite index subgroup of  $\pi$ ,*
- (3) *for any  $k$  the map  $\pi_k \rightarrow \pi_k/\pi_{k+1}$  factors through  $\pi_k \rightarrow H_1(\pi_k; \mathbb{Z})/\text{torsion}$ .*

*Then there exists a finite cover  $q: \tilde{N} \rightarrow N$  such that  $q^*([R])$  is subordinate to a fibered class.*

**5.2. The Virtual Fibering Theorem for 3-manifolds with non-trivial boundary.** We will now give a proof of Theorem 5.1 in the case that  $N$  has non-trivial toroidal boundary. One approach would be to adapt the proof of the previous section. In fact quickly browsing through the proof shows that the only aspect which

needs to be modified is the statement and the proof of Proposition 4.1. This can be done, but the proof of Proposition 4.1 becomes considerably less readable.

We therefore employ a slightly roundabout way which is inspired by the proof of [Ag08, Theorem 6.1]. In the following let  $N$  be an irreducible connected 3-manifold with non-trivial toroidal boundary such that  $\pi_1(N)$  is virtually RFRS. The theorem trivially holds for  $N = S^1 \times D^2$ , we therefore henceforth assume that  $N \neq S^1 \times D^2$ . Let  $\phi \in H^1(N; \mathbb{Q})$  be non-trivial. In light of Lemma 2.1 (4) we can again assume that  $\pi = \pi_1(N)$  is already RFRS. We pick a RFRS filtration  $\{\pi_k\}_{k \in \mathbb{N}} \in \mathbb{N}$  for  $\pi$ .

We denote by  $W$  the double of  $N$  along its boundary, i.e.  $W = N \cup_{\partial N = \partial N'} N'$  where  $N'$  is a copy of  $N$ . We consider the inclusion map  $i: N \rightarrow W$  and the retraction  $r: W \rightarrow N$ . We also consider  $R := \partial N = \partial N' \subset W$  and  $\Phi := r^*\phi \in H^1(W; \mathbb{Z}) = H_2(W; \mathbb{Z})$ . Note that  $R$  is a good surface since  $N$  is irreducible and  $N \neq S^1 \times D^2$ . It follows from Proposition 4.1 and the proof of Proposition 4.2 that there exists a surface  $S$  of the form  $S = F \uplus (R \times -1 \cup R \times 1)$  such that  $[S] = k[R] + \Phi$  for some  $k \in \mathbb{N}$  and such that the guts of  $S$  are contained in  $W \setminus R \times (-1, 1)$ .

Note that the Thurston norm of  $[R]$  is zero, it follows that  $\Phi$  and  $[S]$  lie in the same Thurston cone, in particular  $\Phi$  is subordinate to  $[S]$ . We now apply Lemma 5.4 to the filtration given by  $\text{Ker}(\pi_1(W) \rightarrow \pi_1(N) \rightarrow \pi/\pi_k)$ ,  $k \in \mathbb{N}$  and the surface  $S$ . Since each gut of  $S$  is contained in one of the two copies of  $N$ , and since  $\{\pi_k\}_{k \in \mathbb{N}} \in \mathbb{N}$  is a RFRS filtration it follows that the conditions of Lemma 5.4 are satisfied. There exists therefore a finite cover  $q: \widetilde{W} \rightarrow W$  such that  $q^*([S])$  is subordinate to a fibered class  $\widetilde{\Psi}$ . It follows from Lemma 2.1 that  $\widetilde{\Phi} := q^*\Phi$  is also subordinate to the fibered class  $\widetilde{\Psi}$ .

We now denote by  $\widetilde{N} \subset \widetilde{W}$  a connected component of  $q^{-1}(N)$ . We recycle the above notation by denoting the covering map  $\widetilde{N} \rightarrow N$  by  $q$  and the inclusion map  $\widetilde{N} \rightarrow \widetilde{W}$  by  $i$ . Since  $N \neq S^1 \times D^2$  we can view  $\widetilde{N}$  as a union of JSJ components of  $\widetilde{W}$ . It follows from [EN85, Theorem 4.2] that  $\widetilde{\psi} := i^*\widetilde{\Psi} \in H^1(\widetilde{N}; \mathbb{Q})$  is also fibered.

It remains to show that  $\widetilde{\phi} := q^*\phi$  is subordinate to  $\widetilde{\psi}$ . We first note that the fact that  $\widetilde{\Phi} := q^*\Phi$  is subordinate to  $\widetilde{\Psi}$  implies that

$$(3) \quad x_{\widetilde{W}}(\widetilde{\Phi}) + x_{\widetilde{W}}(\widetilde{\Psi}) = x_{\widetilde{W}}(\widetilde{\Phi} + \widetilde{\Psi}).$$

We denote by  $\widetilde{M}$  the closure of  $\widetilde{W} \setminus \widetilde{N}$ . Note that  $\widetilde{N}$  and  $\widetilde{M}$  are a union of JSJ components of  $\widetilde{W}$ . It now follows immediately from [EN85, Proposition 3.5] that for any class  $\widetilde{\Theta} \in H^1(\widetilde{N}; \mathbb{Q})$  we have

$$x_{\widetilde{W}}(\widetilde{\Theta}) = x_{\widetilde{N}}(\widetilde{\Theta}|_{\widetilde{N}}) + x_{\widetilde{M}}(\widetilde{\Theta}|_{\widetilde{M}}).$$

Since  $x_{\widetilde{M}}$  is a seminorm it follows immediately from (3) that

$$x_{\widetilde{N}}(\widetilde{\phi}) + x_{\widetilde{N}}(\widetilde{\psi}) = x_{\widetilde{N}}(\widetilde{\phi} + \widetilde{\psi}).$$

This shows that  $\widetilde{\phi}$  and  $\widetilde{\psi}$  lie on the closure of a Thurston cone. We now recall that the fact that  $\widetilde{\psi}$  is fibered implies that  $\widetilde{\psi}$  lies in a top dimensional Thurston cone.

Combining these two statements implies that  $\tilde{\phi}$  is in fact subordinate to the fibered class  $\tilde{\psi}$ .

This concludes the proof of Theorem 5.1 in the case that  $N$  has non-trivial boundary.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, GERMANY  
*E-mail address:* `sfriedl@gmail.com`

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO, 606–8502 JAPAN  
*E-mail address:* `kitayama@kurims.kyoto-u.ac.jp`