Loop Groups and the Path Model

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Littelmann path model and MV cycles

- **Littelmann path model** [1997]: Combinatorial model for Lie algebra representations
- Gives: Branching rules, tensor product decomposition (Littelwood-Richardson coefficients), Characters,. . .
- **Mirković-Vilonen cycles** [2006]: Subvarieties of affine Grassmannians
- give geometric construction of Lie algebra representations
- Loop groups are differentiable analogues of affine Grassmannians
- Aim: Find differentiable analogue for Littelmann paths resp. MV-cycle
- Exhibit those to find connections
- . . .
  Profit!

Notation I

G any Lie group.

- $L(G)$ free loop group of G, i.e. maps $S^1 \to G$.
- $\Omega(G) \subseteq L(G)$, (based) loop group of G, i.e. $1 \mapsto 1_G$
- $L(SU_2)$ contains $z \mapsto \begin{pmatrix} iz & 0 \\ 0 & -iz^{-1} \end{pmatrix}$
- $\Omega(SU_2)$ contains $z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$

Notation II

- K simple, compact Lie group, SU_{n+1}
- $S \cong (S^1)^n$ maximal torus of K, diagonal matrices of SU_{n+1}
- $\mathfrak{s}$ Lie algebra of S, purely imaginary, traceless, diagonal matrices
- G complexification of K, $SL_{n+1}(\mathbb{C})$
- with maximal torus T complexification of S, diagonal matrices of $SL_{n+1}(\mathbb{C})$
- $X^*(S) \subseteq \mathfrak{s}$ cocharacters of S, the kernel of the matrix exponential map
- $X^*_+(S) \subseteq \mathfrak{s}$ dominant cocharacters of S, increasing entries

The exponential of a path

- Π paths in $\mathfrak{s}$ starting in 0 ending in $X^*(S)$
- (Littelmann) root operators $\Pi \to \Pi$
- $\exp : \mathfrak{s} \to S$ exponential of Lie group S
- Applying exp pointwise yields $\exp : \Pi \to \Omega(S)$
- Example SU_2

$$\pi(t) = t\alpha^\vee \overset{\exp}{\mapsto} (z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix})$$

A Loop Model

Theorem (R.)

- *Root operators integrate to operators on $\Omega(S)$*
- *Resulting crystal parametrize bases for representations of Langlands dual group of K*
- *Weight computable via winding numbers*
- *Root operators are given by multipilication in the Loop group*

Set $K = PU_3$:

$$
\begin{array}{ccccc}
\gamma := (z,1,1) & \xrightarrow{(-\alpha_1^\vee)\circ\alpha_1\circ\gamma} & \eta := (1,z,1) & \xrightarrow{(-\alpha_2^\vee)\circ\alpha_2\circ\eta} & \mu := (1,1,z) \\
\text{wind}(\alpha_1\circ\gamma)\ \big\downarrow\ \text{wind}(\alpha_2\circ\gamma) & & \text{wind}(\alpha_1\circ\eta)\ \big\downarrow\ \text{wind}(\alpha_2\circ\eta) & & \text{wind}(\alpha_1\circ\mu)\ \big\downarrow\ \text{wind}(\alpha_2\circ\mu) \\
(1,0) & & (-1,1) & & (0,-1)
\end{array}
$$

Birkhoff decomposition

Problem: Extend root operators to $\Omega(K)$.

Theorem (Pressley 1981)

Every $\gamma \in \Omega(K)$ can be written as

$$\gamma = p_- \lambda p_+$$

where $\lambda \in X^*_+(S)$ is unique, $p_- \in L^-(G)$ and $p_+ \in L^+(G)$.

Remark

$L^-(G), L^+(G)$ analogues of parabolic groups.
Also true for $K = S$. Then p_-, p_+ unique up to constant.

Work in progress

- Characterization of maximal element in crystal via Birkhoff decomposition
- Description of root operators for factors of Birkhoff decomposition
- Extend root operators to $\Omega(K)$ via this description

Remark

Uniqueness of p_-, p_+ still given on open,dense subset.

Thank you for your attention