Some remarks on Loop Groups

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December 20, 2017

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Chapter 0

Introduction

Let $G \subseteq GL_n(\mathbb{C})$ be a subgroup. In these notes, almost always we will consider either the group of invertible complex $n \times n$-matrices

$$G = GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det A \neq 0\} \tag{1}$$

or the goup of unitary matrices

$$G = U_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \overline{A}^{\intercal} A = \mathbb{1}\}. \tag{2}$$

The description in (1) implies that $GL_n(\mathbb{C})$ is an open and dense subset in $M_n(\mathbb{C})$. The condition $\overline{A}^{\intercal} A = \mathbb{1}$ in (2) should be interpreted as a collection of n^2-equation which all have to be satisfied by the elements in $U_n(\mathbb{C})$. This implies that $U_n(\mathbb{C})$ is a closed subset of $M_n(\mathbb{C})$, and, since the equations imply that A is invertible, it is indeed a closed subset of $GL_n(\mathbb{C})$. Further, for $A = (a_{i,j})_{i,j=1,\ldots,n}$, the condition $\overline{A}^{\intercal} A = \mathbb{1}$ implies for all $j = 1, \ldots, n$:

$$\sum_{i=1}^{n} |a_{i,j}|^2 = 1,$$

so $U_n(\mathbb{C})$ is a bounded and closed subset of $M_n(\mathbb{C})$, i.e. $U_n(\mathbb{C})$ is a compact set. Recall that a $n \times n$-matrix A lies in $GL_n(\mathbb{C})$ if and only if the column vectors form a basis of $\mathbb{C}^n$, so we can identify $GL_n(\mathbb{C})$ with the set of ordered bases of $\mathbb{C}^n$. Let now

$$\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n, \quad (\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}) \mapsto \sum_{i=1}^{n} \overline{a_i} b_i \tag{3}$$

be the standard inner product on $\mathbb{C}^n$. In terms of ordered bases $\mathbb{C}^n$, $U_n(\mathbb{C})$ can be identified with the set of ordered orthonormal bases (with respect to $\langle , \rangle$, the fixed inner product) of $\mathbb{C}^n$.

We will describe later in more detail the connection between the compact group $U_n(\mathbb{C})$ and the group $GL_n(\mathbb{C})$.

Let now $G = GL_n(\mathbb{C})$ or $G = U_n(\mathbb{C})$. The loop group LG is, as a set,

$$LG := \{\phi : S^1 \to G\}$$

just the set of all parameterized maps from the circle $S^1 \subset \mathbb{C}$ to the group. Pointwise multiplication of the maps endows the set LG with an associative composition law:

$$LG \times LG \to LG, \quad (\phi, \psi) \mapsto \phi \cdot \psi : \left\{ \begin{array}{ccc} S^1 & \to & G \\ z & \mapsto & \phi(z)\psi(z) \end{array} \right. \tag{4}$$

Denote by $1\!\!1 \in GL_n(\mathbb{C})$ the unit matrix. By abuse of notation, we identify the matrix $1\!\!1$ with the constant map

$$1\!\!1(z) = 1\!\!1 \quad \forall\, z \in S^1. \tag{5}$$

This is a neutral element for the composition law in (1.6), and Cramer's rule shows that for a given map $\phi \in LG$, the map

$$\phi^{-1} : S^1 \to G, \quad z \mapsto \frac{1}{\det \phi(z)} \phi(z)^\dagger$$

is a well defined element in LG with the property $\phi^{-1} \cdot \phi = \phi \cdot \phi^{-1} = 1\!\!1$. Here $\phi(z)^\dagger$ denotes the adjugate matrix associated to $\phi(z)$. In other words, (1.6) defines a group law on LG.

Actually, one almost never considers the loop group in the generality as defined above. Depending on the circumstances (for example the applications one has in mind), one puts restrictions on the maps $\phi : S^1 \to G$ and considers only those maps which are continuous, or which are smooth, or which have an absolutely convergent Fourier expansion, or which are algebraic, or....

Accordingly, let $L^0G \subset LG$ be the subset of continuous maps, let $L^\infty G \subset LG$ be the subset of smooth maps, let $L^F G \subset LG$ be the subset of maps admitting an absolutely convergent Fourier series, let $L^{alg}G \subset LG$ be the subset of algebraic maps, and so on (we define later more precisely what it means for a map to be smooth ore algebraic or $\ldots$). Roughly speaking, the map ϕ is given by functions $\phi_{i,j}$, where $\phi(z) = (\phi_{i,j}(z))$, and the map ϕ is continuous or smooth or ... only if the functions $\phi_{i,j}$ are continuous or smooth or .... So it is often (but not always) easy to check that the corresponding subsets like L^0G or $L^\infty G$ are indeed subgroups. Correspondingly, L^0G is called the continuous loop group respectively $L^\infty G$ is called the smooth loop group.

It remains the question why does one want to study these groups. Here some examples:

Example 0.0.1 Birkhoff was looking at a differential equation of the form

$$\frac{dv}{dz} = A(z)v(z),$$

where $v : \mathbb{C} \to \mathbb{C}^n$ is a vector valued function and $A(z)$ is a $n \times n$-matrix valued function. If A is a constant matrix, then there exist standard algorithms to simplify the equation. Birkhoff's aim was to develop similar tools in this more general case, for example to find sufficient conditions for the existence of a coordinate transform $T(z)$ such that for $\tilde{v}(z) = T(z)v(z)$ the equation above reduces to $\frac{d\tilde{v}}{dz} = \tilde{A}\tilde{v}(z)$ such that $\tilde{A}$ is a constant matrix.

Example 0.0.2 A more philosophical reason is that the loop group is an object which is inherent to the group. So if one wants to study the group, its properties, its representations and so on, then it is natural to study as well objects inherent to the group. One of these "natural" objects associated to the groups we are looking at will be the algebraic loop group, also called affine Grassmannian, which can be used to construct finite dimensional representations attached with a canonical basis.

Let us finish the introduction with more examples of loop groups and the various ways one might look at them.

Example 0.0.3 A loop $\phi : S^1 \to G = GL_n(\mathbb{C})$ is called an algebraic loop if there exists Laurent polynomials $\tilde{\phi}_{i,j} \in \mathbb{C}[t, t^{-1}]$ such that ϕ is the restriction of the map

$$\tilde{\phi} : \mathbb{C}^* \to GL_n(\mathbb{C}), \quad z \mapsto (\tilde{\phi}_{i,j}(z)),$$

i.e. $\phi = \tilde{\phi}|_{S^1}$. Next suppose we are given two such algebraic lifts $\tilde{\phi}(t), \tilde{\psi}(t)$ such that $\tilde{\phi}|_{S^1} = \tilde{\psi}|_{S^1}$ as maps. But then $(\tilde{\phi} - \tilde{\psi})|_{S^1} = 0$ is the zero map, which implies that for all i, j the Laurent polynomials $\tilde{\phi}_{i,j}(t) - \tilde{\psi}_{i,j}(t)$ have an infinite zero set. This is only possible if $\tilde{\phi}_{i,j}(t) = \tilde{\psi}_{i,j}(t)$ for all i, j, and hence we have $\tilde{\phi} = \tilde{\psi}$.

Now here is another way of looking at algebraic loops. For simplicity set $n = 2$ and let $R = \mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials in one variable, its quotient field is the field K of rational functions on $\mathbb{C}$. The group

$$GL_2(K) = \{A \in M_2(K) \mid \det A \neq 0\}$$

is well defined, and so is its subgroup

$$GL_2(R) = \{A \in M_2(R) \mid \det A \in R^*\},$$

where R^* denotes the set of units in R. Indeed, the fact that the determinant is a unit in R ensures by Cramer's rule that the inverse of the matrix is again an element in $M_2(R)$. For $R = \mathbb{C}[t, t^{-1}]$ the set of units are just the nonzero complex multiples of powers of t: one has $R^* = \{at^m \mid m \in \mathbb{Z}, a \in \mathbb{C}^*\}$. Now an element of $GL_2(R)$ looks like

$$A = A(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$$

where the $a_{i,j}(t) \in \mathbb{C}[t, t^{-1}]$ are Laurent polynomials. Since the determinant is of the form at^m for some $m \in \mathbb{Z}$ and $a \in \mathbb{C}^*$, for all $z \in \mathbb{C}^*$ the matrix $A(z)$ has a nonzero determinant and is hence an element in $GL_2(\mathbb{C})$. So the restriction of this map to $S^1 \subset \mathbb{C}^*$ provides an algebraic loop

$$A(t) : S^1 \to GL_2(\mathbb{C}), \quad z \mapsto A(z).$$

Vice versa, suppose $\phi : S^1 \to GL_2(\mathbb{C})$ is an algebraic loop, i.e. we know there exist $\tilde{\phi}_{i,j}(t) \in \mathbb{C}[t, t^{-1}]$ such that ϕ is the restriction of the map

$$\tilde{\phi} : \mathbb{C}^* \to GL_2(\mathbb{C}), \quad z \mapsto (\tilde{\phi}_{i,j}(z))$$

to the circle. Since $\phi(z) \in GL_2(\mathbb{C})$ for all $z \in \mathbb{C}^*$, it follows that $\det \phi(z) \neq 0$ for all $z \in \mathbb{C}^*$, and hence $\det \phi(t)$ is a Laurent polynomial having only 0 as pole or as vanishing set. It follows that $\det \phi(t) = at^m$ for some $m \in \mathbb{Z}$ and $a \in \mathbb{C}^*$ and hence $\tilde{\phi}(t) \in GL_2(\mathbb{C}[t, t^{-1}])$.

It follows that this natural construction of matrix groups over Laurent polynomials, like $GL_2(\mathbb{C}[t, t^{-1}])$, provides a construction of the algebraic loop group $L^{alg}GL_2(\mathbb{C})$.

Example 0.0.4 Let $G = U_n(\mathbb{C})$. Here is an attempt to try to understand LG by using the exponential map. Let $\mathfrak{u}_n(\mathbb{C}) \subset M_n(\mathbb{C})$ be the real vector subspace of skew hermitian matrices, i.e. the transpose of the complex conjugate matrix has the property $\overline{A}^\intercal = -A$. It follows that

$$\exp(A) = \sum_{i \geq 0} \frac{1}{i!} A^i$$

has the property that the transpose of the complex conjugate matrix satisfies:

$$\overline{\exp(A)}^\intercal = \sum_{i \geq 0} \frac{1}{i!} (\overline{A}^i)^\intercal = \exp(-A) = (\exp(A))^{-1},$$

so this is a unitary matrix. Note that unitary as well as skew hermitian matrices are normal matrices, i.e., $\overline{A}^\intercal A = A\overline{A}^\intercal$. Recall (Linear algebra course) that complex normal matrices are diagonalizable by conjugation with a unitary matrix, in particular, the diagonal matrix is again unitary respectively skew hermitian. Now a diagonal matrix is skew hermitian if and only if all eigenvalues are purely imaginary, and a diagonal matrix is unitary if and only if all eigenvalues have absolute value one. It follows that the exponential maps sends the diagonal skew hermitian matrices onto the diagonal unitary matrices, and, since $g \exp(A) g^{-1} = \exp(gAg^{-1})$, it follows that the map

$$\exp : \mathfrak{u}_n(\mathbb{C}) \to U_n(\mathbb{C})$$

is a surjective map from the space of skew hermitian matrices onto the unitary group $U_n(\mathbb{C})$. Let $Map(S^1, \mathfrak{u}_n(\mathbb{C}))$ be the set of smooth maps from S^1 to $\mathfrak{u}_n(\mathbb{C})$. By combining such a smooth map with the exponential map, one gets a map

$$Map(S^1, \mathfrak{u}_n(\mathbb{C})) \to LG, \quad \phi \mapsto \exp(\phi),$$

The questions to investigate is whether this is a local homeomorphism near the identity. These kind of constructions show up in theoretical physics (string theory, gauge groups).

Chapter 1

Loop groups - some examples

Before we start with the formal part, let us look at more examples.

1.1 Algebraic version for the unitary group

Let $G = U_n(\mathbb{C})$ be the group of unitary $n \times n$-matrices (see (2)). The set $L^{alg}U_n(\mathbb{C})$ of algebraic loops is the set of maps $\gamma : S^1 \to U_n(\mathbb{C})$ having a Laurent expansion, i.e. there exists some non-negative integer m and $n \times n$ matrices $A_k \in M_n(\mathbb{C})$, $-m \leq k \leq m$, such that

$$\gamma(t) = \sum_{k=-m}^{m} A_k t^k.$$

This looks slightly different from the definition in Example 0.0.3. Let us try to understand why we can relax the definition in the case $U_n(\mathbb{C})$ and how to reconcile the two definitions.

From the definition above it is clear that the pointwise multiplication as in (1.6) defines an associative composition law

$$L^{alg}U_n(\mathbb{C}) \times L^{alg}U_n(\mathbb{C}) \to L^{alg}U_n(\mathbb{C}), \tag{1.1}$$

having the constant map: $\mathbb{1}(z) = \mathbb{1}$ for all $z \in S^1$, as unit element (see (5)).

For $\gamma(t) = \sum_{k=-m}^{m} A_k t^k \in L^{alg}U_n(\mathbb{C})$ set $\overline{\gamma}^{\intercal}(t) := \sum_{k=-m}^{m} \overline{A}_k^{\intercal} t^{-k}$. This defines again an algebraic loop

$$\overline{\gamma}^{\intercal} : S^1 \to U_n(\mathbb{C}), \quad z \mapsto \overline{\gamma}^{\intercal}(z) = \sum_{k=-m}^{m} \overline{A}_k^{\intercal} z^{-k} = \overline{\left(\sum_{k=-m}^{m} A_k z^k\right)}^{\intercal},$$

so $\overline{\gamma}^{\intercal}$ is an element of $L^{alg}U_n(\mathbb{C})$ too. Indeed, note that $\overline{\gamma}^{\intercal}(t) = (\overline{\gamma(t^{-1})})^{\intercal}$, so for $z \in S^1$ we have $\overline{\gamma}^{\intercal}(z) = (\overline{\gamma(z)})^{\intercal} = \gamma(z)^{-1} \in U_n(\mathbb{C})$.

Moreover, since $\gamma(z) \in U_n(\mathbb{C})$ for all $z \in S^1$ we have $\gamma(z) \cdot \overline{\gamma}^{\intercal}(z) = \overline{\gamma}^{\intercal}(z) \cdot \gamma(z) = \mathbb{1}$ for all $z \in S^1$. The determinants $\det(\gamma(t))$ and $\det(\overline{\gamma}^{\intercal}(t))$ are a Laurent polynomials. Since $\overline{\gamma}^{\intercal}(z) \cdot \gamma(z) = \gamma(z) \cdot \overline{\gamma}^{\intercal}(z) = \mathbb{1}$ for all $z \in S^1$, it follows that $\det(\gamma(z)) \det(\overline{\gamma}^{\intercal}(z)) = 1$ for all $z \in S^1$. Again, since $\det(\gamma(t)) \det(\overline{\gamma}^{\intercal}(t)) - 1$ is a Laurent polynomial with infinitely many zeros, it follows $\det(\gamma(t)) \det(\overline{\gamma}^{\intercal}(t)) = 1$. Hence $\det(\gamma(t))$ is a unit in $\mathbb{C}[t, t^{-1}]$, and we have

$$L^{alg}U_n(\mathbb{C}) \subseteq GL_n(\mathbb{C}[t, t^{-1}]). \tag{1.2}$$

More precisely:

$$\Psi = (\psi_{i,j}) : \mathbb{C}^* \to M_n(\mathbb{C}),\ z \to \gamma(z) \cdot \overline{\gamma}^{\intercal}(z) - \mathbb{1}$$

respectively

$$\Psi' = (\psi'_{i,j}) : \mathbb{C}^* \to M_n(\mathbb{C}),\ z \to \overline{\gamma}^{\intercal}(z) \cdot \gamma(z) - \mathbb{1}$$

are given by Laurent polynomials $\psi_{i,j}, \psi'_{i,j} \in \mathbb{C}[t, t^{-1}]$ such that

$$\psi_{i,j}|_{S^1} \equiv 0, \quad \psi'_{i,j}|_{S^1} \equiv 0 \quad \forall\, 1 \leq i, j \leq n.$$

Now nonzero Laurent polynomials have only a finite number of zeros and hence $\psi_{i,j} = \psi'_{i,j} = 0$ for all $1 \leq i, j \leq n$. It follows that

$$\overline{\gamma}^{\intercal}(t) \cdot \gamma(t) = \gamma(t) \cdot \overline{\gamma}^{\intercal}(t) = \mathbb{1}. \tag{1.3}$$

Summarizing :

Proposition 1.1.1 *i) $L^{alg}U_n(\mathbb{C})$ is a subgroup of $LU_n(\mathbb{C})$.*

ii) Every $\gamma \in L^{alg}U_n(\mathbb{C})$ extends to an algebraic map $\gamma : \mathbb{C}^* \to GL_n(\mathbb{C})$.

iii) $L^{alg}U_n(\mathbb{C}) = LU_n(\mathbb{C}) \cap L^{alg}GL_n(\mathbb{C})$.

To get a better understanding of the connection between $L^{alg}G$ for $G = U_n(\mathbb{C})$ the unitary group and $L^{alg}G$ for $G = GL_n(\mathbb{C}[t, t^{-1}])$, we will discuss later the *complexification* $L^{alg}G_{\mathbb{C}}$ of the loop group.

1.2 Loops with convergent Fourier series

We fix a real valued submultiplicative norm $\|\cdot\|$ on the complex vector space of $n \times n$-matrices $M_n(\mathbb{C})$. (Recall that the norm is called submultiplicative if $\|AB\| \leq \|A\|\|B\|$.) Denote by $\hat{\mathcal{F}}$ the set of maps $f : S^1 \to M_n(\mathbb{C})$ of the form

$$f : \ z \mapsto f(z) = \sum_{i=-\infty}^{\infty} A_i z^i$$

where the $A_i \in M_n(\mathbb{C})$ are such that $\sum_{i=-\infty}^{\infty} ||A_i|| < \infty$. The set $\hat{\mathcal{F}}$ is naturally endowed with the structure of a real vector space, elements of $\hat{\mathcal{F}}$ can be added and multiplied by real numbers in the usual way: for $f(t) = \sum_{i=-\infty}^{\infty} A_i t^i$ and $g(t) = \sum_{i=-\infty}^{\infty} B_i t^i$ in $\hat{\mathcal{F}}$ and $\lambda, \mu \in \mathbb{R}$ one has

$$\lambda f(t) + \mu g(t) := \sum_{i=-\infty}^{\infty} (\lambda A_i + \mu B_i) t^i \in \hat{\mathcal{F}}$$

because $\sum_{i=-\infty}^{\infty} \|\lambda A_i + \mu B_i\| \leq |\lambda|(\sum_{i=-\infty}^{\infty} \|A_i\|) + |\mu|(\sum_{i=-\infty}^{\infty} \|B_i\|) < \infty$. If fact, $\hat{\mathcal{F}}$ endowed with the norm:

$$\hat{\mathcal{F}} \ni f(t) = \sum_{i=-\infty}^{\infty} A_i t^i : \quad ||f|| = \sum_{i=-\infty}^{\infty} ||A_i|| < \infty,$$

is a complete real vector space. The pointwise multiplication of the images: $(f \cdot g)(z) := f(z) g(z)$ makes $\hat{\mathcal{F}}$ into an associative algebra:

$$(f \cdot g)(t) := \sum_{i=-\infty}^{\infty} (\sum_{j=-\infty}^{\infty} A_{i-j} B_j) t^i.$$

Indeed, the condition $\sum_{i=-\infty}^{\infty} ||A_i|| < \infty$ implies that

$$\lim_{\ell \to \infty} \max\{||A_m|| \mid |m| \geq \ell\} = 0$$

and hence for all $i \in \mathbb{Z}$ the series $\lim_{k\to\infty} \sum_{j=-k}^{k} A_{i-j} B_j$ is a Cauchy series: let $\ell \geq k \geq 0$, then

$$|| \sum_{j=-\ell}^{\ell} A_{i-j} B_j - \sum_{j=-k}^{k} A_{i-j} B_j || \leq \max\{\|B_j\|, \|B_{-j}\| \mid k+1 \leq j \leq \ell\} (\sum_{i=-\infty}^{\infty} ||A_i||), \tag{1.4}$$

so by moving k to infinity, one can make (1.4) arbitrarily small. Further,

$$\|fg\| = \sum_{i=-\infty}^{\infty} ||(\sum_{j=-\infty}^{\infty} A_{i-j}B_j)|| \leq \sum_{i=-\infty}^{\infty} (\sum_{j=-\infty}^{\infty} \|A_{i-j}\|\|B_j\|) = \|f\|\|g\|$$

so the norm is submultiplicative, i.e. $\|fg\| \leq \|f\|\|g\|$. In other words, the algebra $\hat{\mathcal{F}}$ is a Banach algebra.

We are interested in the loops with values in $GL_n(\mathbb{C})$, denote by

$$L^F G := \{f \in \hat{\mathcal{F}} | \operatorname{Im} f \subset GL_n(\mathbb{C})\}.$$

this subset. To get a description of this set in terms of $\hat{\mathcal{F}}$ note that the condition for f to be an element of $L^F G$ implies that $\det f(z)$ is a non-vanishing continous function admitting an absolutely convergent Fourier series. By a theorem of Wiener, this implies that $(\det f(z))^{-1}$ admits the same property and hence

$$f^{-1} : S^1 \to GL_n(\mathbb{C}), \quad z \mapsto (f(z))^{-1}$$

is again an element of $\hat{\mathcal{F}}$. It follows that:

Proposition 1.2.1 *$L^F G$ is the subgroup of units in $\hat{\mathcal{F}}$. In particular, $L^F G$ is a subgroup of LG.*

1.3 The group $\Omega^\infty G$ of based smooth loops

Let $G = U_n(\mathbb{C})$ be the unitary group. The object we are mostly interested in is the group $\Omega^\infty G$ of based smooth loops, i.e. we consider only the maps

$$\Omega^\infty G := \{\phi : S^1 \to G, \phi \text{ is smooth and } \phi(1) = \mathbb{1}\}.$$

Another way of looking at the group is the following. Given a loop $\gamma \in L^\infty G$, then we think of $\gamma(1) \in G$ as the starting and end point of the loop. We have a natural map

$$ev : L^\infty G \to G, \quad \gamma \mapsto \gamma(1).$$

This map is a group homomorphism, by the definition of the multiplication of elements in $L^\infty G$ one has

$$ev(\gamma_1 \cdot \gamma_2) = (\gamma_1 \cdot \gamma_2)(1) = \gamma_1(1)\gamma_2(1) = ev(\gamma_1)ev(\gamma_2).$$

The map is surjective: given $g \in G$, by abuse of notation identify g with the constant loop $g_\gamma : S^1 \to G$, $z \mapsto g$ for all $g \in G$. This map is smooth and, of course, $ev(g_\gamma) = g$. The kernel of the map is the subgroup of loops such that $\gamma(1) = \mathbb{1}$, so $\ker ev = \Omega^\infty G$. So we can identify $\Omega^\infty G$ with the quotient

$$\Omega^\infty G = L^\infty G/G,$$

where we identify G with the subgroup of constant loops in $L^\infty G$. We will see later, and, in fact, this will be one of the important points in the course, that one has another way of looking at $\Omega^\infty G$. We will show that

$$\Omega^\infty G = \Omega^\infty U_n(\mathbb{C}) = L^\infty GL_n(\mathbb{C})/L^{\infty,+}GL_n(\mathbb{C}),$$

where $L^{\infty,+}GL_n(\mathbb{C})$ is the subgroup of smooth loops which are the boundary value of a holomorphic map:

$$\gamma : \{z \in \mathbb{C} \mid |z| < 1\} \to GL_n(\mathbb{C}).$$

1.4 Loops in the Lie algebra

In Example 0.0.4 we have already seen that the groups $GL_n(\mathbb{C})$ and $U_n(\mathbb{C})$ come associated with certain vector spaces of matrices. In the case $G = GL_n(\mathbb{C})$, we have $\mathfrak{g} = M_n(\mathbb{C})$, for the unitary group $G = U_n(\mathbb{C})$ it is $\mathfrak{g} = \mathfrak{u}_n(\mathbb{C})$, the space of skew hermitian matrices. These vector spaces come endowed with the Lie bracket operation:

$$[A, B] = AB - BA. \tag{1.5}$$

Note that for two skew hermitian matrices the standard matrix product AB is in general not anymore skew hermitian, but the following holds:

$$\overline{([A, B])}^\intercal = \overline{B}^\intercal \overline{A}^\intercal - \overline{A}^\intercal \overline{B}^\intercal = BA - AB = -([A, B]).$$

A complex (or real) vector space L together with a bilinear skew symmetric operation $L \times L \to L$, $(A, B) \mapsto [A, B]$ satisfying the Jacobi identity

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$$

is called a *complex* (respectively *real*) *Lie algebra*. Now the bilinear skew symmetric operation $[,]$ defined on $n \times n$ matrices in (1.5) satisfies the Jacobi identity, so $(M_n(\mathbb{C}), [,])$ and $(\mathfrak{u}_n(\mathbb{C}), [,])$ are examples for Lie algebras. Note that $(M_n(\mathbb{C}), [,])$ is an example for a complex Lie algebra, whereas $(\mathfrak{u}_n(\mathbb{C}), [,])$ is only a Lie algebra over the real numbers (after all, despite the $\mathbb{C}$ in the notation, $\mathfrak{u}_n(\mathbb{C})$ is only a real vector space). We will see that the Lie algebra serves as a linear model for group (see next chapter), and we will see that loops in the Lie algebra will serve as a linear model for the loop group.

Exercise 1.4.1 Show that the operation defined on $M_n(\mathbb{C})$ in (1.5) is bilinear, skew symmetric and satisfies the Jacobi identity.

Now let $\mathfrak{g} = M_n(\mathbb{C})$ or $\mathfrak{g} = \mathfrak{u}_n(\mathbb{C})$. Denote by $L\mathfrak{g}$ the set

$$L\mathfrak{g} := \{\phi : S^1 \to \mathfrak{g}\}$$

of all parameterized maps from the circle $S^1 \subset \mathbb{C}$ into the vector space $\mathfrak{g}$. $L\mathfrak{g}$ has an obvious structure as a vector space (complex vector space in the case $L\mathfrak{g} = M_n(\mathbb{C})$, a real vector space in the case $L\mathfrak{g} = \mathfrak{u}_\mathfrak{n}(\mathbb{C})$). The pointwise multiplication of the maps endows the set $L\mathfrak{g}$ with a Lie algebra structure:

$$\begin{array}{rcl} L\mathfrak{g} \times L\mathfrak{g} & \to & L\mathfrak{g}, \\ (\phi, \psi) & \mapsto & [\phi, \psi] = \phi \cdot \psi - \phi \cdot \psi : \left\{ \begin{array}{rcl} S^1 & \to & \mathfrak{g} \\ z & \mapsto & \phi(z)\psi(z) - \psi(z)\phi(z) \end{array} \right. \end{array} \tag{1.6}$$

As before, we consider special cases like polynomial loops $L^{pol}\mathfrak{g}$, continuous loops $L^0\mathfrak{g}$, smooth loops $L^\infty\mathfrak{g}$ and so on by imposing the corresponding condition of the map to be polynomial, or continuous, or smooth, or ....

The Lie algebra $L\mathfrak{g}$ is called the loop algebra of $\mathfrak{g}$.

1.5 A short outlook

We have now seen several examples of loop groups (and loop algebras). The aim of the next chapters will be to endow this group with the structure of a Lie group. Before doing so, we recall a few definitions and results from the classical case. As in the classical case, where Lie algebras are local models for Lie groups, loop algebras will be local models for loop groups.

Once we have endowed $L^\infty G$ respectively $\Omega^\infty G$ with the structure of a Lie group, we will construct realizations of the Lie groups, i.e. we will construct them as special subset of known infinite dimensional manifolds like certain Grassmann varieties.

Suppose now $G = U_n(\mathbb{C})$. Once $\Omega^\infty G$ is endowed with the structure of a Lie group, then we can start to study special functions (like Morse functions) on this manifold and their critical values. These Morse functions are very helpful to study decompositions of manifolds and their homology. One of the functions we will look at will have as critical values very special loops. Recall that we look at maps from S^1 to our favorite group $G = GL_n(\mathbb{C})$ or $G = U_n(\mathbb{C})$. But $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a Lie group itself, so we have as special subset the Lie group homomorphisms $\mathrm{Hom}\,(S^1, G) \subset \Omega^\infty G$, and it turns out that the so called energy function

$$\gamma \mapsto \frac{1}{4\pi} \int_{S^1} \langle \gamma'(t)\gamma'(t) \rangle d\mu(t)$$

on the loops is a Morse function having precisely $\mathrm{Hom}\,(S^1, G)$ as critical values (here $\langle \cdot, \cdot \rangle$ is a fixed metric on G invariant under left and right translations). This leads first of all to an interesting decomposition of $\Omega^\infty G$ (stable and unstable manifolds), and secondly it turns out that that this decomposition can be used to construct in a homological way finite dimensional representations of G, one for each conjugacy class in $\mathrm{Hom}\,(S^1, G)$. Here the realization of $\Omega^{pol} U_n(\mathbb{C})$ as quotient space $L^{pol} GL_n(\mathbb{C}) / L^{pol,+} GL_n(\mathbb{C})$ plays an important role. This connection had first been noted in the 1990's an has since then lead to an exciting development in representation theory.

Another aspect is the fact that $\Omega^\infty G$ is a Kähler manifold and admits a so called moment map, again a very exciting tool this time for differential geometers to analyze the structure of a manifold. Only rather recently it has been proved, for example, that the preimage of a point for the moment map on based loop groups is always connected.

Of course, we will not be able to address all these points in the course. The aim of the course is to give an introduction, and these short remarks just mention some examples to ensure you that this is an introduction into a relevant research area with many exciting branches.

Chapter 2

The exponential map

2.1 Manifolds, Lie groups and examples

Just for the sake of completeness let us recall some definitions from the calculus class. Though we will not use them in this generality, it is useful to keep in mind the general background while looking at special cases.

Definition 2.1.1 Let $U \subset \mathbb{R}^d$ be an open subset. A function $f : U \to \mathbb{R}$ is called *differentiable of class C^k on U* if all partial derivatives $\frac{\partial^\alpha f}{\partial^\alpha x}$ exist and are continuous for all $\alpha \in \mathbb{N}^d$, $\alpha_1 + \ldots + \alpha_d \leq d$. We say f is smooth and of class C^∞ on U if f is differentiable of class C^k on U for all $k \geq 0$.

A map $f : U \to \mathbb{R}^m$ is called *differentiable of class C^k on U* respectively smooth if all coordinate functions $f_i = y_i \circ f$ are *differentiable of class C^k on U* respectively smooth.

Definition 2.1.2 A *locally Euclidean space M of dimension d* is a Hausdorff topological space M for which each point $p \in M$ has a neighborhood homeomorphic to an open subset of some $\mathbb{R}^d$.

A *differentiable structure $\mathcal{F}$ of class C^k* $(1 \leq k \leq \infty)$ on a locally Euclidean space M is a collection of coordinate systems $\{(U_\beta, \phi_\beta) \mid \beta \in I\}$ (often called an atlas) satisfying the following properties:

- for all $\beta \in I$, U_β is connected, and $\phi_\beta : U_\beta \to \mathbb{R}^d$ is a homeomorphism of U_β onto an open subset of $\mathbb{R}^d$;

- $\bigcup_{\beta \in I} U_\beta = M$;

- $\phi_\beta \circ \phi_{\beta'}^{-1}$ is of class C^k for all $\beta, \beta' \in I$
- the collection $\mathcal{F}$ is maximal with respect to the last point, i.e., if (U, ϕ) is a coordinate system (meaning U is connected and $\phi : U \to \mathbb{R}^d$ is a homeomorphism onto an open subset of $\mathbb{R}^d$) such that $\phi_\beta \circ \phi^{-1}$ and $\phi \circ \phi_\beta^{-1}$ are of class C^k for all $\beta \in I$, then $(U, \phi) \in \mathcal{F}$.

Remark 2.1.1 Given a collection $\mathcal{F}_0 = \{(U_\beta, \phi_\beta) \mid \beta \in I\}$ of coordinate systems satisfying the first three points, one can complete $\mathcal{F}_0$ to a collection $\mathcal{F}$ satisfying all four points:

$$\mathcal{F} = \{(U, \phi) \mid \phi \circ \phi_\beta^{-1} \text{ and } \phi_\beta \circ \phi^{-1} \text{are } C^k \text{ for all } (U_\beta, \phi_\beta) \in I\}$$

Definition 2.1.3 A *(real) C^k-manifold M* of dimension d is a locally Euclidean space M of dimension d having a second countable topology, and which is endowed with differentiable structure $\mathcal{F}$ of class C^k. A C^∞-manifold M is called a smooth manifold.

Definition 2.1.4 A *complex manifold* is a manifold with an atlas of coordinate charts homeomorphic to subsets in $\mathbb{C}^n$, such that the transition maps are holomorphic.

Example 2.1.1 $GL_n(\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and is hence naturally a real manifold. Now $GL_n(\mathbb{C})$ is an open subset of $\mathbb{C}^{n^2}$ and is hence naturally an example for a complex manifold.

Example 2.1.2 Another way to construct manifolds is the following, these manifolds are called *embedded manifolds*. Suppose we are given a smooth function $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$, and let

$$M = \{x \in \mathbb{R}^{n+m} \mid f(x) = 0\}.$$

Assume further that 0 is a regular value, i.e. the Jacobi matrix $Df(x)$ of f is of maximal rank for all points in M. Then the *Implicit Function Theorem* implies that for every point $x = (x_1, x_2) \in M$ ($x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^m$) there exists an open neighborhood $U_{x_1} \subset \mathbb{R}^n$ containing x_1 and an open neighborhood $U_{x_2} \subset \mathbb{R}^m$ containing x_2, and a smooth map $g : U_{x_1} \to U_{x_2}$ such that for all $(y_1, y_2) \in U_{x_1} \times U_{x_2}$ we have

$$f(y_1, y_2) = 0 \Leftrightarrow y_2 = g(y_1).$$

Or, in other words: $(U_{x_1} \times U_{x_2}) \cap M = \{(y_1, g(y_1)) \mid y_1 \in U_{x_1}\}$ is an open neighborhood of x in M, and the projection $\pi : (y, g(y)) \mapsto y$ defines a coordinate chart for this neighborhood of x in M. (For more details see Warner, Foundations of Differentiable Manifolds and Lie Groups, Theorem 1.38).

Example 2.1.3 Recall the definition $U_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \overline{A}^\intercal A = 1\!\!1\}$ of the unitary group (see (2)). Let $\mathfrak{H} \subset M_n(\mathbb{C})$ be the real subspace of hermitian matrices, i.e. $A = \overline{A}^\intercal$. We can look at the defining equations of $U_n(\mathbb{C})$ also as follows. Let f be the smooth map

$$f : M_n(\mathbb{C}) \to \mathfrak{H}, \quad X \mapsto \overline{X}^\intercal X - 1\!\!1, \tag{2.1}$$

then $U_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid f(X) = 0\}$.

Exercise 2.1.1 Verify that the Jacobi matrix $Df(X)$ of f in (2.1) is of maximal rank for all points in $U_n(\mathbb{C})$.

Example 2.1.4 It follows by Example 2.1.3 and Exercise 2.1.1 that $U_n(\mathbb{C})$ is a smooth manifold.

Definition 2.1.5 A *Lie group* is a pair (G, μ) where G is a smooth manifold and $\mu : G \times G \to G$ is a smooth mapping which gives G the structure of a group.

In other words, G is at the same time a smooth manifold and a group, and these two structures are compatible, i.e. the product map and the inversion are smooth maps.

Example 2.1.5 We have already seen that $G = GL_n(\mathbb{C})$ is a complex manifold. The determinant does not vanish and hence its inverse is a holomorphic function on $GL_n(\mathbb{C})$. The product of two matrices is a polynomial map and hence holomorphic. It follows that $GL_n(\mathbb{C})$ is a complex Lie group because the matrix product and the inversion are given by holomorphic maps.

Example 2.1.6 We have already seen that $G = U_n(\mathbb{C})$ is a smooth manifold. The product of two matrices is a polynomial map and hence analytic. The inversion of a unitary matrix is given by $X \mapsto \overline{X}^\intercal$, which is again a smooth map. It follows that $U_n(\mathbb{C})$ is a smooth Lie group.

2.2 Tangent vectors

A point in a manifold comes always equipped with its associated tangent space. The formal definition of a tangent vector is the following. Let M be (as in our running example $GL_n(\mathbb{C})$ und $U_n(\mathbb{C})$) a manifold and let $m \in M$. Let f and g be two C^∞-functions defined on open neighborhoods U_f respectively U_g of m in M. We say f and g have the same *germ* in m if there exists an open neighborhood $U \subset U_f \cap U_g$ of m such that $f|_U = g|_U$. The property of having the same germ defines an equivalence relation. Denote by $\mathbf{F}_m$ the set of all equivalence classes. Note if $\mathbf{f} \in \mathbf{F}_m$, then the value $\mathbf{f}(m)$ is well defined. The operations of addition and multiplication of functions naturally induces operations of addition and multiplication of the corresponding germs. Let $F_m = \{\mathbf{f} \in \mathbf{F}_m \mid \mathbf{f}(m) = 0\}$. Then $F_m \subset \mathbf{F}_m$ is an ideal, and we get a decreasing sequence of ideals

$$\ldots \subset F_m^3 \subset F_m^2 \subset F_m \subset \mathbf{F}_m.$$

Definition 2.2.1 A *tangent vector* v at a point $m \in M$ is a linear derivation of $\mathbf{F}_m$, i.e., it is a map $v : \mathbf{F}_m \to \mathbb{R}$ such that

- $v(\mathbf{f} + \lambda\mathbf{g}) = v(\mathbf{f}) + \lambda v(\mathbf{g})$;
- $v(\mathbf{fg}) = \mathbf{f}(m)v(\mathbf{g}) + v(\mathbf{f})\mathbf{g}(m)$.

Example 2.2.1 Let $M = \mathbb{R}^n$ and $m = 0$. Let $v = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Consider the directional derivative evaluated in 0:

$$\partial_v|_{x=0} : \mathbf{F}_0 \to \mathbb{R}, \quad \mathbf{f} \mapsto \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(0).$$

This map is obviously linear and satisfies the Leibniz rule, so $\partial_v|_{x=0}$ is an example for a tangent vector.

Now if we define for tangent vectors $(v+w)(\mathbf{f}) = v(\mathbf{f})+w(\mathbf{f})$ and $(\lambda v)(\mathbf{f}) = \lambda v(\mathbf{f})$, then it is easy to verify that this defines on the set T_mM of all tangent vectors the structure of a real vector space.

Lemma 2.2.1 *T_mM is naturally isomorphic to $(F_m/F_m^2)^*$.*

Proof. We have a natural map from T_mM to $(F_m/F_m^2)^*$ as follows: v is a linear function on $\mathbf{F}_m$, so its restriction to F_m remains linear. Further, since it is a derivation, it vanishes on products $\mathbf{fg}$ of elements in F_m because $v(\mathbf{fg}) = \mathbf{f}(m)v(\mathbf{g}) + v(\mathbf{f})\mathbf{g}(m) = 0$. Now an element of F_m^2 is a linear combination of products $\mathbf{fg}$ of elements in F_m, so $v|_{F_m^2} \equiv 0$ and hence $v|_{F_m} \in (F_m/F_m^2)^*$.

Conversely, let $\ell \in (F_m/F_m^2)^*$ and define $v_\ell : \mathbf{F}_m \to \mathbb{R}$ by

$$v_\ell(\mathbf{f}) := \ell(\mathbf{f} - \mathbf{f}(m)).$$

(To be more precise, one should take the class of the function $f - f(m)$, ...) It is clear that the map is linear, to prove that it is a derivation one has to verify the Leibniz rule:

$$\begin{aligned} v_\ell(\mathbf{fg}) &= \ell(\mathbf{fg} - \mathbf{f}(m)\mathbf{g}(m)) \\ &= \ell((\mathbf{f} - \mathbf{f}(m))(\mathbf{g} - \mathbf{g}(m)) + \mathbf{f}(m)(\mathbf{g} - \mathbf{g}(m)) \\ &\quad +(\mathbf{f} - \mathbf{f}(m)\mathbf{g}(m)) \\ &= \ell((\mathbf{f} - \mathbf{f}(m))(\mathbf{g} - \mathbf{g}(m))) + \ell(\mathbf{f}(m)(\mathbf{g} - \mathbf{g}(m))) \\ &\quad +\ell((\mathbf{f} - \mathbf{f}(m)\mathbf{g}(m)) \\ &= \mathbf{f}(m)v_\ell(\mathbf{g}) + \mathbf{g}(m)v_\ell(\mathbf{f}) \end{aligned}$$

•

Example 2.2.2 Let $M = \mathbb{R}^n$ and $m = 0$.

Claim: $T_0M \simeq \mathbb{R}^n$ by identifying a vector $u = (a_1, \ldots, a_n) \in \mathbb{R}^n$ with the directional derivative evaluated in 0: $f \mapsto \partial_u|_{x=0}f$.

To prove that this is an isomorphism, note that for all $u \in \mathbb{R}^n$, the map $f \mapsto \partial_u|_{x=0}f$ defines a linear derivation on $\mathbf{F}_0$ by Example 2.2.1, and the map is linear. For $u = \sum_{i=1}^n a_ie_i$ let us look at the image of the coordinate function x_i: we have $\partial_u|_{x=0}x_i = a_i$, so the map $u \mapsto \frac{\partial}{\partial u}|_{x=0}$ is injective.

Now given a tangent vector $v \in T_0M$, let $v(x_i) = a_i$ and set $u := \sum_{i=1}^n a_ie_i$, then $v(x_i) = a_i = \partial_u|_{x=0}x_i$ for all $i = 1, \ldots, n$. If $f \in F_0$ is any function, then we can develop f into a partial Taylor series: $f(x) = f(0) + \ell(x) + h(x)$, where $h \in F_0^2$. So v is completly determined by its values on linear functions and hence $v = \partial_u|_{x=0}$. It follows that the map $\mathbb{R}^n \to T_0M$, $u \mapsto \partial_u|_{x=0}$, is an isomorphism of vector spaces.

Example 2.2.3 The same arguments prove: for $M = \mathbb{R}^n$ and $m \in M$ one has $T_mM \simeq \mathbb{R}^n$ by identifying a vector $u \in \mathbb{R}^n$ with the directional derivative evaluated in m: $f \mapsto \partial_u|_{x=m}$.

Example 2.2.4 $T_{\mathbb{I}}(GL_n(\mathbb{C})) = M_n(\mathbb{C})$.

Example 2.2.5 Let M be an embedded manifold as in Example 2.1.2, given as a set as $f^{-1}(0)$ for a smooth map $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$. For $p \in M$ let $Df|_p : T_p\mathbb{R}^{n+m} \to T_0\mathbb{R}^m$ be the total differential of the map f in the point p, then

$$T_pM = \ker Df|_p.$$

Indeed, by Example 2.1.2, we can choose locally coordinates around p such that $p = (0, \dots, 0, 0, \dots, 0)$, and a neighborhood $U_p \subset M$ of p such that $U_p \subset \mathbb{R}^n$ is an open subset and $U_p \times \mathbb{R}^m \subset \mathbb{R}^{n+m}$ is a neighborhood of p in $\mathbb{R}^{n+m}$, and f is locally in these coordinates just the projection onto the last m coordinates, so the kernel $\ker Df|_p$ is just $\mathbb{R}^n \times (0, \dots, 0)$, the tangent space T_pU_p.

Exercise 2.2.1 Show that $T_{\mathbb{I}}(U_n(\mathbb{C})) = \mathfrak{u}_n(\mathbb{C})$

2.3 The exponential map in the classical case

A special feature of Lie groups is the exponential map, which provides a diffeomorphism between a neighborhood of 0 in the tangent space and a neighborhood of the identity in the group. In the cases we are interested in, the exponential map is the ordinary exponential map for matrices (see Example 0.0.4) known from a linear algebra course or a course on differential equations.

The purpose of this section is to recall some basic facts about the exponential map in the classical case. We will extend this later to the loop case.

Example 2.3.1 The exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is analytic, it is surjective, and the logarithm

$$\log z := \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m}$$

is an inverse function which is analytic inside a circle of radius 1 around 1.

Definition 2.3.1 *Let $A \in M_n(\mathbb{C})$ be a complex matrix. Define* $\log A$ *by*

$$\log A := \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - \mathbb{1})^m}{m}$$

whenever the series converges.

Remark 2.3.1 The classical logarithm converges for all $z \in \mathbb{C}$ such that $|z - 1| < 1$. Since $\|(A - \mathbb{1})^m\| \leq \|A - \mathbb{1}\|^m$, it follows that $\log A$ converges for all $A \in M_n(\mathbb{C})$ such that $\|A - \mathbb{1}\| < 1$. But this just a sufficient, not a necessary condition. Recall that a matrix is called unipotent if 1 is the only eigenvalue. So if A is unipotent, then $(A - \mathbb{1})$ is a matrix having only zero as eigenvalue, i.e. $(A - \mathbb{1})$ is nilpotent. It follows that $\log A$ is just a finite sum and the series is hence convergent, independent of $\|A - \mathbb{1}\|$. It is easy to see that if A is unipotent, then $\log A$ is nilpotent, and if A is nilpotent, then $\exp A$ is unipotent.

Theorem 2.3.1 *The exponential map*

$$\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C}), \quad A \mapsto \exp(A) = \sum_{i \geq 0} \frac{1}{i!} A^i$$

is a surjective, continuously differentiable map, and it is a diffeomorphism in a neighborhood of 0 *with* $\log$ *as inverse map in a neighborhood of* $\mathbb{1}$.

Proof. Let $\Upsilon \subset M_n(\mathbb{C})$ be a bounded region, so there exists $\mu \in \mathbb{R}$ such that $|x_{i,j}(A)| \leq \mu$ for all $1 \leq i, j \leq n$ and all $A \in \Upsilon$. By induction one shows that hence $|x_{i,j}(A^k)| \leq n^{k-1}\mu^k$. It follows by Weierstrass M-test that for all $1 \leq i, j \leq n$ the series

$$\sum_{k=0}^{\infty} \frac{x_{i,j}(A^k)}{k!}$$

converges absolutely and uniformly for all $A \in \Upsilon$, and hence the series

$$\exp(A) = \sum_{i \geq 0} \frac{1}{i!} A^i$$

converges uniformly for A in Υ. It follows that the map $A \to \exp(A)$ is continous.

Let $S_j(A)$ be the partial sum $\sum_{k=0}^{j} \frac{A^k}{k!}$. The multiplication by a matrix is a continuous map, so $\lim_{j\to\infty}(BS_j(A)) = B(\lim_{j\to\infty} S_j(A))$, which in turn implies for $B \in GL_n(\mathbb{C})$:

$$B(\exp A)B^{-1} = \exp(BAB^{-1}). \tag{2.2}$$

So to calculate $\det\exp(A)$, we can, if necessary, replace A by a conjugate matrix which is upper triangular. It follows that if $\lambda_1, \dots, \lambda_n$ are the entries on the diagonal of A, then the diagonal entries in $\exp A$ are $e^{\lambda_1}, \dots, e^{\lambda_n}$, and $\exp A$ is upper triangular too. Hence $\det\exp(A) = \exp(tr(A))$. In particular, $\det\exp(A) \neq 0$ and hence $\exp(A) \in GL_n(\mathbb{C})$.

Next consider $\|\frac{d}{dt}(X+tY)^m|_{t=0}\|$, where X and Y are arbitrary complex $n\times n$ matrices. Now the two do not necessarily commute, but still we get

$$\|\frac{d}{dt}(X+tY)^m|_{t=0}\| = \|\sum_{\ell=1}^{m} X^{\ell-1}YX^{m-\ell}\| \leq m\|X\|^{m-1}\|Y\|.$$

This implies for the series

$$\exp(X+tY) = \sum_{i\geq 0}\frac{1}{i!}(X+tY)^i$$

and the norm of the series of term-by-term directional derivatives:

$$\|\sum_{i\geq 0}\frac{d}{dt}(\frac{1}{i!}(X+tY)^i)|_{t=0}\| = \sum_{i\geq 0}\|(\frac{1}{i!}(X+tY)^i)|_{t=0}\| \leq \|Y\|\sum_{i\geq 1}\frac{\|X\|^{i-1}}{i-1!} < \infty$$

so the latter is absolutely convergent for all X (and Y) which are elements of a bounded subset of the form $\{A \in M_n(\mathbb{C}) \mid \|A\| < r\}$ for some $r > 0$. Now this implies that $\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is continuously differentiable. (See, for example, Rudin, *Principles of Mathematical Analysis*, Theorem 7.17. Here is the version for functions: if (f_n) is a sequences of differentiable functions on an interval $[a,b]$ and $\lim_{n\mapsto\infty} f_n(x_0)$ exists for some $x_0 \in [a,b]$ and the sequence (f_n') converges uniformely on $[a,b]$, then (f_n) converges uniformely on $[a,b]$ to a function f and $f'(x) = \lim_{n\mapsto\infty} f_n'(x)$ for $x \in [a,b]$.)

Now let us look at the differential in 0. For $X = 0$ we get by the above that the differential of the exponential map at this point is the identity map: $Y \mapsto Y$. In particular, it follows that the exponential map is a diffeomorphism of a neighborhood of 0 in $M_n(\mathbb{C})$ onto a neighborhood around $\mathbb{1}$ in $GL_n(\mathbb{C})$.

It remains to prove that the map is surjective. By Example 2.3.1, the map is a surjective map from the set of diagonal matrices in $M_n(\mathbb{C})$ onto the diagonal matrices in $GL_n(\mathbb{C})$. By (2.2), It follows that the map is surjective from the set of diagonalizable matrices in $M_n(\mathbb{C})$ onto the set of diagonalizable matrices in $GL_n(\mathbb{C})$.

If A is not diagonalizable, then we know that A is conjugate to a block matrix such that each block is of the form $\lambda\mathbb{1} + N_\lambda$, where $\lambda \in \mathbb{C}^*$ and N_λ is a strictly upper triangular nilpotent matrix (Jordan decomposition). We can hence reduce the calculation to the case where A has just one Jordan block, so A is of the form $\lambda\mathbb{1} + N_\lambda$. Since $\lambda \neq 0$, one can rewrite the sum into a product: $(\lambda\mathbb{1})(\mathbb{1} + \frac{1}{\lambda}N_\lambda)$, where the first is a diagonal matrix and the second is a unipotent matrix. Hence, by Exercicse 2.3.1 below, we can find a nilpotent matrix N' such that $\exp(N') = \mathbb{1} + \frac{1}{\lambda}N_\lambda$, and we can find $\mu \in \mathbb{C}$ such that $\exp\mu = \lambda$. Since N' and $\mu\mathbb{1}$ commute, we get

$$A = (\lambda\mathbb{1})(\mathbb{1} + \frac{1}{\lambda}N_\lambda) = \exp(\mu\mathbb{1})\exp N' = \exp(\mu\mathbb{1} + N').$$

It follows that $\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is a surjective map. •

Exercise 2.3.1 Show that $\exp(\log(A)) = A$ for unipotent matrices, and $\log(\exp(A)) = A$ for nilpotent matrices.

Exercise 2.3.2 i) Show that $\exp(A+B) = \exp(A)\exp(B)$ if $AB = BA$.

ii) Show that $(\exp(A))^{-1} = \exp(-A)$.

Now keep in mind that one has an inverse map for the exponential map: the logarithm, for a neighborhood around $\mathbb{1}$ (see Remark 2.3.1). We have already seen (see Example 0.0.4) that $\exp(A) \in U_n(\mathbb{C})$ if $A \in \mathfrak{u}_n(\mathbb{C})$, the real space of skew hermitian matrices, i.e. the transpose of the complex conjugate matrix has the property $\overline{A}^\intercal = -A$. Recall that all elements of $U_n(\mathbb{C})$ are diagonalizable and the eigenvalues have modulus 1. Now let $\mathfrak{T}_\mathbb{R} \subset M_n(\mathbb{R})$ be the space of diagonal real matrices, then $i\mathfrak{T}_\mathbb{R} \subset \mathfrak{u}_n(\mathbb{C})$ and $\exp : i\mathfrak{T}_\mathbb{R} \to U_n(\mathbb{C})$ is surjective onto the subgroup of diagonal matrices in $U_n(\mathbb{C})$. By conjugation, it follows that $\exp : \mathfrak{u}_n(\mathbb{C}) \to U_n(\mathbb{C})$ is surjective. One shows as above:

Theorem 2.3.2 *The exponential map*

$$\exp : \mathfrak{u}_n(\mathbb{C}) \to U_n(\mathbb{C}), \quad A \mapsto \exp(A) = \sum_{i \geq 0} \frac{1}{i!} A^i$$

is a surjective, continuously differentiable map, and it is a diffeomorphism in a neighborhood of 0*, with* $\log$ *as inverse map around* $\mathbb{1}$.

Chapter 3

Lie groups and Loop groups

3.1 Lie groups: basics and examples

Recall that a *Lie group* is a pair (G, μ) where G is a smooth (complex or real) manifold and $\mu : G \times G \to G$ is a smooth mapping which gives G the structure of a group (see Definition 2.1.5). The tangent space $T_{\mathbb{I}}G$ is referred to as the Lie algebra of G. In other words, G is at the same time a smooth manifold and a group, and these two structures are compatible, i.e. the product map and the inversion are smooth maps.

Example 3.1.1 The first example we have in mind is $GL_n(\mathbb{C})$, which, as an open subset of a complex vector space is a smooth complex manifold. The multiplication is a polynomial map and hence smooth. The determinant is (on $GL_n(\mathbb{C})$) a nowhere vanishing holomorphic function and hence so is $\frac{1}{\det}$, which implies that the inversion $g \mapsto g^{-1}$ is a holomorphic map of $GL_n(\mathbb{C})$ into itself. The Lie algebra of $GL_n(\mathbb{C})$ is $M_n(\mathbb{C})$. Note that this is a complex vector space, and, in particular, we see directly that this space is a complex Lie algebra in the sense of section 1.4. (See Example 2.1.1 and Example 2.2.4.)

Example 3.1.2 The second example we have in mind is the unitary group $U_n(\mathbb{C})$. The multiplication is a polynomial map and hence smooth. The inverse map is the map $g \mapsto \overline{g}^\intercal$ which is obviously a smooth map. The Lie algebra is $\mathfrak{u}_n(\mathbb{C})$, the space of skew hermitian matrices, i.e. the transpose of the complex conjugate matrix has the property $\overline{A}^\intercal = -A$, and, in particular, we see directly that this space is a real Lie algebra in the sense of section 1.4. (See Example 2.1.4 and Exercise 2.2.1.)

Complexification: The group $GL_n(\mathbb{C})$ is often referred to as a complexification of the compact group $U_n(\mathbb{C})$. The background is the following: the group $U_n(\mathbb{C})$ is a smooth (real) manifold with Lie algebra $\mathfrak{u}_n(\mathbb{C})$, which is a real vector space. Note that $\mathfrak{u}_n(\mathbb{C}) \subset M_n(\mathbb{C})$ is just a real subspace. Indeed, it is a subspace with quite a special property:

$$M_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) \oplus i(\mathfrak{u}_n(\mathbb{C})),$$

or, to state it in a more fancy way: the inclusion $\mathfrak{u}_n(\mathbb{C}) \subset M_n(\mathbb{C})$ induces an isomorphism

$$\mathfrak{u}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \to M_n(\mathbb{C}).$$

In general (this is a simplified description), having a connected complex Lie group G with complex Lie algebra $\mathfrak{g}$ and a real connected Lie group $H \subset G$ with Lie algebra $\mathfrak{h}$, one says that *G is a complexification of H* if the inclusion $\mathfrak{h} \subset \mathfrak{g}$ of $\mathfrak{h}$ as a real subspace of $\mathfrak{g}$ provides a decomposition $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{h}$ (or, in terms of field extensions, an isomorphism $\mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{g}$). The inclusions $U_n(\mathbb{C}) \subset GL_n(\mathbb{C})$ on the level of groups and $\mathfrak{u}_n(\mathbb{C}) \subset M_n(\mathbb{C})$ on the level of Lie algebras is an example of such an instance.

3.2 Loop groups

3.2.1 Introduction

We would like to make the groups $L^\infty G$ and $\Omega^\infty G$ into "infinite dimensional" Lie groups in a similar way, and to get the loop groups $L^\infty GL_n(\mathbb{C})$ and $L^\infty U_n(\mathbb{C})$ connected in a similar way, meaning one is the complexification of the other. To do so, we have to make a short excursion into the theory of infinite dimensional manifolds.

In the finite dimensional case, the role model for a manifold is an open subset in some finite dimensional complex respectively real vector space. The finite dimensional vector space will be replaced by a complete separable metrizable topological vector space E. Here *metrizable* means the topology is induced by a metric, *complete* means that every Cauchy sequence converges to an element in E, *separable* means that E contains a countable, dense subset.

3.2.2 The role model

In our case, E is the vector space $L^\infty\mathfrak{g}$, where $\mathfrak{g} = M_n(\mathbb{C})$ or $\mathfrak{g} = \mathfrak{u}_n(\mathbb{C})$. We think of the circle S^1 as consisting interchangeably of real numbers θ modulo 2π or as complex numbers $z = \exp i\theta \in S^1$ of modulus one.

If $\phi \in E$ is a loop with matrix entries $(\phi_{i,j})_{i,j}$, then its derivative $\frac{\partial\phi}{\partial\theta}$ is again a matrix, the entries being $(\frac{\partial\phi_{i,j}}{\partial\theta})_{i,j}$. The same holds for the higher derivatives. So for $\phi \in E$, $\phi : S^1 \to M_n(\mathbb{C})$, all its derivatives of all orders (recall that ϕ is smooth) provide again maps $\frac{\partial^n\phi}{\partial\theta^n} : S^1 \to M_n(\mathbb{C})$.

We endow E with the topology of uniform convergence, meaning that a sequence $\{\phi_i\}_{i\in\mathbb{N}} \subset E$ is a Cauchy sequence if the maps ϕ_i as well as all their derivatives $\{\phi_i^{(n)} = \frac{\partial^n\phi_i}{\partial\theta}\}_{i\in\mathbb{N}}$ of all orders n are uniformly convergent sequences of maps on the circle. The condition on the ϕ_i implies that $\phi := \lim_{i\mapsto\infty} \phi_i$ is a continuous loop (because the ϕ_i converge uniformly). The condition on the derivations $\frac{\partial^n\phi_i}{\partial\theta}$ implies that they converge uniformly towards $\frac{\partial^n\phi}{\partial\theta}$. In particular, it turns out that $\phi \in E$, implying that E is complete with respect to this topology.

To translate these ideas a bit more precisely into a metric on E, we fix on $M_n(\mathbb{C})$ as norm the Frobenius norm: for $A = (a_{i,j})_{1\leq i,j\leq n}$ we set

$$\|A\|_F = \sqrt{(\mathrm{tr}(\overline{A}^\intercal A))} = \sqrt{\sum_{1\leq i,j\leq n} |a_{i,j}|^2} \tag{3.1}$$

Given a smooth loop $\gamma \in E$, set

$$\|\gamma\|_\infty := \sup_{z\in S^1} \|\gamma(z)\|_F. \tag{3.2}$$

Now uniform convergence for a sequence $\{\phi_i\}_{i\in\mathbb{N}}$ of loops means that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|\phi_i - \phi_j\|_\infty < \epsilon$ for all $i, j > N$. To ensure that all the derivations converge uniformly too, we need an additional function $f(s) := \frac{s}{s+1}$ for $s \in \mathbb{R}_{\geq 0}$. We define a metric on E as follows: given two loops γ and ϕ, we define as the distance between the two loops

$$d(\gamma, \phi) := \|\gamma - \phi\|_\infty + \sum_{k\geq 1} \frac{1}{2^k} f(\|\gamma^{(k)} - \phi^{(k)}\|_\infty)$$

This defines a metric on $L^\infty\mathfrak{g}$: since $0 \leq f(\|\gamma^{(k)} - \phi^{(k)}\|_\infty) < 1$ for all k and all $\gamma, \phi \in E$, $d(\gamma, \phi)$ is a non-negative real number by definition. Further,

$d(\gamma,\phi)$ is obviously definite and symmetric, and the triangle inequality holds. Indeed, suppose we have three loops γ,ϕ,δ. Since f is increasing on $\mathbb{R}_{\geq 0}$, we get

$$\begin{array}{rcl} d(\gamma,\phi) & = & \|\gamma-\phi\|_\infty + \sum_{k\geq 1}\frac{1}{2^k} f(\|\gamma^{(k)}-\phi^{(k)}\|_\infty) \\ & \leq & \|\gamma-\delta\|_\infty + \|\delta-\phi\|_\infty \\ & & \quad + \sum_{k\geq 1}\frac{1}{2^k}(f(\|\gamma^{(k)}-\delta^{(k)}\|_\infty + \|\delta^{(k)}-\phi^{(k)}\|_\infty)) \\ & \leq & \|\gamma-\delta\|_\infty + \|\delta-\phi\|_\infty \\ & & \quad + \sum_{k\geq 1}\frac{1}{2^k}(f(\|\gamma^{(k)}-\delta^{(k)}\|_\infty) + f(\|\delta^{(k)}-\phi^{(k)}\|_\infty)) \\ & = & d(\gamma,\delta)+d(\delta,\phi). \end{array}$$

To justify the last inequality, recall that f is increasing, so for $a,b\in\mathbb{R}_{\geq 0}$ we have

$$\begin{array}{rcl} f(a+b)\leq f(ab+a+b) & = & \frac{ab+a+b}{ab+a+b+1} \\ & \leq & \frac{2ab+a+b}{ab+a+b+1} = \frac{a}{a+1}+\frac{b}{b+1} = f(a)+f(b). \end{array}$$

Now suppose $\{\phi_i\}_{i\in\mathbb{N}}$ is a sequence of loops such that the maps ϕ_i as well as all their derivatives $\{\frac{\partial^n\phi_i}{\partial\theta}\}_{i\in\mathbb{N}}$ of all orders n are uniformly convergent sequences. Given $\epsilon>0$, fix N such that $\sum_{\ell>N}\frac{1}{2^\ell}<\frac{\epsilon}{2}$, and let $N'\geq N$ be such that for all $i,j>N'$ and all $0\leq k\leq N'$ we have $\|\frac{\partial^k\phi_i}{\partial\theta^k}-\frac{\partial^k\phi_j}{\partial\theta^k}\|_\infty<\frac{\epsilon}{4}$. It follows that for all $i,j>N'$ one has

$$\begin{array}{rcl} d(\phi_i,\phi_j) & = & \|\phi_i-\phi_j\|_\infty + \sum_{k\geq 1}\frac{1}{2^k} f(\|\frac{\partial^k\phi_i}{\partial\theta^k}-\frac{\partial^k\phi_j}{\partial\theta^k}\|_\infty) \\ & \leq & \frac{1}{2}\epsilon(\sum_{k\geq 1}^{N'}\frac{1}{2^k}) + \sum_{k>N'}\frac{1}{2^k} < \epsilon. \end{array}$$

So this gives a reformulation of the uniform convergence of the loops and their derivatives in terms of the metric. Vice versa, suppose $\{\phi_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence with respect to $d(\cdot,\cdot)$. Fix $k\in\mathbb{N}$, then for all $1>\epsilon>0$ there exists an $N\in\mathbb{N}$ such that $d(\phi_i,\phi_j)<\frac{\epsilon}{2^{k+1}}$ for all $i,j>N$ and hence

$$d(\phi_i,\phi_j) = \|\phi_i-\phi_j\|_\infty + \sum_{\ell\geq 1}\frac{1}{2^\ell} f(\|\phi_i^{(\ell)}-\phi_j^{(\ell)}\|_\infty) < \frac{\epsilon}{2^{k+1}}.$$

Since all summands are non-negative, it follows that $\frac{1}{2^k} f(\|\phi_i^{(k)}-\phi_j^{(k)}\|_\infty) < \frac{\epsilon}{2^{k+1}}$ and hence $f(\|\phi_i^{(k)}-\phi_j^{(k)}\|_\infty)<\frac{\epsilon}{2}$. Since $1>\epsilon>0$, one gets $\forall i,j>N$:

$$\frac{\|\phi_i^{(k)}-\phi_j^{(k)}\|_\infty}{\|\phi_i^{(k)}-\phi_j^{(k)}\|_\infty+1} < \frac{\epsilon}{2} \Leftrightarrow \|\phi_i^{(k)}-\phi_j^{(k)}\|_\infty < \frac{\epsilon}{2-\epsilon} < \epsilon.$$

So the convergence with respect to $d(\cdot,\cdot)$ implies uniform convergence for the loops and all their derivatives. So the topology is the one induced by the metric.

It remains to comment on the fact that E is separable, but this is an immediate consequence of the Stone-Weierstraß approximation theorem (approximation by Fourier series with rational coefficients).

Remark 3.2.1 One could try to make E into a Banach space. To do so, we need to define a norm on E. Given a metric on a vector space, the usual trick to define a norm $\|\phi\|_d$ of an element as the distance of the element from the origin of the ambient vector space:

$$\|\phi\|_d := d(\phi, 0).$$

But if α is a real number, then, in general, for the metric we have choosen one gets $\|\alpha\phi\|_d \neq |\alpha|\|\phi\|_d$ because f is not homogeneous. In fact, one can show that it is impossible to make the space E of smooth loops into a Banach space.

Before we come to the loop group:

3.2.3 First some generalities

A possibly infinite dimensional manifold X is a paracompact (which means any cover admits a locally finite refinement) topological space modelled on some topological vector space. More precisely, this means that X admits an atlas of open set $\{U_\alpha\}$, and each U_α is homeomorphic $\phi_\alpha : U_\alpha \to E_\alpha$ to some open subset $E_\alpha \subset E$, E a topological vector space. In the following we will always assume that E is separable, metrizable and complete. The transition functions

$$\phi_\alpha(U_\alpha \cap U_\beta) \to^{\phi_\alpha^{-1}} U_\alpha \cap U_\beta \to^{\phi_\beta} \phi_\beta(U_\alpha \cap U_\beta)$$

are supposed to be smooth, i.e. infinitely differentiable. Here we use the following convention: Let $U \subset E$ be an open subset and let $f : U \to E$ be a map. Then ϕ is called continuously differentiable on U if the limit

$$DF(u; v) := \lim_{t\to 0} \frac{f(u + tv) - f(u)}{t}$$

exists for all $u \in U$ and $v \in E$ and is continuous as a map

$$Df : U \times E \to E.$$

The second derivative is the map

$$D^2 f : U \times E \times E \to E$$

defined by $D^2 f(u; v, w) := \lim \frac{Df(u+tw;v)-Df(u;v)}{t}$ (if this limit exists), and so on.

If E is a complex vector space and the transition functions are holomorphic (i.e. smooth and $Df : U \times E \to E$ is complex linear in the second variable), then X is called a complex manifold.

As in the classical finite dimensional case, a possibly infinite dimensional Lie group Γ is a smooth manifold such that the group law $\Gamma \times \Gamma \to \Gamma$ and the inversion $\Gamma \to \Gamma$ are given by smooth maps.

3.2.4 $L^\infty G$ as a topological group

We make $L^\infty G$ first into a topological space by endowing it with the topology of uniform convergence. Another way of viewing the topology is by using the embedding of $G \subset M_n(\mathbb{C})$. In this way we may view $L^\infty G$ as a subset of $L^\infty M_n(\mathbb{C})$. Now the latter is a metrizable space, the topology induced by the metric is the topology of uniform convergence on $L^\infty M_n(\mathbb{C})$, and we take the induced topology on $L^\infty G$. Recall that we have a metric on $L^\infty M_n(\mathbb{C})$, so $L^\infty M_n(\mathbb{C})$ is a Hausdorff space, and hence, as a subspace, $L^\infty G$ is Hausdorff too, and so is $L^\infty G \times L^\infty G$. So to prove that that $L^\infty G \times L^\infty G \to L^\infty G$ is continuous at (ϕ, ψ), it is sufficient to show that for all sequences $(\phi_i)_{i\in\mathbb{N}}$ and $(\psi_i)_{i\in\mathbb{N}}$ such that $\lim_{t\to\infty} \phi_i = \phi$ respectively $\lim_{t\to\infty} \psi_i = \psi$ we have $\lim_{t\to\infty} \phi_i\psi_i = \phi\psi$ in the topology of uniform convergence on $L^\infty G$ defined above. We get for the ℓ-th derivative:

$$\begin{aligned}
\| \quad & (\phi_i\psi_i)^{(\ell)} - (\phi\psi)^{(\ell)}\|_\infty \\
= \quad & \|(\textstyle\sum_{j=0}^{\ell} \begin{pmatrix} \ell \\ j \end{pmatrix} \phi_i^{(j)}\psi_i^{(n-j)}) - (\textstyle\sum_{j=0}^{\ell} \begin{pmatrix} \ell \\ j \end{pmatrix} \phi^{(j)}\psi^{(n-j)})\|_\infty \\
= \quad & \|(\textstyle\sum_{j=0}^{\ell} \begin{pmatrix} \ell \\ j \end{pmatrix} (\phi_i^{(j)}\psi_i^{(n-j)} - \phi_i^{(j)}\psi^{(n-j)} + \phi_i^{(j)}\psi^{(n-j)} - \phi^{(j)}\psi^{(n-j)})\|_\infty \\
\leq \quad & \textstyle\sum_{j=0}^{\ell} \begin{pmatrix} \ell \\ j \end{pmatrix} \big(\|\phi_i^{(j)}\|_\infty\|\psi_i^{(n-j)} - \psi^{(n-j)}\|_\infty + \|\phi_i^{(j)} - \phi^{(j)}\|_\infty\|\psi^{(n-j)}\|_\infty.
\end{aligned}$$

Now $\begin{pmatrix} \ell \\ j \end{pmatrix}$, $\|\phi_i^{(j)}\|_\infty$ and $\|\psi^{(n-j)}\|_\infty$ are fixed constants, so for every $\epsilon > 0$ and $0 \leq j \leq \ell$ we can find an integer $M(\ell, j)$ such that for all $i > M(\ell, j)$:

$$\|\phi_i^{(j)} - \phi^{(j)}\|_\infty < \frac{\epsilon}{\begin{pmatrix} \ell \\ j \end{pmatrix} \|\psi^{(n-j)}\|_\infty (2\ell + 1)}$$

and

$$\|\psi_i^{(n-j)} - \psi^{(n-j)}\|_\infty < \frac{\epsilon}{\begin{pmatrix} \ell \\ j \end{pmatrix} \|\phi^{(j)}\|_\infty (2\ell + 1)}.$$

Now set $M(\ell) = \max\{M(\ell, j) \mid 0 \leq j \leq \ell\}$, then

$$\|(\phi_i \psi_i)^{(\ell)} - (\phi\psi)^{(\ell)}\|_\infty < \epsilon$$

for all $i > M(\ell)$. This implies the desired property of uniform convergence for the sequence $(\phi_i \psi_i)_{i \in \mathbb{N}}$ and all its derivatives. We leave the proof that the inversion is a homeomorphism to the reader.

3.2.5 $L^\infty G$ as a Lie group

Let us first look at the classical case. We know G is a Lie group and hence has an atlas, i.e. there exists a collection of open subsets $\{(U_\alpha, \phi_\alpha)\}$ covering G, together with maps $\phi_\alpha : U_\alpha \to \mathbb{R}^d$ which are homeomorphisms onto an open subset, and which satisfy the transition condition that $\phi_\beta \circ \phi_\alpha^{-1}$ is smooth (wherever it is defined). Let us try to construct an such an atlas in an explicit way, and then to copy the construction to the loop group.

We have seen that there exist open neighborhoods $U_0 \subset \mathfrak{g}$ (of the origin) and $U_{\mathbf{I}} \subset G$ (of the identity) such that the exponential map $\exp : U_0 \to U_{\mathbf{I}}$ is a homeomorphism (in fact, it is a diffeomorphism), having the logarithm $\log : U_{\mathbf{I}} \to U_0$ as inverse map.

So the first open subset of our atlas is $U_{\mathbf{I}}$ with $\phi_{\mathbf{I}} = \log$ as homeomorphism. G is a Lie group, so multiplication by an element from the right or the left induces a homeomorphism from G into itself. It follows that for every g, the set $U_g := \{ug \mid u \in U_{\mathbf{I}}\}$, is an open subset of G, it is in fact an open neighborhood of the element g in G, and we have a homeomorphism

$$\phi_g : U_g \to U_0 \subset \mathfrak{g},\ v \mapsto \log(vg^{-1}).$$

So it follows that the collection $\{(U_g, \phi_g) \mid g \in G\}$ is a good candidate for an atlas. It remains to check that the transition maps $\phi_g \circ \phi_h^{-1}$ are smooth maps from $\phi_g(U_g \cap U_h)$ to $\phi_h(U_g \cap U_h)$. But we have for $\phi_g \circ \phi_h^{-1}$ the following explicit description:

$$\begin{array}{ccccc} U_0 & & & & U_0 \\ \cup & & & & \cup \\ \log(U_{\mathbb{1}} \cap U_{hg^{-1}}) & \xrightarrow{\exp} U_{\mathbb{1}} \cap U_{hg^{-1}} \xrightarrow{\cdot g} U_g \cap U_h \xrightarrow{\cdot h^{-1}} U_{\mathbb{1}} \cap U_{gh^{-1}} \xrightarrow{\log} & & & \log(U_{\mathbb{1}} \cap U_{gh^{-1}}) \\ || & & & & || \\ \phi_g(U_g \cap U_h) & & & & \phi_h(U_g \cap U_h) \end{array}$$

We see that the transition map is a combination of diffeomorphisms: exp, log and linear maps like multiplying with an invertible matrix.

To copy this approach, the first object we need is a replacement of the open neighborhood $U_0 \subset \mathfrak{g}$ by an open neighborhood $\mathcal{U}_0 \subset L^\infty\mathfrak{g}$.

Our candidate is to replace U_0 in this setting by $\mathcal{U}_0 = L^\infty(S^1, U_0) \subset L^\infty\mathfrak{g}$, the set of smooth maps from S^1 into U_0. We want to show that $\mathcal{U}_0$ is an open neighborhood of 0 (= the constant zero map) in $L^\infty\mathfrak{g}$.

Indeed, $0 \in \mathcal{U}_0$, and let $\phi \in \mathcal{U}_0$, so $\operatorname{Im}\phi \subset U_0$. We claim there exists an $\epsilon > 0$ such that

$$U_{p,\epsilon} = \{g \in \mathfrak{g} \mid \|g - p\|_F < \epsilon\} \subset U_0 \quad \forall p \in \operatorname{Im}\phi \subset U_0.$$

Suppose such an ϵ does not exist, then one could define a sequence of points in the image $\{p_n \mid n \in \mathbb{N}\} \subset \operatorname{Im}\phi$ such that $U_{p_n, \frac{1}{n}} \not\subset U_0$ for all $n \gg 0$. By replacing the sequence by a convergent subsequence if necessary, we may assume that $\lim_{n\to\infty} p_n = p$ exists. But this would imply that $p \in \operatorname{Im}\phi$ but p has no open neighborhood contained in U_0, which is not possible. It follows there exists an $\epsilon > 0$ such that $U_{p,\epsilon} \subset U_0$ for all $p \in \operatorname{Im}\phi$, and hence the open subset

$$\mathcal{U}_{\phi,\epsilon} := \{\Psi \in L^\infty(S^1, \mathfrak{g}) \mid d(\Psi, \phi) < \epsilon\} \subset \mathcal{U}_0,$$

is contained in $\mathcal{U}_0$ and is an open neighborhood for ϕ, and thus $\mathcal{U}_0$ is open.

By Theorem 2.3.1 respectively Theorem 2.3.2, there exists an open neighborhood $U_{\mathbf{1}}$ of $\mathbb{1}$ in G which is diffeomorphic to an open neighborhood U_0 of 0 in $\mathfrak{g}$. Then $\mathcal{U}_0 = L^\infty(S^1, U_0)$ is an open neighbourhood of 0 in $L^\infty\mathfrak{g}$, and the exponential map gives a map

$$\exp : \mathcal{U}_0 = L^\infty(S^1, U_0) \to \mathcal{U}_{\mathbf{1}} = L^\infty(S^1, U_{\mathbf{1}}), \tag{3.3}$$

with log as inverse map. Since $\exp : \mathfrak{g} \to G$ is a diffeomorphism around 0 respectively $\mathbb{I}$, a sequence of maps $\{\phi_j\}_{j\in\mathbb{N}}$ contained in $\mathcal{U}_0$ is uniform convergent if and only if the sequence $\{\exp \phi_j\}_{j\in\mathbb{N}} \subset \mathcal{U}_{\mathbf{I}}$ is uniform convergent. It follows that (3.3) is a homeomorphism.

We use these open sets to define an atlas: given $\phi \in L^\infty G$ and $\mathcal{U}_{\mathbf{I}}$ a neighborhood of $\mathbb{I}$ as above, define $\mathcal{U}_\phi = \mathcal{U}_{\mathbf{I}} \cdot \phi$, this is again a subset homeomorphic $\mathcal{U}_{\mathbf{I}}$. The collection $\{\mathcal{U}_\phi\}_{\phi\in L^\infty G}$ defines a covering of $L^\infty G$ of open neighborhoods, each of them homeomorphic to $\mathcal{U}_{\mathbf{I}}$. We leave it as an exercise to show that the transfer maps are smooth respectively holomorphic.

Exercise 3.2.1 Show that the atlas $\{\mathcal{U}_\phi\}_{\phi\in L^\infty G}$ makes $L^\infty G$ into a Lie group.

We have seen that the exponential map $\exp : \mathfrak{g} \to G$ is surjective for $G = GL_n(\mathbb{C})$ and $G = U_n(\mathbb{C})$. This does not hold necessarily for the corresponding loop groups.

Example 3.2.1 Consider the map

$$\phi : S^1 \to U_2(\mathbb{C}), \quad z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

Suppose $\phi = \exp \Psi$ for some $\Psi \in L^\infty(S^1, \mathfrak{u}_2(\mathbb{C}))$, so

$$\Psi(z)\phi(z) = \Psi(z)(\exp \Psi(z)) = (\exp \Psi(z))\Psi(z) = \phi(z)\Psi(z),$$

and hence $\phi(z)$ and $\Psi(z)$ commute for all z. But this implies that $\Psi(z)$ is a diagonal matrix for all $z \in S^1$. Now there exist no smooth real valued function f on the circle such that $\exp(if(z)) = z$ for all all $z \in S^1$.

Tangent vectors and vector fields can be defined as in the finite dimensional case, correspondingly the Lie algebra of a Lie group will be, as in the finite dimensional case, defined as the tangent space at $\mathbb{I}$. So in our case, not surprisingly, we refer to $L^\infty(S^1, \mathfrak{g})$ as the Lie algebra of $L^\infty G$. Note that, as in the finite dimensional case, a neighborhood of 0 in the Lie algebra serves as the model for a neighborhood in the Lie group.

Chapter 4

The algebraic case

4.1 Affine varieties

We recall some basic notions from algebraic geometry over the complex numbers. In the following V is always the complex vector space $\mathbb{C}^n$, and $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$ is the algebra of polynomial functions on V.

The main object of affine algebraic geometry is the common set of zeros of some subset $\mathcal{S} \subset \mathbb{C}[V]$, we denote this set by $\mathcal{V}(\mathcal{S})$:

$$\mathcal{V}(\mathcal{S}) := \{v \in V \mid f(v) = 0 \ \forall\, f \in \mathcal{S}\}.$$

One can reduce the description of such a set always to a finite set as follows. First, given a subset $\mathcal{S} \subset \mathbb{C}[V]$, let $I(\mathcal{S}) \subset \mathbb{C}[V]$ be the ideal generated by $\mathcal{S}$:

$$I(\mathcal{S}) = \{f \in \mathbb{C}[V] \mid \exists f_1, \ldots, f_r \in \mathcal{S},\ q_1, \ldots, q_r \in \mathbb{C}[V] :\ f = \sum_{i=1}^{r} q_i f_i\}.$$

Then we have obviously: $\mathcal{V}(\mathcal{S}) = \mathcal{V}(I(\mathcal{S}))$. The algebra $\mathbb{C}[V]$ is noetherian, so every ideal is finitely generated, and we can find a finite set of polynomials $\{h_1, \ldots, h_t\} \subset \mathcal{S}$ (exercise!) such that

$$\mathcal{V}(\mathcal{S}) = \mathcal{V}(\{h_1, \ldots, h_t\}).$$

In the following, most of the time we will start with an ideal I and look at its common set of zeros $X = \mathcal{V}(I)$. The reason why this seems to be the right language (compared to starting with a finite set of polynomials) is the following. Let now $Z \subseteq V$ be a subset and set

$$\mathcal{I}(X) := \{f \in \mathbb{C}[V] \mid f(z) = 0 \ \forall\, z \in Z\},$$

then $\mathcal{I}(X)$ is an ideal in $\mathbb{C}[V]$. Now starting with an ideal I and looking at its common zero set $Z = \mathcal{V}(I)$, it is natural to ask how I and $\mathcal{I}(Z) = \mathcal{I}(\mathcal{V}(I))$ are connected. Recall that the radical of an ideal $I \subset \mathbb{C}[V]$ is the ideal

$$\sqrt{I} = \{f \in \mathbb{C}[V] \mid \exists m \geq 1 : f^m \in I\}.$$

We have obviously $I \subseteq \sqrt{I} \subseteq \mathcal{I}(\mathcal{V}(I))$, and I is called a radical ideal if $I \subseteq \sqrt{I}$. We present the following without proof:

Theorem 4.1.1 (Hilbert's Nullstellensatz) $\sqrt{I} = \mathcal{I}(\mathcal{V}(I))$

This tells us also something about the existence of a common set of zeros, the following is an immediate consequence of Hilbert's theorem.

Corollary 4.1.1 $\mathcal{V}(I) = \emptyset \Leftrightarrow \sqrt{I} = \mathbb{C}[V]$.

A map $\Psi : \mathbb{C}^n \to \mathbb{C}^m$ is called a polynomial map if there exist polynomials $p_1, \ldots, p_m \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\psi(v) = (p_1(v), \ldots, p_m(v))$.

The descriptive version of a definition of affine varieties and morphisms between them is the following:

Definition 4.1.1 A subset $X \subseteq \mathbb{C}^n$ is called an *affine variety* if there exists an ideal $I \subset \mathbb{C}[V]$ such that $X = \mathcal{V}(I)$. The coordinte ring of X is the algebra $\mathbb{C}[X] := \mathbb{C}[V]/\sqrt{I} = \mathbb{C}[V]/\mathcal{I}(X)$.

Example 4.1.1 The group $SL_n(\mathbb{C})$ of complex $n \times n$ matrices with determinant one is an example for an affine variety:

$$SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det A - 1 = 0\}.$$

Definition 4.1.2 A *morphism* between two affine varieties $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ is a map $\phi : X \to Y$ which can be extended to a polynomial map $\tilde{\phi} : \mathbb{C}^n \to \mathbb{C}^m$ such that $\tilde{\phi}|_X = \phi$.

Proposition 4.1.1 *A morphism $\phi : X \to Y$ between two affine varieties $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ induces an algebra homomorphism $\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ defined by $f \mapsto f \circ \phi$.*

Proof. The poynomial map $\tilde{\phi} : \mathbb{C}^n \to \mathbb{C}^m$ induces via $f \mapsto f \circ \tilde{\phi}$ an algebra homomorphism $\mathbb{C}[y_1, \ldots, y_m] \to \mathbb{C}[x_1, \ldots, x_n]$. Indeed, if $\tilde{\phi}$ is given by $v \mapsto (f_1(v), \ldots, f_m(v))$, then the map $\tilde{\phi}^*$ is the unique algebra homomorphism that sends y_i to f_i. By combining the latter map with the quotient map $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[X]$, which means nothing but restricting the function to X, we get an algebra homomorphism

$$\tilde{\phi}^*|_X : \mathbb{C}[y_1, \ldots, y_m] \to \mathbb{C}[X].$$

Now if $f \in \mathcal{I}(Y)$, then $(\tilde{\phi}^*|_X(f))(u) = f(\tilde{\phi}(u)) = 0$ for all $u \in X$ because $\tilde{\phi}(u) = \phi(u) \in Y$. Hence $\mathcal{I}(Y) \subseteq \ker \tilde{\phi}^*|_X$, and we get an induced algebra homomorphism

$$\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X], \quad f \mapsto f \circ \phi.$$

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Definition 4.1.3 An *affine algebraic group* G is an affine variety, which at the same time is a group and the maps

$$\begin{array}{lcl} G \times G \to G & \text{and} & G \to G \\ (g, h) \mapsto gh & & g \mapsto g^{-1} \end{array}$$

are morphisms of affine varieties.

Example 4.1.2 The group $SL_n(\mathbb{C})$ of complex $n \times n$ matrices with determinant one is an example for an affine algebraic group:

The group multiplication is induced by the matrix product:

$$\begin{array}{ccccc} M_n(\mathbb{C}) & \times & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \\ \cup & & \cup & & \cup \\ SL_n(\mathbb{C}) & \times & SL_n(\mathbb{C}) & \to & SL_n(\mathbb{C}), \end{array}$$

and the matrix multiplication is a polynomial map:

$$\Big((x_{i,j})_{1 \le i,j \le n}, (y_{k,\ell})_{1 \le k,\ell \le n} \Big) \mapsto \Big((\sum_{j=1}^{n} x_{i,j} y_{j,\ell}) \Big)_{1 \le i,\ell \le n}$$

For the calculation of the inverse recall the definition of the adjugate matrix. Given a matrix $A \in M_n(\mathbb{C})$, let $A_{i,j}$ be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and j-th column, and let $m_{i,j} = \det A_{i,j}$. The cofactor matrix is the matrix

$$C = (c_{i,j})_{1\le i,j\le n} \quad \text{where } c_{i,j} = (-1)^{i+j} m_{i,j}$$

and the adjugate matrix $A^\dagger$ is the transpose of C: $A^\dagger := C^\intercal$. Recall that $A \cdot A^\dagger = (\det A)\mathbb{1}$, so for $A \in SL_n(\mathbb{C})$ we have $A^{-1} = A^\dagger$. It follows that the map sending a matrix to its adjugate matrix

$$\begin{array}{ccccccc} M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) & \qquad & A & \mapsto & A^\dagger \\ \cup & & \cup & & & & \\ SL_n(\mathbb{C}) & \to & SL_n(\mathbb{C}) & & g & \mapsto & g^{-1} \end{array}$$

is a polynomial map which is a lift to $M_n(\mathbb{C})$ of the inverse map on $SL_n(\mathbb{C})$.

The example shows that in general it seems rather awkward to work with this definition of an affine variety because everything seems to depend on the embedding $X \subseteq \mathbb{C}^n$ and the lucky choice of a lift $\tilde{\phi} : \mathbb{C}^n \to \mathbb{C}^m$ for a map $\phi : X \to Y$. A first hint that this is not the case is given by the fact that the reverse direction of Proposition 4.1.1 holds too:

Proposition 4.1.2 *Let $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ be two affine varieties. If $\Psi : \mathbb{C}[Y] \to \mathbb{C}[X]$ is an algebra homomorphism, then there exists a unique morphism $\phi : X \to Y$ such that $\phi^* = \Psi$.*

Proof. Let $\Psi : \mathbb{C}[Y] \to \mathbb{C}[X]$ be an algebra homomorphism and denote by $\Psi' : \mathbb{C}[y_1, \dots, y_m] \to \mathbb{C}[X]$ the composition of the quotient map $\mathbb{C}[y_1, \dots, y_m] \to \mathbb{C}[Y]$ with Ψ. Fix polynomials $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ such that $\bar{f}_j = \Psi'(y_j) \mod \mathcal{I}(X)$. We get an induced algebra homomorphism

$$\tilde{\Psi} : \mathbb{C}[y_1, \dots, y_m] \to \mathbb{C}[x_1, \dots, x_n] \quad \text{defined by } \tilde{\Psi}(y_j) = f_j.$$

We can use these polynomials also to define a morphism

$$\tilde{\phi} : \mathbb{C}^n \to \mathbb{C}^m, \quad v \mapsto (f_1(v), \dots, f_m(v)).$$

We have obviously $\tilde{\phi}^*(y_j) = f_j = \tilde{\Psi}(y_j)$ and hence $\tilde{\phi}^* = \tilde{\Psi}$. Suppose now $h \in \mathcal{I}(Y)$ and $v \in \mathbb{C}^n$, then

$$h(\tilde{\phi}(v)) = (h \circ \tilde{\phi})(v) = (\tilde{\phi}^*(h))(v) = (\tilde{\Psi}(h))(v).$$

In particular, if $v \in X$, then

$$h(\tilde{\phi}(v)) = (\tilde{\Psi}(h))(v) = (\Psi'(h))(v) = 0$$

because $\Psi'(h) = 0$ (recall, $\Psi' : \mathbb{C}[y_1, \dots, y_m] \to \mathbb{C}[X]$ is the composition of the quotient map $\mathbb{C}[y_1, \dots, y_m] \to \mathbb{C}[Y]$ with Ψ). It follows that $\tilde{\phi}(X) \subseteq Y$ and hence we get an induced map $\phi = \tilde{\phi}|_X$:

$$\phi : X \to Y \quad \text{ such that } \phi^* = \Psi.$$

Comparing the proof above with the one of Proposition 4.1.1, we see that the two are inverse to each other. •

Corollary 4.1.2 *Two affine varieties X and Y are isomorphic to each other if and only if $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are isomorphic to each other as algebras.*

This suggests that all informations about an affine variety can be recovered from the coordinate ring $\mathbb{C}[X]$. Before we start to explore this further, note the following consequence of Hilbert's Theorem:

Proposition 4.1.3 *Every maximal ideal J in $\mathbb{C}[x_1, \dots, x_n]$ is of the form $J_u = \langle x_1 - u_1, x_2 - u_2, \dots, x_n - u_n \rangle$ for a unique $u = (u_1, \dots, u_n) \in \mathbb{C}^n$. In addition, J_u can be described as $J_u = \mathcal{I}(u)$, the ideal of all polynomials vanishing in u.*

Proof. Given a point $u \in \mathbb{C}^n$, let $u = (u_1, \dots, u_n)$ be its coordinates and let $J_u \subseteq \mathbb{C}[x_1, \dots, x_n]$ be the ideal generated by the set of functions $x_i - u_i$, $i = 1, \dots, n$:

$$J_u = \langle x_1 - u_1, x_2 - u_2, \dots, x_n - u_n \rangle.$$

We have $\mathcal{V}(J_u) = u$, and, in addition, J_u is a maximal ideal. Indeed, since $\mathcal{V}(J_u) \neq \emptyset$, we know that J_u is a proper ideal. Clearly, the direct sum of subspaces $W := \mathbb{C} \cdot 1 \oplus J_u \subseteq \mathbb{C}[x_1, \dots, x_n]$ contains for all i the constant function u_i and the linear function $x_i - u_i$, and hence $x_1, \dots, x_n \in W$, and hence for all i, j we have $u_i u_j$, $u_i x_j$, $u_j x_i$ and $(x_i - u_i)(x_j - u_j)$ are elements of W, which implies that $x_i x_j \in W$ for all i, j, and so on. By induction we see $W := \mathbb{C} \cdot 1 \oplus J_u = \mathbb{C}[x_1, \dots, x_n]$ and hence $\mathbb{C}[x_1, \dots, x_n]/J_u \simeq \mathbb{C}$, which implies the ideal is maximal. It follows by Hilbert's Nullstellensatz and the maximality of the ideal:

$$J_u = \sqrt{J_u} = \mathcal{I}(\mathcal{V}(J_u)) = \mathcal{I}(u)$$

Now, vice versa, let $J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a maximal ideal and set $X = \mathcal{V}(J)$. Now J is maximal and hence proper and radical, so by Corollary 4.1.1 we have $X \neq \emptyset$. Let $u \in X$, then (recall that J is maximal and hence a radical ideal)

$$J_u = \mathcal{I}(u) \supseteq \mathcal{I}(X) = \mathcal{I}(\mathcal{V}(J)) = \sqrt{J} = J$$

which, by the maximality of J implies $J = J_u$. •

We reformulate the result as follows:

Corollary 4.1.3 *Let* $\operatorname{Mspec}(\mathbb{C}[x_1, \ldots, x_n])$ *be the set of all maximal ideals in* $\mathbb{C}[x_1, \ldots, x_n]$*. The maps*

$$\begin{array}{rcl} \mathbb{C}^n & \longleftrightarrow & \operatorname{Mspec}(\mathbb{C}[x_1, \ldots, x_n]) \\ u & \mapsto & \mathcal{I}(u) \\ \mathcal{V}(J) & \leftarrow\!\shortmid & J \end{array}$$

induce bijections between points in $\mathbb{C}^n$ *and elements of* $\operatorname{Mspec}(\mathbb{C}[x_1, \ldots, x_n])$*.*

So instead of talking of points in $\mathbb{C}^n$ we can also talk about maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$. This correspondence *points versus maximal ideals* extends to the case of affine varieties.

Definition 4.1.4 Let $X \subseteq \mathbb{C}^n$ be an affine variety with coordinate ring $\mathbb{C}[X]$. For a subset $Y \subset X$ let $\mathcal{I}_X(Y)$ be the ideal

$$\{f \in \mathbb{C}[X] \mid f(y) = 0 \,\forall y \in Y\},$$

and for an ideal $J \subseteq \mathbb{C}[X]$ let $\mathcal{V}_X(J) = \{u \in X \mid f(u) = 0 \,\forall f \in J\}$.

Proposition 4.1.4 *Let* $X \subseteq \mathbb{C}^n$ *be an affine variety. Denote by* $\operatorname{Mspec}(\mathbb{C}[X])$ *the set of all maximal ideals in the algebra* $\mathbb{C}[X]$*. The maps*

$$\begin{array}{rcl} X & \longleftrightarrow & \operatorname{Mspec}(\mathbb{C}[X]) \\ u & \mapsto & \mathcal{I}_X(u) \\ \mathcal{V}_X(J) & \leftarrow\!\shortmid & J \end{array}$$

induce bijections between points in $\mathbb{C}^n$ *and elements of* $\operatorname{Mspec}(\mathbb{C}[x_1, \ldots, x_n])$*.*

Proof. Given a point $u \in X \subseteq \mathbb{C}^n$, let $u = (u_1, \ldots, u_n)$ be its coordinates and let $\tilde{J}_u \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the maximal ideal (see Proposition 4.1.3)

$$\tilde{J}_u = \langle x_1 - u_1, x_2 - u_2, \ldots, x_n - u_n \rangle.$$

Let $I = \mathcal{I}(X) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the vanishing ideal for X. Since $u \in X$, we have $I \subseteq \tilde{J}_u$, let

$$J_u = \tilde{J}/I \subset \mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I$$

be the corresponding ideal (for surjective ring homomorphisms the image of an ideal is an ideal), which is a maximal ideal because

$$\mathbb{C}[X]/J_u \simeq (\mathbb{C}[x_1, \ldots, x_n]/I)/(\tilde{J}_u/I) \simeq \mathbb{C}[x_1, \ldots, x_n]/\tilde{J}_u \simeq \mathbb{C}.$$

By construction, $J_u := \{f \in \mathbb{C}[X] \mid f(u) = 0\} = \mathcal{I}_X(u)$.

Vice versa, let $J \in \operatorname{Mspec}(\mathbb{C}[X])$ and denote by $\tilde{J} \subset \mathbb{C}[x_1, \ldots, x_n]$ the ideal obtained as preimage with respect to the surjective natural map $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]/I$. Then $\tilde{J}$ is a maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]$ (same argument as above), and, by Proposition 4.1.3, $\mathcal{V}(\tilde{J}) = u$ is just a point. Since $I \subset \tilde{J}$ we have in addition $u \in X$, and hence

$$u = \{v \in \mathbb{C}^n \mid f(v) = 0 \, \forall f \in \tilde{J}\} = \{v \in X \mid f(v) = 0 \, \forall f \in J\} = \mathcal{V}_X(J).$$

Now as in the $\mathbb{C}^n$ case, the two constructions are inverse to each other. •

Remark 4.1.1 Bijective morphisms are in general not isomorphisms. As an example consider $X = \mathbb{C}$ with coordinate ring $\mathbb{C}[x]$ and

$$Y := \{(u, v) \in \mathbb{C}^2 \mid u^2 - v^3 = 0\}.$$

with coordinate ring $\mathbb{C}[u, v]/\langle u^2 - v^3 \rangle$ (Exercise: prove that the ideal is a radical ideal!) The map

$$X \to \mathbb{C}^2, \quad a \mapsto \begin{pmatrix} a^3 \\ a^2 \end{pmatrix}$$

is polynomial and hence a morphism, and since $(a^3)^2 - (a^2)^3 = 0$ for all $a \in \mathbb{C}$ we get in fact a morphism

$$\phi : X \to Y \quad a \mapsto \begin{pmatrix} a^3 \\ a^2 \end{pmatrix}.$$

The map is injective since we can recover a from $\phi(a)$: $a = \frac{a^3}{a^2}$. To show that it is surjective fix an element $(r, t) \in Y$ and set $a = \frac{r}{t}$ for $t \neq 0$ and set $a = 0$

for $t = 0$. Now $(r, t) \in Y$, so we know $r^2 - t^3 = 0$ and hence $r^2 = t^3$. It follows for $t \neq 0$:

$$a^3 = \frac{r^3}{t^3} = r \quad \text{and} \quad a^2 = \frac{r^2}{t^2} = t\frac{r^2}{t^3} = t,$$

and for $t = 0$ we have of course $r = 0$ and $a = 0$, so the equalities $r = a^3$ and $t = a^2$ hold for all $(r, t) \in Y$. Hence the map is surjective too.

So ϕ is a bijective morphism. But ϕ is not an isomorphism. To prove this, consider the subalgebra $\mathbb{C}[q^2, q^3] \subseteq \mathbb{C}[q]$. We have a natural map $\Psi : \mathbb{C}[u, v] \to \mathbb{C}[q^2, q^3]$ defined by sending u to q^3 and $v \mapsto q^2$. It follows that $\Psi(u^2 - v^3) = q^6 - q^6 = 0$, and hence we get an induced surjective morphism (Exercise: show that this is an isomorphism):

$$\bar{\Psi} : \mathbb{C}[Y] \to \mathbb{C}[q^2, q^3]$$

Now the algebra $\mathbb{C}[q^2, q^3]$ needs obviously at least two generators, and so does the algebra $\mathbb{C}[Y]$. It follows that the latter can not be isomorphic to $\mathbb{C}[x]$, and hence X and Y can not be isomorphic.

This bijection in Proposition 4.1.4 between points and maximal ideals leads to the following approach:

Definition 4.1.5 An *affine variety* X is the set $\operatorname{Mspec} R$ of maximal ideals in a finitely generated algebra R over $\mathbb{C}$, which contains no non-trivial nilpotent elements (i.e. if $f \in R$ and $f^k = 0$ for some k, then $f = 0$). We write $X = \operatorname{Mspec} R$. The coordinate ring of X is the algebra R.

Definition 4.1.6 Let X and Y be two affine varieties in the sense of Definition 4.1.5. A *morphism* between X and Y is a map

$$\phi : X = \operatorname{Mspec} R \to Y = \operatorname{Mspec} Q$$

induced by an algebra homomorphism $\phi^* : Q \to R$. The map ϕ sends a maximal ideal $J \in X = \operatorname{Mspec} R$ to the maximal ideal $J' = (\phi^*)^{-1}(J) \in Y = \operatorname{Mspec} Q$.

Remark 4.1.2 Let R and Q be finitely generated algebras over $\mathbb{C}$ and let $\phi^* : Q \to R$ be an algebra homomorphism (sending 1 to 1). If $I \subset R$ is an ideal, then $(\phi^*)^{-1}(I) = \{f \in Q \mid \phi^*(f) \in I\}$ is an ideal: as the preimage of a subgroup, $(\phi^*)^{-1}(I)$ is a subgroup of Q. Further, if $q \in (\phi^*)^{-1}(I)$ and $q' \in Q$, then $\phi^*(qq') = \phi^*(q)\phi^*(q') \in I$ because I is an ideal and $\phi^*(q) \in I$ by assumption. So $qq' \in (\phi^*)^{-1}(I)$, which implies $(\phi^*)^{-1}(I)$ is an ideal. Note that $1 \in (\phi^*)^{-1}(I)$ implies $1 \in I$ and vice versa, so I is a proper ideal if and only if $(\phi^*)^{-1}(I)$ is a proper ideal.

Suppose $I \subset R$ is a maximal ideal, which is equivalent to say $R/I \simeq \mathbb{C}$. The map $\phi^* : Q \to R$ induces a map $\bar{\phi}^* : Q \to R/I$. Since $\phi(1) = 1$ and $1 \notin I$, the map $\bar{\phi}^*$ is surjective, with kernel $(\phi^*)^{-1}(I)$, so $Q/(\phi^*)^{-1}(I) \simeq \mathbb{C}$ and hence $(\phi^*)^{-1}(I)$ is a maximal ideal.

To reconcile the intuition provided by the Definitions 4.1.1/4.1.2 with the Definitions 4.1.5/4.1.6 let us try to explain how to translate one into the other.

(I) Definition 4.1.1$\mapsto$Definition 4.1.5. Given an affine variety $X \subseteq \mathbb{C}^n$ according to Definition 4.1.1, its coordinate ring $R = \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}(X)$ is finitely generated (indeed, it is generated by the images of $x_1, \ldots, x_n$ in R). Now $\mathcal{I}(X)$ is a radical ideal (Hilbert Nullstellensatz) and hence R has no non-trivial nilpotent elements: indeed, let $\bar{f} \in R$ and let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a representative. Then $(\bar{f})^k = 0$ in R means $f^k \in \mathcal{I}(X)$, but this implies already $f \in \mathcal{I}(X)$ and hence $\bar{f} = 0$ in R. So an affine variety in the sense of Definition 4.1.1 provides an algebra R satisfying the conditions in Definition 4.1.5, and thanks to Proposition 4.1.4 we can identify the points in X with the points in $\operatorname{Mspec} \mathbb{C}[X]$.

(II) Definition 4.1.5 $\mapsto$ Definition 4.1.1. Given a finitely generated algebra R as in Definition 4.1.5, fix a set of generators, say $f_1, \ldots, f_r$. Let $\psi : \mathbb{C}[x_1, \ldots, x_r] \to R$ be the map that sends x_i to the generators f_i. This is an algebra homomorphism, it is surjective, the kernel is an ideal, and $R \simeq \mathbb{C}[x_1, \ldots, x_r]/\operatorname{Ker} \psi$. Since R contains no non-trivial nilpotent elements, $\operatorname{Ker} \psi$ is a radical ideal. Set $X_R = \mathcal{V}(\operatorname{Ker} \psi) \subseteq \mathbb{C}^r$, then X_R is an affine variety in the sense of Definition 4.1.1, with coordinate ring isomorphic to R. And, thanks to Proposition 4.1.4, we can identify the points in X_R with the points in $\operatorname{Mspec} R$.

As a next step we have to compare the notion of a morphism in Definition 4.1.2 with the concept in Definition 4.1.6.

(III) Definition 4.1.2$\mapsto$Definition 4.1.6. Suppose we are given affine varieties $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$, and a map $\phi : X \to Y$ which can be extended to a polynomial map $\tilde{\phi} : \mathbb{C}^n \to \mathbb{C}^m$ such that $\tilde{\phi}|_X = \phi$. By Proposition 4.1.1

we know that we get an induced algebra homomorphism

$$\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X], \quad f \mapsto f \circ \phi.$$

Let $u \in X$ and set $v = \phi(u)$. Denote by $J_u \subset \mathbb{C}[X]$ the corresponding maximal ideal, we want to understand the induced map

$$\tilde{\phi} : \mathrm{Mspec}(\mathbb{C}[X]) \to \mathrm{Mspec}(\mathbb{C}[Y])$$

sending a maximal ideal $J_u \subset \mathbb{C}[X]$ to the maximal ideal $J' := (\phi^*)^{-1}(J_u) \subset \mathbb{C}[Y]$. Now

$$f \in J' \Leftrightarrow \phi^*(f) \in J_u \Leftrightarrow (f \circ \phi)(u) = 0 \Leftrightarrow f(\phi(u)) = 0 \Leftrightarrow f \in J_{\phi(u)},$$

which implies that $\phi : X \to Y$ and $\tilde{\phi} : \mathrm{Mspec}(\mathbb{C}[X]) \to \mathrm{Mspec}(\mathbb{C}[Y])$ are, in view of **(I)** and **(II)**, the same maps, up to a slight difference in the notation:

$$\phi(u) = y \Leftrightarrow \tilde{\phi}(J_u) = J_y.$$

(IV) Definition 4.1.6 $\mapsto$ Definition 4.1.2. Vice versa, suppose we are given two finitely generated algebras R and Q, both having no non-trivial nilpotent elements, and we are given an algebra homomorphism $\Psi : Q \to R$. Let $X \subseteq \mathbb{C}^r$ and $Y \subseteq \mathbb{C}^s$ be corresponding affine varieties as in **(II)**, then $\mathbb{C}[X] \simeq R$ and $\mathbb{C}[Y] \simeq Q$. Now by Proposition 4.1.2, there exists a morphism of affine varieties $\phi : X \to Y$ such that $\phi^* = \Psi$.

We want to compare ϕ with the induced map

$$\tilde{\Psi} : \mathrm{Mspec}(\mathbb{C}[X]) \to \mathrm{Mspec}(\mathbb{C}[Y])$$

sending a maximal ideal $J_u \subset \mathbb{C}[X]$ to the maximal ideal $J' := \Psi^{-1}(J_u) \subset \mathbb{C}[Y]$. Now

$$f \in J' \Leftrightarrow \Psi(f) \in J_u \Leftrightarrow \phi^*(f) \in J_u \Leftrightarrow (f \circ \phi)(u) = 0 \Leftrightarrow f(\phi(u)) = 0 \Leftrightarrow f \in J_{\phi(u)},$$

and hence

$$\phi(u) = y \Leftrightarrow \tilde{\Psi}(J_u) = J_y.$$

(V) functions versus functions. In Definition 4.1.1 it is obvious what the coordinate ring means: this are functions on X coming as restrictions from polynomial functions on the ambient space $\mathbb{C}^n$.

In the Definition 4.1.5 the meaning of the coordinate ring as functions on Mspec R is a priori not so obvious, but we will see that it is indeed the same as above. So let R and X_R be as in (**II**). Let $J_u \in \operatorname{Mspec} R$ be a maximal ideal corresponding a point $u \in X_R$ (by Proposition 4.1.4). Given $f \in R$, we have to define what we mean by $f(J_u)$. Now J_u is a maximal ideal, so R/J_u is isomorphic to $\mathbb{C}$. This isomorphism is an algebra homomorphism, in particular it sends the class "$\bar{1}$" of $1 \in R$ to the "1" in $\mathbb{C}$. So the isomorphism is completely fixed, and the class $\bar{f}$ of f in $R/J_u = \mathbb{C}$ is a well defined complex number.

Now we have a well defined algebra homomorphism $ev_u : R = \mathbb{C}[X_R] \to \mathbb{C}$, it is the evaluation in u: given a function $g \in R = \mathbb{C}[X_R]$, $g(u)$ is a complex number. The kernel of this map in J_u, which gives us hence an induced map $\bar{ev}_u : R/J_u \to \mathbb{C}$. This map is an algebra homomorphism, it sends 1 to 1, and hence has to be the same as above. It follows:

$$f(u) = \bar{f} \text{ in } \mathbb{C} = R/J_u.$$

In the following we will often freely jump between these various points of view of an affine variety, its functions and the morphisms between them.

4.2 $L^{alg}GL_n(\mathbb{C})$ as algebraic ind-group

We want to endow the algebraic loop group $L^{alg}GL_n(\mathbb{C})$ with the structure of an affine variety, or, more precisely, a generalized version of an affine variety.

The following is only a quasi-example because we haven't introduced all necessary tools yet. But it should to show how the algebraic geometric language connects with the theory of loop groups.

We have seen that $L^{alg}GL_n(\mathbb{C}) = GL_n(\mathbb{C}[t, t^{-1}])$, the group of invertible matrices (over $\mathbb{C}[t, t^{-1}]$) with entries in the algebra of Laurent polynomials.

For $d \geq 0$ let $\mathbb{C}[t, t^{-1}]_d$ be the subset of Laurent polynomials of the form $p(t) = a_{-d}t^{-d} + \ldots + a_d t^d$, where $a_{-d}, \ldots, a_d \in \mathbb{C}$. This is a complex vector space of dimension $2d + 1$.

Correspondingly let $M_n(\mathbb{C}[t, t^{-1}]_d)$ be the vector space of $n \times n$-matrices, with entries in $\mathbb{C}[t, t^{-1}]_d$. This is a vector space of dimension $n^2(2d + 1)$. If $A \in M_n(\mathbb{C}[t, t^{-1}]_d)$, then the determinant of such a matrix is a Laurent polynomial of the form

$$\det A = f_{-dn}(A)t^{-dn} + f_{-dn+1}(A)t^{-dn+1} + \ldots + f_0(A) + \ldots + f_{dN}t^{dn},$$

where $f_{-dn}, \dots, f_{dn} \in \mathbb{C}[M_n(\mathbb{C}[t,t^{-1}]_d)]$ are polynomial functions on this $n^2(2d+1)$-dimensional vector space. Note that if

$$A \in M_n(\mathbb{C}[t,t^{-1}]_d) \cap GL_n(\mathbb{C}[t,t^{-1}]),$$

then $\det A = at^m$ for some $-dn \leq m \leq dm$ and $a \in \mathbb{C}^*$. For $-dn \leq \ell \leq dn$ let

$$GL_n(\mathbb{C}[t,t^{-1}])_{d,\ell} = \{A \in M_n(\mathbb{C}[t,t^{-1}]_d) \mid f_\ell(A) \neq 0, f_k(A) = 0 \forall k \neq \ell\}.$$

So up to the fact that we assume one function to be nonzero, this looks like a perfect affine variety. We will look into this problem with a non-vanishing function soon (see the section about $GL_n(\mathbb{C})$), so let us assume for the moment that the above is an affine variety.

The union of a finite number of affine varieties is an affine variety (see section about Zariski topology), so

$$GL_n(\mathbb{C}[t,t^{-1}])_d = \bigcup_\ell GL_n(\mathbb{C}[t,t^{-1}])_{d,\ell}$$

is an affine variety. We get hence an increasing sequence of affine varieties:

$$GL_n(\mathbb{C}[t,t^{-1}])_1 \subseteq GL_n(\mathbb{C}[t,t^{-1}])_2 \subset \cdots \subseteq GL_n(\mathbb{C}[t,t^{-1}])_d \subseteq \cdots \tag{4.1}$$

and

$$GL_n(\mathbb{C}[t,t^{-1}]) = \bigcup_{d \in \mathbb{N}} GL_n(\mathbb{C}[t,t^{-1}])_d$$

So our group $L^{alg}GL_n(\mathbb{C})$ of algebraic loops can be described as a union of affine varieties. In addtion, the filtration in (4.1) has the nice property that the inclusions

$$GL_n(\mathbb{C}[t,t^{-1}])_d \subseteq GL_n(\mathbb{C}[t,t^{-1}])_{d+1}$$

are closed embeddings: it is of course injective, it is a morphism in the sense of affine varieties, and the image is closed (Zariski topology): the image are exactly those elements in $GL_n(\mathbb{C}[t,t^{-1}])_{d+1}$ such that the entries are Laurent polynomials subject to the additional condition that the coefficient of t^{d+1} and t^{-d-1} are equal to zero. This means we cut $GL_n(\mathbb{C}[t,t^{-1}])_{d+1}$ with a finite number of hyperplanes.

Having a filtration with these properties, one says that $GL_n(\mathbb{C}[t,t^{-1}])$ is an *affine ind-variety*. The inclusion $GL_n(\mathbb{C}[t,t^{-1}])_d \subseteq GL_n(\mathbb{C}[t,t^{-1}])_{d+1}$ induces (since the image is closed) a surjective morphism

$$\mathbb{C}[GL_n(\mathbb{C}[t,t^{-1}])_{d+1}] \to \mathbb{C}[GL_n(\mathbb{C}[t,t^{-1}])_d].$$

More generally, for all $k > d$ we get algebra homomorphsims

$$\pi_{k,d} : \mathbb{C}[GL_n(\mathbb{C}[t,t^{-1}])_k] \to \mathbb{C}[GL_n(\mathbb{C}[t,t^{-1}])_d]$$

such that $\pi_{k,d} \circ \pi_{d,b} = \pi_{k,b}$ for all $k > d > b$. The coordinate ring $\mathbb{C}[GL_n(\mathbb{C}[t,t^{-1}])]$ of this affine ind-variety is defined as the inverse limit

$$\lim_{\substack{\leftarrow \\ d}} \mathbb{C}[GL_n(\mathbb{C}[t,t^{-1}])_d].$$

of these algebras. This means that an element in the coordinate ring is a sequence $(f_i)_{i \in \mathbb{N}}$ such that $f_d \in \mathbb{C}[GL_n(\mathbb{C}[t,t^{-1}])_d]$ and $\rho_{k,d}(f_k) = f_d$ for all $k > d$.

Remark 4.2.1 An example for an inductive limit is the case of $\mathbb{C}^\infty$, the infinite dimensional vector space with basis $\mathbb{B} = \{e_1, e_2, e_3, \ldots\}$ obtained as limit by the inclusions

$$\mathbb{C} = \mathbb{C}e_1 \subset \mathbb{C}^2 = \langle e_1, e_2 \rangle \subset \mathbb{C}^3 = \langle e_1, e_2, e_3 \rangle \subset \ldots$$

The inclusions are morphisms of affine varieties, the images are closed subvarieties in the Zariski topology, the union is $\mathbb{C}^\infty$, so we get induced algebra homomorphisms

$$\mathbb{C}[x_1] \leftarrow \mathbb{C}[x_1, x_2] \leftarrow \mathbb{C}[x_1, x_2, x_3] \leftarrow \mathbb{C}[x_1, x_2, x_3, x_4] \leftarrow \cdots$$

Let $\mathbb{C}[[x_1, x_2, \ldots, x_d, \ldots]]$ be the algebra of formal power series in infinitely many variables. The inverse limit

$$\mathbb{C}[\widehat{x_1, x_2, \ldots}] := \lim_{\substack{\leftarrow \\ d}} \mathbb{C}[x_1, x_2, \ldots, x_d]$$

can be identified with a subalgebra of $\mathbb{C}[[x_1, x_2, \ldots, x_d, \ldots]]$, it can viewed as a kind of completion of the polynomial ring $\mathbb{C}[x_1, x_2, \ldots]$ in infinitely many variables. To describe the subalgebra more precisely, for $f \in \mathbb{C}[[x_1, x_2, \ldots]]$ and a positive integer k, let f_k be the power series obtained by omitting all monomials in f involving a variable x_j for $j > k$, so $f_k \in \mathbb{C}[[x_1, \ldots, x_k]]$. Let $\rho_k : \mathbb{C}[[x_1, x_2, \ldots, x_d, \ldots]] \to \mathbb{C}[[x_1, x_2, \ldots, x_k]]$ be the map defined by $\rho(f) = f_k$, the map $\rho_{m,k} : \mathbb{C}[[x_1, x_2, \ldots, x_m]] \to \mathbb{C}[[x_1, x_2, \ldots, x_k]]$ for $m \geq k$ is defined in the obvious way. Set

$$R(\mathbb{C}^\infty) := \{f \in \mathbb{C}[[x_1, x_2, \ldots, x_d, \ldots]] \mid \rho_k(f) \in \mathbb{C}[x_1, x_2, \ldots, x_d] \, \forall k \geq 1\}.$$

The maps ρ_k are algebra homomorphisms, so one can check directly that $R(\mathbb{C}^\infty)$ is indeed a subalgebra.

Starting with an element $f \in R(\mathbb{C}^\infty)$, the collection $(\rho_k(f))_{k\in\mathbb{Z}_{>0}}$ satisfies the conditions for an element in the inductive limit. So we get an algebra homomorphism

$$R(\mathbb{C}^\infty) \to \mathbb{C}[\widehat{x_1, x_2, \ldots}], \quad f \mapsto (\rho_k(f))_{k\in\mathbb{Z}_{>0}}.$$

Vice versa, given $(f_k)_{k\in\mathbb{Z}_{>0}}$, the formal power series $f := f_1 + \sum_{i\geq 2}(f_i - f_{i-1})$ is an element in $R(\mathbb{C}^\infty)$ because $\rho_k(f) = f_k$ for all $k \geq 1$, so this is in fact an isomorphism.

Let now $v \in \mathbb{C}^\infty$ and $f \in R(\mathbb{C}^\infty) = \mathbb{C}[\widehat{x_1, x_2, \ldots}]$. Then the evaluation $f(v)$ is well defined: v is a finite linear combination of the basis vectors, so all but a finite number of monomials occurring in f will vanish. Indeed, since there exists a k such that $v \in \mathbb{C}^k$, we have $f(v) = \rho_k(f)(v)$.

We can view in the same way the $\mathbb{C}$-vector space $M_n(\mathbb{C}[t,t^{-1}])$ as an inductive limit of the finte dimensional vectors spaces $M_n(\mathbb{C}[t,t^{-1}]_d)$. Keeping this in mind, one might think of a function f in $\mathbb{C}[GL_n(\mathbb{C}[t,t^{-1}])]$ as the restriction of an element $\tilde{f} \in R(M_n(\mathbb{C}[t,t^{-1}])$ to $GL_n(\mathbb{C}[t,t^{-1}])$.

A morphism between two such affine ind-varieties $X = \bigcup_{i\in\mathbb{N}} X_i$, $Y = \bigcup_{j\in\mathbb{N}} Y_j$ is a map $\phi : X \to Y$ such that for all $i \in \mathbb{N}$ there exists a $j(i) \in \mathbb{N}$ such that $\phi(X_i) \subset Y_{j(i)}$, and the induced map $\phi : X_i \to Y_{j(i)}$ is a morphism of affine varieties. For example, let us look at the case discussed above: the inversion $GL_n(\mathbb{C}[t,t^{-1}]) \to GL_n(\mathbb{C}[t,t^{-1}])$, $g \mapsto g^{-1}$, can be broken down into maps between affine varieties. Using the standard argument that $g^{-1} = \frac{1}{\det g} g^\dagger$, a simple guess about the possible powers of t occurring in the formula gives that the inversion induces maps of affine varieties

$$GL_n(\mathbb{C}[t,t^{-1}])_d \to GL_n(\mathbb{C}[t,t^{-1}])_{(2n-1)d}$$

Now as usual one endows $GL_n(\mathbb{C}[t,t^{-1}]) \times GL_n(\mathbb{C}[t,t^{-1}])$ with the structure of an affine ind-variety, the filtration given by the affine varieties

$$GL_n(\mathbb{C}[t,t^{-1}])_d \times GL_n(\mathbb{C}[t,t^{-1}])_d,$$

and then one sees that the product map

$$GL_n(\mathbb{C}[t,t^{-1}]) \times GL_n(\mathbb{C}[t,t^{-1}]) \to GL_n(\mathbb{C}[t,t^{-1}])$$

breaks down into a sequence of morphisms of affine varieties

$$GL_n(\mathbb{C}[t,t^{-1}])_d \times GL_n(\mathbb{C}[t,t^{-1}])_d \to GL_n(\mathbb{C}[t,t^{-1}])_{2d}$$

It follows that the product and the inversion are morphisms of affine ind-varieties. In such a case the group, which in our case is $GL_n(\mathbb{C}[t,t^{-1}])$, is called an algebraic ind-group.

4.3 Zariski topology

We have to fill in some gaps. As a first step we want to see an affine variety $X \subset \mathbb{C}^n$ not with the induced $\mathbb{C}$-topology but with a topology that makes sense on $\operatorname{Mspec} R$ too. This topology is called the *Zariski-topology*. Let us first consider the case of an affine variety $X \subseteq \mathbb{C}^n$ embedded in some $\mathbb{C}^n$ and $R = \mathbb{C}[X]$. We keep the notation as in Definition 4.1.4.

Definition 4.3.1 A subset $Y \subseteq X$ is called *closed* if there exists an ideal $J \subset \mathbb{C}[X]$ such that $Y = \mathcal{V}_X(J)$. The subset is called *open* if it is the complement of a closed subset.

Remark 4.3.1 We keep the notation as above. Let $\tilde{J} \subset \mathbb{C}[x_1,\ldots,x_n]$ be the preimage of the ideal J with respect to the natural quotient $\mathbb{C}[x_1,\ldots,x_n] \to \mathbb{C}[X]$ induced by the embedding $X \subseteq \mathbb{C}^n$ as affine variety. Since $\mathcal{I}(X) \subseteq \tilde{J}$ we see

$$\mathcal{V}(\tilde{J}) = \{v \in \mathbb{C}^n \mid f(v) = 0 \,\forall f \in \tilde{J}\} = \{v \in X \mid f(v) = 0 \,\forall f \in J\} = \mathcal{V}_X(J).$$

In particular, a closed subset of X is always an affine variety. This is not true in general for open subsets.

Exercise 4.3.1 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}_i, i \in J$ be ideals in $\mathbb{C}[X]$. Show that:

(i) $\mathfrak{a} \subset \mathfrak{b} \Longrightarrow \mathcal{V}_X(\mathfrak{a}) \supset \mathcal{V}_X(\mathfrak{b})$

(ii) $\bigcap_{i\in J} \mathcal{V}_X(\mathfrak{a}_i) = \mathcal{V}_X(\sum_{i\in J} \mathfrak{a}_i)$.

(iii) $\mathcal{V}_X(\mathfrak{a}) \cup \mathcal{V}_X(\mathfrak{b}) = \mathcal{V}_X(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{V}_X(\mathfrak{a} \cdot \mathfrak{b})$.

The properties (i)–(iii) imply that the subsets of the form $\mathcal{V}_X(I) \subset X$ fulfill the axioms of a system of closed subsets of a topology.

Definition 4.3.2 The topology on X having as closed subsets the sets of the form $\mathcal{V}_X(I)$, $I \subset \mathbb{C}[X]$ ideal, is called the *Zariski-topology* on X.

Let us now look at the more abstract version, which is essentially the same (after a second thought). Let R be a finitely generated algebra with no non-trivial nilpotent elements. Given an ideal $I \subset R$, set

$$\mathcal{V}_R(I) = \{\mathfrak{m} \in \mathrm{Mspec\,R} \mid I \subset \mathfrak{m}\}$$

As in Exercise 4.3.1 one shows that these sets fulfill the axioms of a system of closed subsets of a topology.

Definition 4.3.3 The topology on $\mathrm{Mspec\,R}$ having as closed subsets the sets of the form $\mathcal{V}_R(I)$, $I \subset R$ ideal, is called the *Zariski-topology* on Y.

Now by **(I)**–**(IV)** we know that given an algebra R as above, there exists an embedded affine variety $X \subset \mathbb{C}^n$ such that $R \simeq \mathbb{C}[X]$ and bijections

$$X \leftrightarrow \mathrm{Mspec\,R}, \quad \left\{ \begin{array}{rcl} u & \mapsto & J_u = \mathcal{I}_X(u) \\ \mathcal{V}_X(J) & \leftarrow\!\shortmid & J \end{array} \right.$$

Lemma 4.3.1 *The bijections above respect the Zariski-topology on X and* $\mathrm{Mspec\,R}$*, i.e. the maps in both directions send closed subsets to closed subsets. In other words: these maps are homeomorphisms.*

Proof. Let $I \subseteq R \simeq \mathbb{C}[X]$ be an ideal and let $u \in X$ be a point with associated maximal ideal $J_u \subset R$. Then

$$u \in \mathcal{V}_X(I) \Leftrightarrow f(u) = 0\,\forall f \in I \Leftrightarrow I \subset J_u \Leftrightarrow J_u \in \mathcal{V}_R(I)$$

•

Exercise 4.3.2 *i)* Show that a proper subset of $\mathbb{C}$ is closed if and only if it is finite.

ii) Show that a morphism between affine varieties is continuous in the Zariski-topology

The closure of a subset $U \subset X$ of an affine variety is the intersection of all closed subsets containing U. Another way of defining the closure is: $\overline{U} = \mathcal{V}_X(\mathcal{I}_X(U))$.

Definition 4.3.4 An affine variety Z is called *irreducible* if $Z = X \cup Y$ with $X, Y \subset Z$ closed implies $X = Z$ or $Y = Z$. By taking the complements, an equivalent version is the following: Z is called *irreducible* if the intersection of any two open subsets is not empty. Another equivalent formulation: X is *irreducible* if and only if $X = \overline{U}$ for every non-empty open subset of X.

Exercise 4.3.3 Prove the equivalence of the three definitions of irreducibility above.

It is expected that all important properties of an affine variety can be discovered from $\mathbb{C}[X]$, so what about irreducibility:

Proposition 4.3.1 *An affine variety X is irreducible if an only if $\mathbb{C}[X]$ has no zero divisors.*

Proof. If $f, g \in \mathbb{C}[X] - \{0\}$ are such that $fg = 0$, then $X = \mathcal{V}_X(f) \cup \mathcal{V}_X(g)$ is a decomposition of X into closed subsets, non of which is equal to X, so X is not irreducible.

Next suppose $X = Z_1 \cup Z_2$, where $Z_i = \mathcal{V}(\mathfrak{a}_i) \subset X$ are proper closed subsets and $\mathfrak{a}_1, \mathfrak{a}_2$ are ideals. Let $f_1 \in \mathfrak{a}_1, f_2 \in \mathfrak{a}_2$ be non-zero elements, then $f_1 f_2 = 0$ but $f_1, f_2 \neq 0$, and hence $\mathbb{C}[X]$ has zero divisors. •

Exercise 4.3.4 Let X be an irreducible affine variety and let $f, g \in [X]$. Let $U \subset Z$ be open and not empty. Show: $f|_U = g|_U \Longrightarrow f = g$

The property of being irreducible is stable under morphisms:

Proposition 4.3.2 *Let $\phi : X \to Y$ be a morphism between affine varieties. If X is irreducible, then so is $\overline{\phi(X)} \subseteq Y$.*

Proof. Without loss of generality we may assume $\overline{\phi(X)} = Y$. Let $U_1, U_2 \subset Y$ be open and not empty subsets. Since ϕ is continuous, $\phi^{-1}(U_1)$ and $\phi^{-1}(U_2)$ are open too. The preimages are not empty because $U_i \cap \phi(X) = \emptyset \Leftrightarrow \phi(X) \subseteq$ *compl*$(U_i) \Leftrightarrow U_i \cap \overline{\phi(X)} = \emptyset$. It follows that $\phi^{-1}(U_1) \cap \phi^{-1}(U_2) \neq \emptyset$ in X, and hence $U_1 \cap U_2 \neq \emptyset$, which implies $\overline{\phi(X)}$ is irreducible. •

Exercise 4.3.5 Show that $O_n(\mathbb{C})$ has at least two irreducible components. Hint: use Proposition 4.3.2 and the determinant.

4.4 $GL_n(\mathbb{C})$ as an affine algebraic group

To finish the proof in section 4.2, recall that we defined for $d \geq 0$ and $-dn \leq \ell \leq dn$ the subset $GL_n(\mathbb{C}[t, t^{-1}])_{d,\ell}$ as

$$\{A \in M_n(\mathbb{C}[t, t^{-1}]_d) \mid f_\ell(A) \neq 0, f_k(A) = 0 \forall\, k \neq \ell\} \tag{4.2}$$

and claimed that this is an affine variety. The conditions $f_k(A) = 0$ for $k \neq \ell$ look appropriate and have exactly the form demanded in Definition 4.1.1, so they describe an affine variety, let us call it Z. But the additional condition $f_\ell(A) \neq 0$ describes an open subset in this affine variety Z, and it is not at all clear why this should be an affine variety. We start with a slight generalization of Definition 4.1.5.

Definition 4.4.1 A set Z endowed with an algebra of $\mathbb{C}$-valued functions $\mathcal{O}(Z)$ is called an *affine variety* if there exists an affine variety $X \subseteq \mathbb{C}^n$ in the sense of Definition 4.1.1 and a bijection $\phi : Z \to X$ such that the comorphism $\phi^* : \mathbb{C}[X] \to \mathcal{O}(Z)$ is an isomorphism (i.e., for all $h \in \mathbb{C}[X]$ the function $\phi^*(h) := h \circ \phi$ is an element in $\mathcal{O}(Z)$, and the map ϕ^* is an isomorphism of $\mathbb{C}$-algebras).

Remark 4.4.1 Clearly, an affine variety in the sense of Definition 4.1.1 is affine in the sense of Definition 4.4.1. For an affine variety in the sense of Definition 4.1.5, the discussion in section 4.1 (**I**), (**II**) constructs the desired bijection and identification of regular functions.

In the following let X be an irreducible affine variety, which by Proposition 4.3.1 is equivalent to say that the coordinate ring $\mathbb{C}[X]$ has no zero divisors.

Definition 4.4.2 A *quasiaffine* variety Y is an open subset $Y \subset X$ of an irreducible affine variety, endowed with the induced Zariski topolgy from X.

A special class of quasiaffine varieties are the following.

Definition 4.4.3 A *special open subset* of an irreducible affine variety X is a quasiaffine variety obtained as the complement of the vanishing set of an element $f \in \mathbb{C}[X]$, i.e.,

$$X_f = \{x \in X \mid f(x) \neq 0\}.$$

Lemma 4.4.1 *The special open sets form a basis for the Zariski topology on X, i.e. if $U \subset X$ is open, then there exists an $f \in \mathbb{C}[X]$ such that $X_f \subseteq U$.*

Proof. Let $U \subset X$ be open, let V be its complement and set $\mathfrak{a} := \mathcal{I}_X(V)$. Let $u \in U$ and let J_u be the associated maximal ideal. Since $u \notin V$ we have $\mathfrak{a} \not\subset J_u$. So there exists an $f \in \mathfrak{a}$ such that $f \notin J_u$, and hence by construction $X_f \neq \emptyset$. Since $V \subseteq \mathcal{V}((f))$ implies $V \cap X_f = \emptyset$, one has $X_f \subseteq U$. •

Example 4.4.1 There are three examples one should have in mind throughout the following: the special open subset $GL_n(\mathbb{C}[t, t^{-1}])_{d,\ell}$ described above in (4.2), the group $GL_n(\mathbb{C})$, which is the special open subset of $M_n(\mathbb{C})$ where the determinant does not vanish (the case $d = 0$), and the quasiaffine variety $\mathbb{C}^2 - \{0\}$.

The functions on X are the elements of $\mathbb{C}[X]$, but what should be the functions on a quasiaffine variety $Y \subset X$? We need to introduce the notion of a regular function on a quasiaffine variety. The approach is the same as always, we first say what it means for a function f to be regular at a point, and then f is regular on Y if it is regular everywhere. Let us first recall the notion of a quotient field.

The algebra $\mathbb{C}[X]$ has no zero divisors, let $\mathbb{C}(X)$ be its quotient field, it is the field of rational functions. We have the following rules: every element in $\mathbb{C}(X)$ is of the form $\frac{f}{g}$ with $f, g \in \mathbb{C}[X]$. This description is not unique, we have

$$\frac{f}{g} = \frac{p}{q} \quad \text{in } \mathbb{C}(X) \Leftrightarrow fq = gp \quad \text{in } \mathbb{C}[X]. \tag{4.3}$$

One has a natural injective algebra homomorphism $\iota : \mathbb{C}[X] \to \mathbb{C}(X)$, $p \mapsto \frac{p}{1}$. Without loss of generality, we identify $\mathbb{C}[X]$ with its image in $\mathbb{C}(X)$.

Definition 4.4.4 A function f on a quasi affine variety $Y \subseteq X$ is called *regular* at a point $p \in Y$ if there exists an open neighborhood $U \subseteq Y$ of p (in the sense of the Zariski topology) and $g, h \in \mathbb{C}[X]$, such that h vanishes nowhere on U and $f|_U = \frac{g}{h}|_U$. A function is called *regular on* Y if it is regular at all points of Y. The set of regular functions form in a natural way an algebra, called the *algebra of regular functions*, which is denoted by $\mathcal{O}(Y)$.

If f is regular in $p \in Y$ and $f(p) \neq 0$, then there exists a neighborhood U such that $f|_U = \frac{g}{h}|_U$ and h vanishes nowhere on U. Since $g(p) \neq 0$, the subset

$U' = \{u \in U \mid g(u) \neq 0\}$ is open and nonempty, and $\frac{1}{f}|_{U'} = \frac{h}{g}|_{U'}$ is regular on U'. So regular and non-vanishing functions can be locally inverted.

Consider now the set of all classes of pairs (U, f) where $U \subseteq Y$ is open, f is a regular function on U, and (U, f) and (U', f') are in the same class if $f|_{U\cap U'} = f'|_{U\cap U'}$. This set can be endowed in the obvious way with the structure of field, called the *function field of* Y.

Definition 4.4.5 The function field of Y is denoted by $\mathcal{K}(Y)$.

Lemma 4.4.2 *$\mathcal{K}(Y) = \mathcal{K}(X) = \mathbb{C}(X)$.*

Proof. The first equality follows from the fact that if (U, f) is a representative of a class in $\mathcal{K}(Y)$, then U is also open in X and hence (U, f) is a representative of a class in $\mathcal{K}(X)$. Vice versa, if (U, f) is a representative of a class in $\mathcal{K}(X)$, then $(U \cap Y, f)$ is a representative of a class in $\mathcal{K}(Y)$. Since open sets in X are always dense (i.e. $\overline{U} = X$), it follows that the map which sends the of (U, f) in $\mathcal{K}(Y)$ to its class in $\mathcal{K}(X)$ induces a well-defined field bijection between $\mathcal{K}(Y)$ and $\mathcal{K}(X)$.

Given a class in $\mathbb{C}(X)$, by definition one can find elements $p, q \in \mathbb{C}[X]$ such that $\frac{p}{q}$ is a representative for the given class. Let X_q be the open subset in X where q does not vanish, then $(X_q, \frac{p}{q})$ is a regular function on the open set $X_q \subset X$. Note that $\frac{p}{q} = \frac{p'}{q'}$ in $\mathbb{C}(X)$ implies that $pq'|_{X_q\cap X_{q'}} = p'q|_{X_q\cap X_{q'}}$, which in turn implies $\frac{p}{q}|_{X_q\cap X_{q'}} = \frac{p'}{q'}|_{X_q\cap X_{q'}}$. It follows that members of the same class in $\mathbb{C}(X)$ are sent to members of the same class in $\mathcal{K}(X)$.

So $\frac{p}{q} \mapsto (X_q, \frac{p}{q})$ induces a well defined map $\mathbb{C}(X) \to \mathcal{K}(X)$, which is injective, it is an algebra homomorphism (exercise!), it remains to show it is surjective. But by the defintion of a regular function, we can find in its class a representative which is of the form $(U, \frac{p}{q})$, where p and q are elements of the coordinate ring $\mathbb{C}[X]$, which is in the same class as $(U_q, \frac{p}{q})$. •

To better understand the connection between $\mathcal{O}(X)$ and $\mathbb{C}[X]$, we attach now to every point in X two algebras. Given $u \in X$, let

$$\mathbb{C}[X]_{(u)} = \left\{ \frac{p}{q} \in \mathbb{C}(X) \mid \begin{array}{c} \text{class has a representative} \\ \frac{\tilde{p}}{\tilde{q}} \text{ such that } {\scriptstyle \tilde{q} \notin J_u (\Leftrightarrow \tilde{q}(u) \neq 0)} \end{array} \right\}.$$

The other algebra is given by

$$\mathcal{O}(X)_{(u)} := \{ f \in \mathcal{K}(X) \mid \ f \text{ is regular in } u \}.$$

Because everything is local and we care only about open subsets containing u, the same arguments as above show:

Lemma 4.4.3 *If $u \in Y$, then $\mathcal{O}(Y)_{(u)} = \mathcal{O}(X)_{(u)} = \mathbb{C}[X]_{(u)}$.*

Now we are ready to prove:

Lemma 4.4.4 *$\mathcal{O}(X) = \mathbb{C}[X]$.*

Proof. By definition and by the lemma above we have:

$$\mathcal{O}(X) = \bigcap_{u \in X} \mathcal{O}(X)_{(u)} = \bigcap_{u \in X} \mathbb{C}[X]_{(u)}$$

It remains to show that the latter is equal to $\mathbb{C}[X]$. Now given an element $f \in \mathbb{C}(X)$, let $N(f) = \{q \in \mathbb{C}[X] \mid qf \in \mathbb{C}[X]\}$. The set $N(f)$ is in fact an ideal. It can be viewed as the ideal of denominators for f, i.e. $q \in N(f)$ is equivalent to say that for f one can find a representative in its class of the form $\frac{p}{q}$ for some $p \in \mathbb{C}[X]$. To say that $f \in \mathbb{C}[X]$ is equivalent to say that $N(f) = \mathbb{C}[X]$. Now for $u \in X$ we have

$$f \in \mathbb{C}[X]_{(u)} \Leftrightarrow \exists\, p, q \in \mathbb{C}[X], q(u) \neq 0 : \; f = \frac{p}{q} \Leftrightarrow N(f) \not\subseteq J_u.$$

Therefore, if $f \notin \mathbb{C}[X]$, then $N(f)$ is a proper ideal, which is contained in some maximal ideal $N(f) \subset J_u$ for some $u \in X$, and hence $f \notin \mathbb{C}[X]_{(u)}$. It follows that $\bigcap_{u \in X} \mathbb{C}[X]_{(u)} = \mathbb{C}[X]$. •

Let $g_1, \ldots, g_r$ be a generating system for $\mathbb{C}[X]$. To show that X_f is an affine variety, let $\mathbb{C}[X]_f$ be the subalgebra of $\mathbb{C}(X)$ generated by $\mathbb{C}[X]$ and $\frac{1}{f}$:

$$\mathbb{C}[X]_f = \left\{ \frac{g}{f^m} \mid m \in \mathbb{N}, g \in \mathbb{C}[X] \right\} = \mathbb{C}[g_1, \ldots, g_r, \frac{1}{f}] \subseteq \mathbb{C}(X).$$

The algebra is finitely generated, has no non-trivial nilpotent elements and no zero divisors, so $\operatorname{Mspec} \mathbb{C}[X]_f$ is an irreducible affine variety.

Proposition 4.4.1 *The set X_f together with the algebra of regular functions $\mathcal{O}(X_f)$ is an affine variety. Moreover, $\mathcal{O}(X_f) \simeq \mathbb{C}[X]_f$, so one can identify X_f with $\operatorname{Mspec} \mathbb{C}[X]_f$.*

Proof. We want to define a natural bijection between X_f and $\operatorname{Mspec} \mathbb{C}[X]_f$. The inclusion $\mathbb{C}[X] \hookrightarrow \mathbb{C}[X]_f$ induces a map between the sets of maximal ideals:

$$\operatorname{Mspec} \mathbb{C}[X]_f \ni \tilde{J} \mapsto J := \tilde{J} \cap \mathbb{C}[X] \in \operatorname{Mspec} \mathbb{C}[X] = X. \tag{4.4}$$

The map in (4.4) is well defined: since $\tilde{J} \not\supseteq \mathbb{C}[X]$, the intersection is a proper ideal, and the ideal is maximal because $\mathbb{C}[X]/J \hookrightarrow \mathbb{C}[X]_f/\tilde{J} \simeq \mathbb{C}$.

The map in (4.4) is injective because one gets $\tilde{J}$ back as the ideal in $\mathbb{C}[X]_f$ generated by J: every element in $\tilde{J}$ is of the form $h = \frac{g}{f^m}$ for some $g \in \mathbb{C}[X]$, and hence $g = f^m h \in J = \tilde{J} \cap \mathbb{C}[X]$, which in turn implies $h = \frac{g}{f^m} \in \langle J \rangle$.

The image of the map in (4.4) is X_f: let $\tilde{J} \in \operatorname{Mspec} \mathbb{C}[X]_f$ be a maximal ideal and let $u \in X$ be the point corresponding to the maximal ideal $J = \tilde{J} \cap \mathbb{C}[X]$. Since $\tilde{J}$ is a proper ideal one has necessarily $f \notin \tilde{J}$ and hence $f \notin J$, which implies $f(u) \neq 0$ and therefore $u \in X_f$. Vice versa, let $J = J_u \subset \mathbb{C}[X]$ for some $u \in X_f$ and let $\tilde{J} = \langle J \rangle$ be the ideal generated by J in $\mathbb{C}[X]_f$. The evaluation map $\mathbb{C}[X]_f \to \mathbb{C}$, $\frac{g}{f^m} \mapsto \frac{g(u)}{f^m(u)}$ is well defined, surjective, the kernel is $\tilde{J}$, so the latter is a maximal ideal with image J_u with respect to the map in (4.4). So (4.4) defines a natural bijection between X_f and $\operatorname{Mspec} \mathbb{C}[X]_f$.

Next we have to understand what happens with the functions. The bijection induces a natural map $\mathbb{C}[X]_f \to \mathcal{O}(X_f)$, which is well defined because $\frac{g}{f^m} \in \mathbb{C}[X]_f$ is a regular function on X_f. The map is injective because $\frac{g}{f^m} = \frac{p}{f^k}$ on X_f implies $gf^k = pf^m$ on X_f, and hence on all of X, and hence $\frac{g}{f^m} = \frac{p}{f^k}$ in $\mathbb{C}[X]_f$. It remains to see that the map is surjective.

For $u \in X_f$ we write $J_u \subset \mathbb{C}[X]$ and $\tilde{J}_u \subset \mathbb{C}[X]_f$ for the corresponding maximal ideals. Note that in $\mathbb{C}(X)$ we have

$$\begin{aligned}
\mathbb{C}[X]_{(u)} &= \left\{ \tfrac{p}{q} \in \mathbb{C}(X) \mid \begin{matrix} \text{class has a representative} \\ \frac{p'}{q'} \text{ such that } q' \notin J_u \end{matrix} \right\} \\
&= \left\{ \tfrac{p}{q} \in \mathbb{C}(X) \mid \begin{matrix} \text{class has a representative} \\ \frac{(p'/f^m)}{(q'/f^m)} \text{ such that } \frac{q'}{f^m} \notin \tilde{J}_u \end{matrix} \right\} = (\mathbb{C}[X]_f)_{(u)}
\end{aligned}$$

The bijection in (4.4) together with Lemma 4.4.4 implies hence:

$$\mathcal{O}(X_f) = \bigcap_{u \in X_f} \mathcal{O}(X)_{(u)} = \bigcap_{u \in X_f} \mathbb{C}[X]_{(u)} = \bigcap_{u \in X_f} (\mathbb{C}[X]_f)_{(u)} = \mathbb{C}[X]_f.$$

•

Exercise 4.4.1 Let $X \subseteq \mathbb{C}^n$ be an irreducible affine variety, let $I = \mathcal{I}(X)$ be its vanishing ideal, denote by $\mathbb{C}[X]$ its coordinate ring and let $f \in \mathbb{C}[X] - \{0\}$. Consider the affine variety

$$Z := \{(v, c) \in \mathbb{C}^n \oplus \mathbb{C} \mid \forall\, g \in I : g(v) = 0,\ c \cdot f(v) - 1 = 0\}$$

i) Show that the projection $\pi : \mathbb{C}^n \oplus \mathbb{C} \to \mathbb{C}^n$ on the first summand induces a bijection $\pi|_Z : Z \to X_f$.

ii) Show that $\mathbb{C}[Z] \simeq \mathbb{C}[X]_f$.

iii) Show that π is a homeomorphism between Z (Zariski topology) and X_f (induced Zariski topology from X).

Corollary 4.4.1 $\mathbb{C}^2 - \{0\}$ *is not an affine variety.*

Proof. Set $X = \mathbb{C}^2$ and $Y = \mathbb{C}^2 - \{0\}$. Denote by $\mathbb{C}[x, y]$ the coordinate ring of X, and let X_x respectively X_y be the two special open subsets where the first respectively the second coordinate is not 0. Note that $Y = X_x \cup X_y$.

Let $f \in \mathcal{O}[Y]$ be a regular function on Y, then $f|_{X_x} = \frac{g}{x^m}$ for some $g \in \mathbb{C}[x, y]$ and $m \in \mathbb{N}$, and $f|_{X_y} = \frac{h}{y^k}$ for some $h \in \mathbb{C}[x, y]$ and $k \in \mathbb{N}$. It follows

$$f|_{X_x \cap X_y} = \frac{g}{x^m}|_{X_x \cap X_y} = \frac{h}{y^k}|_{X_x \cap X_y}$$

The intersection $X_x \cap X_y$ is open and dense in $\mathbb{C}^2$, and hence we get the equation $hx^m = gy^k$ in $\mathbb{C}[x, y]$. Now the latter is a unique factorization ring and hence h is divisible by y^k respectively g is divisible by x^m. In other words, f can be representated by a polynomial. As a consequence we see that the natural restriction map $\mathbb{C}[X] \to \mathcal{O}[Y]$ is a bijection. Now if $Y = \operatorname{Mspec} \mathbb{C}[Y]$ is an affine variety, then

$$\mathbb{C}^2 - \{0\} = Y = \operatorname{Mspec} \mathbb{C}[Y] = \operatorname{Mspec} \mathcal{O}[Y] = \operatorname{Mspec} \mathbb{C}[X] = \mathbb{C}^2,$$

which is a contradiction. •

Corollary 4.4.2 $GL_n(\mathbb{C})$ *is an affine algebraic group, i.e.* $GL_n(\mathbb{C})$ *is an affine variety such that the product map and the inversion are maps of affine algebraic varieties.*

Proof. As a special open subset of a vector space, $GL_n(\mathbb{C})$ is an affine variety. The rest of the proof is more or less the same as in Example 4.1.2, keeping in mind that the inversion is given by the map which sends a matrix to its adjunct matrix (a polynomial map) multiplied by $\frac{1}{\det}$, which is a regular function on $GL_n(\mathbb{C})$. •

4.5 Two decompositions

The aim of this section is to prove two decomposition results. The first concerns the group of algebraic loops $L^{alg}GL_n(\mathbb{C})$, which we can identify with $GL_n(\mathbb{C}[t,t^{-1}])$. Given a vector $\underline{\lambda}$ with integral coefficients, say $\underline{\lambda} = (\lambda_1,\dots,\lambda_n) \in \mathbb{Z}^n$, denote by $\underline{t}^{\underline{\lambda}}$ the following algebraic loop:

$$\underline{t}^{\underline{\lambda}} : \mathbb{C}^* \to GL_n(\mathbb{C}), \quad t \mapsto \begin{pmatrix} t^{\lambda_1} & 0 & 0 & 0 \\ 0 & t^{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t^{\lambda_n} \end{pmatrix}.$$

Theorem 4.5.1 *With respect to the left and right operation of the group $GL_n(\mathbb{C}[t]) \times GL_n(\mathbb{C}[t])$, $GL_n(\mathbb{C}[t,t^{-1}])$ has the following orbit decomposition:*

$$GL_n(\mathbb{C}[t,t^{-1}]) = \dot{\bigcup_{\substack{\underline{\lambda}\in\mathbb{Z}^n \\ \lambda_1\le\lambda_2\le\ldots\le\lambda_n}}} GL_n(\mathbb{C}[t]) \cdot \underline{t}^{\underline{\lambda}} \cdot GL_n(\mathbb{C}[t]).$$

Remark 4.5.1 The irreducible, finite dimensional representation of $GL_n(\mathbb{C})$ respectively $U_n(\mathbb{C})$ are in one-to-one correspondence with n-tuples $\underline{\lambda} \in \mathbb{Z}^n$ such that $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$. So what is the connection between this fact known from representation theory and the decomposition above? This question has intrigued many mathematicians for quite some time, and was only solved about 20 years ago. We will not have time to go into the details (we need intersection cohomology for this, and this is another interesting story), but hopefully be we will get to see a shadow of this.

Proof. Recall first the situation over a field K. In this case, for an arbitrary matrix $A \in M_n(K)$ there exists invertible matrices $g, h \in GL_n(\mathbb{K})$, such that gAh is in standard form:

$$gAh = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where the number of 1's is the rank of the matrix. The latter can be calculated using the minors of A: for a nonzero matrix A, its rank is the maximum of all $1 \leq r \leq n$ such that there exists a non-vanishing minor of A of size r.

Now $\mathbb{C}[t, t^{-1}]$ is not a field, but recall that $\mathbb{C}[t]$ is a Euclidean ring. This means in this case that the degree function

$$\deg : \mathbb{C}[t] - \{0\} \longrightarrow \mathbb{N}, \quad p(t) = a_0 + a_1 t + \ldots + a_\ell t^\ell \mapsto \ell \quad \text{for } a_\ell \neq 0$$

can be used to make the Euclidean algorithm work: given any pair of polynomials $a(t), b(t) \in \mathbb{C}[t]$, $a(t) \neq 0$, there exist $q(t), r(t) \in \mathbb{C}[t]$ such that

$$b(t) = q(t)a(t) + r(t), \quad \text{and, if } r(t) \neq 0, \text{ then} \quad \deg r(t) < \deg a(t).$$

The same algorithm (Gaussian elimination, multiplication from left and right by elementary matrices) as in the case of a fields yields: given $A \in M_n(\mathbb{C}[t])$, there exists $g, h \in GL_n(\mathbb{C}[t])$ such that

$$gAh = \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & f_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.5}$$

where f_1 is a divisor of f_2, f_2 is a divisor of f_3, etc. Now in our case we have a matrix $g \in GL_n(\mathbb{C}[t, t^{-1}])$. The multiplication by t commutes with the matrix multiplication, so after multiplication with an appropriate power of t we may assume without loss of generality that $g \in M_n(\mathbb{C}[t]) \cap GL_n(\mathbb{C}[t, t^{-1}])$.

The matrix is invertible, so in (4.5) we have $r = n$. Further, the determinant has to be a complex multiple of a power of t, so $f_1, \ldots, f_n$ are just complex multiples of a power of t. Using the matrix multiplication by an appropriate complex diagonal matrix from the right, we can assume without loss of generality that all entries on the diagonal are just powers of t.

Since f_1 is a divisor of f_2, f_2 is a divisor of f_3, etc, the powers are weakly increasing along the diagonal, i.e., $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$.

The condition $0 \leq \lambda_1$ holds if we start with $g \in M_n(\mathbb{C}[t]) \cap GL_n(\mathbb{C}[t, t^{-1}])$. If we start with an arbitrary $g \in GL_n(\mathbb{C}[t, t^{-1}])$ and replace it by $t^b g \in$

$M_n(\mathbb{C}[t]) \cap GL_n(\mathbb{C}[t,t^{-1}])$ for some $b > 0$, then we have to shift the λ_i accordingly and hence get the desired decomposition

$$GL_n(\mathbb{C}[t,t^{-1}]) = \bigcup_{\substack{\underline{\lambda}\in\mathbb{Z}^n \\ \lambda_1\leq\lambda_2\leq\ldots\leq\lambda_n}} GL_n(\mathbb{C}[t]) \cdot \underline{t}^{\underline{\lambda}} \cdot GL_n(\mathbb{C}[t]). \tag{4.6}$$

It remains to show that the union is disjoint. Again, let us first assume $g \in M_n(\mathbb{C}[t]) \cap GL_n(\mathbb{C}[t,t^{-1}])$. The entries are then elements in $\mathbb{C}[t]$, and so are all minors of g. Denote by $I_d(g) \subset \mathbb{C}[t]$ the ideal generated by all minors of g of size d:

$$I_d(g) = \langle \text{all } d \times d\text{-minors of } g \rangle \subset \mathbb{C}[t].$$

Now $\mathbb{C}[t]$ is a principle ideal domain, so $I_d(g)$ has a unique monic generator which we denote by $f_d(g)$.

Example 4.5.1 Let us look at the case $n = 3$ and the matrix

$$g = \begin{pmatrix} t^{\lambda_1} & 0 & 0 \\ 0 & t^{\lambda_2} & 0 \\ 0 & 0 & t^{\lambda_3} \end{pmatrix},$$

where $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Then

$$I_1(g) = \langle t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3} \rangle,\ I_2(g) = \langle t^{\lambda_1+\lambda_2}, t^{\lambda_1+\lambda_3}, t^{\lambda_2+\lambda_3} \rangle,\ I_3(g) = \langle t^{\lambda_1+\lambda_2+\lambda_3} \rangle.$$

The assumption on the ordering of the λ_i implies:

$$f_1(g) = t^{\lambda_1}, \quad f_2(g) = t^{\lambda_1+\lambda_2}, \quad f_3(g) = t^{\lambda_1+\lambda_2+\lambda_3}.$$

So we can recover g from the generators: the entries on the diagonal are

$$f_1(g), \frac{f_2(g)}{f_1(g)}, \frac{f_3(g)}{f_2(g)}.$$

Exercise 4.5.1 For $\underline{\lambda} \in \mathbb{Z}^n$, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, determine the ideals $I_d(\underline{t}^{\underline{\lambda}})$ for all $1 \leq d \leq n$, and determine the monic generators $f_1(\underline{t}^{\underline{\lambda}}), \ldots, f_n(\underline{t}^{\underline{\lambda}})$. Show how to recover $\underline{t}^{\underline{\lambda}}$ from this set of generators.

Let us examine how the ideals $I_g(d)$ behave with respect to multiplication with elementary matrices (over the ring $\mathbb{C}[t]$):

i) multiplication of a row or column with a complex number does not change an ideal $I_g(d)$;

ii) it is easy to see that switching rows or columns does not change an ideal $I_g(d)$;

iii) given $1 \leq i_1 < \ldots < i_d \leq b$ and $1 < j_1, \ldots, j_d \leq n$ let $g_{\underline{i},\underline{j}}$ be the submatrix consisting of the entries in the columns $j_1, \ldots, j_d$ and rows $i_1, \ldots, i_d$. We write for short $\underline{i}, \underline{j}$ for $(i_1, \ldots, i_d)$ and $(j_1, \ldots, j_d)$. Let $m_{\underline{i};\underline{j}}(g) = \det g_{\underline{i},\underline{j}}$ be the corresponding minor. Let g' be the matrix obtained from g by adding a multiple of the i-th row to the j-th row, $i \neq j$. If j is not in $\underline{i}$ or i and j are in $\underline{i}$, then $m_{\underline{i};\underline{j}}(g) = m_{\underline{i};\underline{j}}(g')$. Suppose only $j \in \{i_1, \ldots, i_d\}$, say $j = i_k$, and let $\underline{i}'$ be obtained from $\underline{i}$ by repacing j by i. We have

$$m_{\underline{i}\underline{j}}(g') = m_{\underline{i};\underline{j}}(g) + r m_{\underline{i}';\underline{j}}(g)$$

for some $r \in \mathbb{C}[t]$. Now $m_{\underline{i}';\underline{j}}(g) = m_{\underline{i}';\underline{j}}(g')$, which implies for the ideals:

$$\langle m_{\underline{i};\underline{j}}(g), m_{\underline{i}';\underline{j}}(g) \rangle = \langle m_{\underline{i};\underline{j}}(g'), m_{\underline{i}';\underline{j}}(g') \rangle,$$

and hence $I_d(g) = I_d(g')$. The same arguments show $I_d(g) = I_d(g')$ if g' is obtained from g by adding a multiple of a column to a different column.

Summarizing: if $g \in GL_n(\mathbb{C}[t]) \cdot \underline{t}^{\lambda} \cdot GL_n(\mathbb{C}[t])$ for some $\underline{\lambda} \in \mathbb{Z}^n_{\geq 0}$, then $I_d(g) = I_d(\underline{t}^{\lambda})$. The same arguments as above show the general case $\underline{\lambda} \in \mathbb{Z}^n$ can be reduced to the special case $\underline{\lambda} \in \mathbb{Z}^n_{\geq 0}$, and hence: $I_d(g) = I_d(\underline{t}^{\lambda})$ for $g \in GL_n(\mathbb{C}[t]) \cdot \underline{t}^{\lambda} \cdot GL_n(\mathbb{C}[t])$ and $\underline{\lambda} \in \mathbb{Z}^n$. By Exercise 4.5.1 we know how to recover $\underline{\lambda}$ from the ideals $I_d(\underline{t}^{\lambda})$, $1 \leq d \leq n$, so the union in (4.6) is disjoint.
•

The second decomposition shows the connection between algebraic loops in $GL_n(\mathbb{C})$ and algebraic loops in $U_n(\mathbb{C})$.

Theorem 4.5.2 *The algebraic loop group is the direct product of two of its subgroups: the subgroups of based algebraic loops with image in $U_n(\mathbb{C})$, and the subgroup of polynomial loops:*

$$GL_n(\mathbb{C}[t, t^{-1}]) = \Omega^{alg}(U_n(\mathbb{C})) \cdot GL_n(\mathbb{C}[t]).$$

Proof. Clearly $GL_n(\mathbb{C}[t])$ is a subgroup of $GL_n(\mathbb{C}[t, t^{-1}])$, and by Proposition 1.1.1 we know that $L^{alg}U_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C}[t, t^{-1}])$, and hence so is $\Omega^{alg}(U_n(\mathbb{C}))$, the group of based algebraic loops. To see that the product is direct let g be an element of the intersection $GL_n(\mathbb{C}[t]) \cap \Omega^{alg}(U_n(\mathbb{C}))$.

Then g^{-1} is an element of the intersection too. But $g \in GL_n(\mathbb{C}[t])$ implies $g^{-1} \in GL_n(\mathbb{C}[t])$ and $g \in \Omega^{alg}(U_n(\mathbb{C}))$ implies $g^{-1} = \overline{g}^{\intercal} \in GL_n(\mathbb{C}[t^{-1}])$. It follows that $g \in GL_n(\mathbb{C}) \cap \Omega^{alg}(U_n(\mathbb{C}))$ and hence $g = \mathbb{I}$. •

Before we continue with the proof, let us introduce some notation. We regard $(\mathbb{C}[t, t^{-1}])^n$ as an infinite dimensional complex vector space, endowed with the following hermitian form:

$$\langle \cdot, \cdot \rangle : (\mathbb{C}[t, t^{-1}])^n \times (\mathbb{C}[t, t^{-1}])^n \to \mathbb{C}, \; (f(t), g(t)) \mapsto [\overline{f(t)}^{\intercal} \cdot g(t)]_0. \quad (4.7)$$

(Recall that $\overline{f(t)}^{\intercal}$ means to take the complex conjugate of all complex coefficients, replace t by t^{-1}, and to take the transpose of the resulting vector.) The notation $[\ldots]_0$ means that inside the bracket we have a Laurent polynomial and we take the coefficient of t^0. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. One checks easily that

$$\mathbb{B} = \{\ldots, t^{-2}e_1, \ldots, t^{-2}e_n, t^{-1}e_1, \ldots, t^{-1}e_n, e_1, \ldots, e_n, te_1, \ldots, te_n, t^2e_1, \ldots\}$$

is an orthonormal basis for $(\mathbb{C}[t, t^{-1}])^n$ with respect to this form. We enumerate the basis elements by integers, the basis vector $t^k e_j$ has the number $nk + j$.

Example 4.5.2 We get for $n = 2$:

basis element	...	$t^{-2}e_2$	$t^{-1}e_1$	$t^{-1}e_2$	e_1	e_2	te_1	te_2	...
enumeration	...	-2	-1	0	1	2	3	4	...

Next let $A(t) \in GL_n(\mathbb{C}[t, t^{-1}])$, there exists some $m \in \mathbb{N}$ such that

$$A(t) = \sum_{j=-m}^{m} A_j t^j, \quad A_{-m}, \ldots, A_m \in M_n(\mathbb{C}).$$

With respect to the basis $\mathbb{B}$ we can represent the invertible linear map

$$(\mathbb{C}[t, t^{-1}])^n \to (\mathbb{C}[t, t^{-1}])^n, \quad f(t) \mapsto A(t)f(t),$$

as a matrix in $M_\infty(\mathbb{C})$:

$$\widehat{A} := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & A_0 & A_{-1} & A_{-2} & A_{-3} & A_{-4} & A_{-5} & \cdots \\ \cdots & A_1 & A_0 & A_{-1} & A_{-2} & A_{-3} & A_{-4} & \cdots \\ \cdots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ \cdots & A_3 & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & \cdots \\ \cdots & A_4 & A_3 & A_2 & A_1 & A_0 & A_{-1} & \cdots \\ \cdots & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.8}$$

Note that in each row and each column there are only a finite number of nonzero entries, so the multiplication of these type of matrices is well defined. Further, the entries in the columns respectively rows are repetitive, after a shift of n columns to the right and n rows down.

Example 4.5.3 Consider the matrices $A(t)$ and $A(t)^{-1}$ in $GL_2(\mathbb{C}[t, t^{-1}])$, where

$$A(t) = \begin{pmatrix} t & a_0 + a_1 t \\ 0 & t \end{pmatrix} \quad A(t)^{-1} = \begin{pmatrix} t^{-1} & -a_0 t^{-2} - a_1 t^{-1} \\ 0 & t^{-1} \end{pmatrix}$$

Rewrite both as matrices in $M_\infty(\mathbb{C})$ as in (4.8) and multiply them.

Example 4.5.4 Consider the matrix

$$A(t) = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in GL_2(\mathbb{C}[t, t^{-1}]),$$

Then

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and $A_j = 0$ for all $j \neq 0, -1$. The matrix $\widehat{A}$ has hence the form

$$\widehat{A} := \left(\begin{array}{cccccccc|c}
\cdots & \text{-3} & \text{-2} & \text{-1} & 0 & 1 & 2 & \cdots & \\
\hline
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots \\
\cdots & 1 & 0 & 0 & 1 & & & \cdots & \text{-3} \\
\cdots & 0 & 1 & 0 & 0 & & & \cdots & \text{-2} \\
\cdots & & & 1 & 0 & 0 & 1 & \cdots & \text{-1} \\
\cdots & & & 0 & 1 & 0 & 0 & \cdots & 0 \\
\cdots & & & & & 1 & 0 & \cdots & 1 \\
\cdots & & & & & 0 & 1 & \cdots & 2 \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{array}\right) \tag{4.9}$$

The top row and the last column indicate the numbering of the rows and columns according to the enumeration of the basis $\mathbb{B}$. So e_1 (the 1-st basis vector) is just mapped to itself, and e_2 (the 2-nd basis vector) is mapped to itself plus $t^{-1}e_1$ (which is the (-1)-st basis vector), and so on.

Now the arguments in section 1.1, in particular (1.3), imply that (by restricting the map to S^1) $A(t) \in L^{alg}U_n(\mathbb{C})$ if and only if $\overline{A(t)}^{\intercal} A(t) = \mathbb{1}$, or, equivalently

$$\left(\sum_{j=-m}^{m} \overline{A}_j^{\text{transpose}} t^{-j}\right)\left(\sum_{\ell=-m}^{m} A_\ell t^\ell\right) = \sum_{k=-2m}^{2m}\left(\sum_{-j+\ell=k} \overline{A}_j^{\text{transpose}} A_\ell\right)t^k = \mathbb{1}. \tag{4.10}$$

For $k = 0$ this results in the condition:

$$\sum_{\ell=-m}^{m} \overline{A}_\ell^{\text{transpose}} A_\ell = \mathbb{1} \tag{4.11}$$

Let us translate this into a condition on the matrix of $A(t)$ in (4.8). Note that the entries in (4.8) are $n \times n$-matrices, we collect the columns of $A(t)$ in (4.8) accordingly into groups: a group of columns consist exactly of the images of the basis vectors $t^k e_1, \ldots, t^k e_n$ for a fixed k. Then (4.11) implies: if $A(t) \in L^{alg}U_n(\mathbb{C})$, then all columns of this matrix consist of vectors of norm one, and different vectors in the same group are orthogonal to each other.

For $k = 1$, equation (4.10) results in the condition:

$$\sum_{\ell=-m}^{m-1} \overline{A}_\ell^{\text{transpose}} A_{\ell+1} = 0, \tag{4.12}$$

which again can be translated into a property of the matrix of $A(t)$. If $A(t) \in L^{alg}U_n(\mathbb{C})$, then the columns of adjoining groups of columns are orthogonal to each other. We leave it as an exercise to generalize this and to prove:

Exercise 4.5.2 Show that $A(t) \in L^{alg}U_n(\mathbb{C})$ if and only if the matrix $\widehat{A}$ in (4.8) has as columns an orthonormal basis of $(\mathbb{C}[t,t^{-1}])^n$ with respect to the form defined in (4.7)

Example 4.5.5 We continue with Example 4.5.4 and we apply the Gram-Schmidt process to the matrix in (4.9). Fix an odd number $\ell = 2k+1 \in \mathbb{Z}$ and let $(\mathbb{C}[t,t^{-1}])^2_\ell$ be the subspace spanned by the column vectors of the matrix $\widehat{A}$ with index $\geq \ell$. In the subspace spanned by these columns we find the vectors

$$t^k e_1, t^{k+1}e_1, t^{k+2}e_1, \ldots$$

and the vectors

$$t^{k+1}e_2 = (t^{k+1}e_2 + t^k e_1) - t^k e_1, t^{k+2}e_2 = (t^{k+2}e_2 + t^{k+1}e_1) - t^{k+1}e_1, \ldots.$$

To apply the Gram-Schmidt process in an infinite dimensional case, we have to guarantee that we have an orthogonal projection

$$\pi_\ell : (\mathbb{C}[t,t^{-1}])^2 \to (\mathbb{C}[t,t^{-1}])^2_\ell,$$

i.e. given $v \in (\mathbb{C}[t,t^{-1}])^2$ one has $v = v_1 + v_2$, where $v_2 = \pi_\ell(v) \in (\mathbb{C}[t,t^{-1}])^2_\ell$ and $v_2 = v - \pi_\ell(v)$ is orthogonal to $(\mathbb{C}[t,t^{-1}])^2_\ell$. If we can prove that the subspace has an orthonormal basis, then such a projection exists.

Now from the above we see that $(\mathbb{C}[t,t^{-1}])^2_\ell$, $\ell = 2k+1$, has a basis given by the vectors

$$\{t^k e_1, t^{k-1}e_1 + t^k e_2\} \cup \{t^m e_1, t^m e_2 \mid m \geq k+1\}$$

This is already an orthogonal basis, by rescaling the vector $t^{k-1}e_1 + t^k e_2$ to $\frac{1}{\sqrt{2}}(t^{k-1}e_1 + t^k e_2)$, we get an orthonormal basis of this subspace, which implies the existence of the orthogonal projection.

Let now $\widehat{A}_{2k-1}, \widehat{A}_{2k}$ be the next two columns to the left of $\widehat{A}_\ell$. Consider the images $\pi_\ell(\widehat{A}_{2k-1}), \pi_\ell(\widehat{A}_{2k})$ of the two in $(\mathbb{C}[t,t^{-1}])^2_\ell$. These are finite linear combinations of the columns $\widehat{A}_\ell, \widehat{A}_{\ell+1}, \widehat{A}_{\ell+2}, \ldots$.

So to replace in the matrix $\widehat{A}$ the columns $\widehat{A}_{2k-1}$ and $\widehat{A}_{2k}$ by $\widehat{A}_{2k-1} - \pi_\ell(\widehat{A}_{2k-1})$ and $\widehat{A}_{2k} - \pi_\ell(\widehat{A}_{2k})$ amounts to multiply $\widehat{A}$ from the right by a matrix of the form:

$$\widetilde{C} := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \mathbb{1} & & & & & & \cdots \\ \cdots & & \mathbb{1} & & & & & \cdots \\ \cdots & & & \mathbb{1} & & & & \cdots \\ \cdots & & & C_1 & \mathbb{1} & & & \cdots \\ \cdots & & & C_2 & & \mathbb{1} & & \cdots \\ \cdots & & & C_3 & & & \mathbb{1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.13}$$

where C_1, C_2, ... are 2×2-matrices (only a finite number of them are nonzero!), filled in the matrix along the $(2k-1)$-th and the $2k$-th column. So the new matrix $\widehat{A}\widetilde{C}$ has the following three properties: 1) all columns are the same as before, except for the $(2k-1)$-th and the $2k$-th column; 2) the new two columns are orthogonal to the subspace $(\mathbb{C}[t,t^{-1}])^2_\ell$, and 3) the subspace $(\mathbb{C}[t,t^{-1}])^2_{\ell-2}$ is spanned by $(\mathbb{C}[t,t^{-1}])^2_\ell$ and the columns $\widehat{A}_{2k-1}, \widehat{A}_{2k}$, or, alternatively, by $(\mathbb{C}[t,t^{-1}])^2_\ell$ and the columns $\widehat{A}_{2k-1} - \pi_\ell(\widehat{A}_{2k-1})$ and $\widehat{A}_{2k} - \pi_\ell(\widehat{A}_{2k})$.

Recall that the entries in the columns are repetitive, after a shift of 2 columns to the right and 2 rows down, we have the same pair of columns. So in a next step we perform this substitution on the columns all at once, i.e. we multiply $\widehat{A}$ by the matrix $\widehat{C}$ from the right, where

$$\widehat{C} := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \mathbb{1} & & & & & & \cdots \\ \cdots & C_1 & \mathbb{1} & & & & & \cdots \\ \cdots & C_2 & C_1 & \mathbb{1} & & & & \cdots \\ \cdots & C_3 & C_2 & C_1 & \mathbb{1} & & & \cdots \\ \cdots & C_4 & C_3 & C_2 & C_1 & \mathbb{1} & & \cdots \\ \cdots & C_5 & C_4 & C_3 & C_2 & C_1 & \mathbb{1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.14}$$

Recall, only a finite number of the C_i are nonzero matrices, so $\widehat{C}$ is the matrix in $M_\infty(\mathbb{C})$ associated to a matrix $C(t) \in GL_2(\mathbb{C}[t])$.

The new matrix $\widehat{G} = \widehat{A}\widehat{C}$ (corresponding to the matrix $G(t) = A(t)C(t)$ in $GL_2(\mathbb{C}[t,t^{-1}])$) has the property that each pair of columns $\widehat{G}_{2k-1}, \widehat{G}_{2k}$ is orthogonal to all other pairs of columns $\widehat{G}_{2m-1}, \widehat{G}_{2m}$, $m \neq k$, of the matrix. In our example the new matrix has the form

$$\widehat{G} := \left(\begin{array}{cccccccc|c} \cdots & \text{-3} & \text{-2} & \text{-1} & 0 & 1 & 2 & \cdots & \\ \hline \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \cdots & 1 & 0 & 0 & 1 & 0 & 0 & \cdots & \text{-3} \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & \text{-2} \\ \cdots & 0 & 0 & 1 & 0 & 0 & 1 & \cdots & \text{-1} \\ \cdots & \text{-1} & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 1 \\ \cdots & 0 & 0 & \text{-1} & 0 & 0 & 1 & \cdots & 2 \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right) \tag{4.15}$$

So get at the end a unitary loop $U(t)$, all what remains to do is to replace each pair of columns $\widehat{G}_{2k-1}, \widehat{G}_{2k}$ by an orthonormal basis of the subspace spanned by the pair of columns. This amounts to normalize $\widehat{G}_{2k}$, so $\widehat{U}_{2k} = \frac{1}{\|\widehat{G}_{2k}\|}\widehat{G}_{2k}$, and to replace G_{2k-1} by a linear combination of $\widehat{G}_{2k-1}$ and $\widehat{G}_{2k}$, so that the result is orthogonal to $\widehat{G}_{2k}$ and of norm one. Again doing this for all pairs of columns at once, this amounts to multiply $\widehat{G}$ from the right by a matrix of the form

$$\widehat{D} := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & D_0 & & & & & & \cdots \\ \cdots & & D_0 & & & & & \cdots \\ \cdots & & & D_0 & & & & \cdots \\ \cdots & & & & D_0 & & & \cdots \\ \cdots & & & & & D_0 & & \cdots \\ \cdots & & & & & & D_0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.16}$$

where D_0 is a lower tringular matrix in $GL_2(\mathbb{C})$. So the matrix $\widehat{D}$ corresponds to a constant loop matrix $D(t) \in GL_2(\mathbb{C}[t])$, and the resulting matrix

$$U(t) = A(t)C(t)D(t)$$

is a unitary loop. It follows that for the matrix $A(t)$ in (4.9) one can find a matrix $H(t) \in GL_2(\mathbb{C}[t])$ such that $U(t) = A(t)H(t)$ is a unitary loop. In our example the resulting unitary loop is

$$\widehat{U} := \left(\begin{array}{cccccccc|c} \cdots & \text{-3} & \text{-2} & \text{-1} & 0 & 1 & 2 & \cdots & \\ \hline \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \cdots & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & \text{-3} \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & \text{-2} \\ \cdots & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & \cdots & \text{-1} \\ \cdots & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 1 \\ \cdots & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & \cdots & 2 \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right) \tag{4.17}$$

or, formulated in terms of matrices contained respectively in $GL_2(\mathbb{C}[t,t^{-1}])$, $\Omega^{alg}U_2(\mathbb{C})$ and $GL_2(\mathbb{C}[t])$:

$$\begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{t^{-1}}{\sqrt{2}} \\ -\frac{t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \\ \frac{t}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1+t^{-1}}{2} & \frac{-1+t^{-1}}{2} \\ \frac{-t+1}{2} & \frac{t+1}{2} \end{pmatrix} \begin{pmatrix} \frac{1+t}{2} & 1 \\ \frac{-1+t}{2} & 1 \end{pmatrix}$$

Example 4.5.6 In the same way one shows

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{t}{\sqrt{2}} \\ \frac{t^{-1}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{t}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+t}{2} & \frac{1-t}{2} \\ \frac{t^{-1}-1}{2} & \frac{t^{-1}+1}{2} \end{pmatrix} \begin{pmatrix} 1 & \frac{1-t}{2} \\ 1 & \frac{1+t}{2} \end{pmatrix} \in \Omega^{alg}U_2(\mathbb{C}) \cdot GL_2(\mathbb{C}[t]). \end{aligned}$$

Exercise 4.5.3 Describe explicitly the decomposition into a product of a unitary and a polynomial loop for the algebraic loops

$$\begin{pmatrix} 1 & t^{-k} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t^{-k} & 1 \end{pmatrix}, \quad k \in \mathbb{N}.$$

Continuation of the proof of the theorem. Except for the difference that one considers groups of columns of size n instead of just two columns, the strategy of the proof is the same as in Example 4.5.5.

Fix a number $\ell = nk + 1$ for some $k \in \mathbb{Z}$ and let $(\mathbb{C}[t,t^{-1}])^n_\ell$ be the span of all columns of index $\geq \ell$. To show that an orthogonal projection

$$\pi_\ell : (\mathbb{C}[t,t^{-1}])^n \to (\mathbb{C}[t,t^{-1}])^n_\ell$$

exists, one has to show that $(\mathbb{C}[t,t^{-1}])^n_\ell$ admits an orthonormal basis. To prove the existence of an orthonormal basis, consider

$$A(t)^{-1} = \sum_{p=-m'}^{m'} A'_p t^p.$$

The $jn+i$-th column of the corresponding matrix $\widehat{A}' \in M_\infty(\mathbb{C})$ can be viewed as an array describing the basis vector $t^j e_i$ as a linear combination of the basis elements given by the columns of the matrix $\widehat{A}$. Now the special form of the matrices of type (4.8) implies hence that $(\mathbb{C}[t,t^{-1}])^n_\ell \cap \mathbb{B}$ spans a subspace of finite codimension. So after a finite number of steps (using Gram Schmidt) one can complete the orthonormal system given by $(\mathbb{C}[t,t^{-1}])^n_\ell \cap \mathbb{B}$ to an orthonormal basis of $(\mathbb{C}[t,t^{-1}])^n_\ell$. This implies the existence of an orthogonal projection:

$$\pi_\ell : (\mathbb{C}[t,t^{-1}])^n \to ((\mathbb{C}[t,t^{-1}])^n)_\ell.$$

Now one can proceed as in the example: consider the next n columns to the left of the column $\widehat{A}_\ell$, and replace the columns $\widehat{A}_{n(k-1)+1}$, $\widehat{A}_{n(k-1)+2}$, ..., $\widehat{A}_{nk}$ by the columns

$$\widehat{A}_{n(k-1)+1} - \pi_\ell(\widehat{A}_{n(k-1)+1}), \widehat{A}_{n(k-1)+2} - \pi_\ell(\widehat{A}_{n(k-1)+2}), \dots, \widehat{A}_{nk} - \pi_\ell(\widehat{A}_{nk}).$$

Recall that the entries in the columns are repetitive, after a shift of n columns to the right and n rows down, we have the same group of n columns. So in a next step one performs this substitution on the columns all at once, i.e. one multiplies $\widehat{A}$ by a matrix $\widehat{C}$ from the right, which has the same form as in (4.16). Only this time the matrices C_i are complex $n \times n$-matrices. But as before, only a finite number of these matrices are nonzero. In particular, $\widehat{C}$ is the matrix associated to a polynomial loop $C(t) \in GL_n(\mathbb{C}[t])$.

The resulting matrix $\widehat{G} = \widehat{A}\widehat{C}$ has the following property: for all $k \in \mathbb{N}$, the group of columns $\widehat{A}_{kn+1}, \dots, \widehat{A}_{kn+n-1}$ is orthogonal to all other columns $\widehat{A}_j$ for $j \notin \{kn+1, \dots, kn+n-1\}$. So to turn $\widehat{G}$ into a unitary matrix, it remains to apply the Gram-Schmidt procedure each of these groups. As in Example 4.5.5, this amounts to multiply $\widehat{G}$ from the right by a matrix $\widehat{D}$ as in Example 4.5.5, where this time D_0 is a lower tringular matrix in $GL_n(\mathbb{C})$. So the matrix $\widehat{D}$ corresponds to a constant loop matrix $D(t) \in GL_n(\mathbb{C}[t])$, and the resulting matrix

$$U(t) = A(t)C(t)D(t)$$

is a unitary loop. To turn it into a based loop, one multiplies from the right by $U^{-1}(1)$ and obtains

$$A(t) = \underbrace{\big(U(t)U^{-1}(1)\big)}_{\text{based unitary loop}}\underbrace{\big(U(1)D^{-1}(t)C^{-1}(t)\big)}_{\text{polynomial loop}}$$

•

Chapter 5

Lattice realization of $\Omega^{alg}U_n(\mathbb{C})$

Another way of formulating Theorem 4.5.2 is to say that it induces a bijection

$$GL_n(\mathbb{C}[t,t^{-1}])/GL_n(\mathbb{C}[t]) \overset{1:1}{\longleftrightarrow} \Omega^{alg}U_n(\mathbb{C}).$$

We want to use this to give a new description of $\Omega^{alg}U_n(\mathbb{C})$.

5.1 Lattices and quotients: an example

Let us start with $\mathbb{Z}$-lattices in $\mathbb{Q}^n$. Given a basis $\mathbb{B} = \{v_1, \ldots, v_n\}$ of $\mathbb{Q}^n$, denote by

$$\mathcal{L}_{\mathbb{B}} = \{\sum_{i=1}^{n} a_i v_i \mid a_1, \ldots, a_n \in \mathbb{Z}\}.$$

the set of **integral** linear combinations of the elements of $\mathbb{B}$. This subset has the following properties:

1) it is a subgroup of $\mathbb{Q}^n$, i.e. for all $\ell_1, \ell_2 \in \mathcal{L}_{\mathbb{B}}$ we have $\ell_1 - \ell_2 \in \mathcal{L}_{\mathbb{B}}$;

2) it is stable under multiplication with integers.

These two properties are just the definition of a $\mathbb{Z}$-module. Recall that a vector space over a field has always a basis, but, in general, modules over a ring R do not have a basis.

Definition 5.1.1 If an R-module has a basis, then the module is called a *free module*. If the module admits a finite basis, then all bases have the same number of elements, the number is called the *rank of the free module*.

Remark 5.1.1 So for bases of a free module of finite rank we can talk about base change matrices. These matrices are of course quadratic matrices, with entries in the ring, and they are invertible over this ring. So as in the case of a vector space of a field, for a free module of rank n over R, we have a bijection between ordered bases and elements of

$$GL_n(R) = \{A \in M_n(R) \mid \det A \in R^\times\},$$

i.e. the determinant is an element of the group of units $R^\times$ in R.

Let us go back to the example above:

3) $\mathcal{L}_{\mathbb{B}}$ is a free $\mathbb{Z}$-module with basis $\mathbb{B}$, and the $\mathbb{Z}$-basis for $\mathcal{L}_{\mathbb{B}}$ is also a $\mathbb{Q}$-basis for $\mathbb{Q}^n$.

The properties 1)–3) are a charcterization of $\mathbb{Z}$-lattices in $\mathbb{Q}^n$. (Usually one finds in the books a definition via tensor product, but for the moment lets stay with the characterization above.) Let $\mathcal{L}(\mathbb{Q}^n, \mathbb{Z})$ be the set of all $\mathbb{Z}$-lattices in $\mathbb{Q}^n$. A lattice is determined by a basis, which, by collecting the basis elements column wise, corresponds to an invertible matrix. Hence we have a surjective map $GL_n(\mathbb{Q}) \to \mathcal{L}(\mathbb{Q}^n, \mathbb{Z})$. Now two matrices $g, g' \in GL_n(\mathbb{Q})$ correspond to the same lattice if and only if the columns of g and the columns of g' span the same lattice, which is equivalent to say that $g^{-1}g'$ is a basis transformation matrix corresponding to two bases of the standard lattice $\mathbb{Z}^n = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$. Or, in other words, $g^{-1}g' \in GL_n(\mathbb{Z})$, so we get a bijection

$$GL_n(\mathbb{Q})/GL_n(\mathbb{Z}) \xrightarrow{1:1} \mathcal{L}(\mathbb{Q}^n; \mathbb{Z}^n).$$

Now we are ready to try out the procedure with the groups $GL_n(\mathbb{C}[t, t^{-1}])$ and $GL_n(\mathbb{C}[t])$.

5.2 $\mathbb{C}[t]$-lattices in $\mathbb{C}(t)^n$ and $\mathbb{C}[t, t^{-1}]^n$

We start with the definition of $\mathbb{C}[t]$-lattices in $\mathbb{C}[t, t^{-1}]^n$ and in $\mathbb{C}(t)^n$:

Definition 5.2.1 A $\mathbb{C}[t]$-*lattice* $\mathcal{L} \subset \mathbb{C}(t)^n$ is a free $\mathbb{C}[t]$-submodule such that one (and hence every) $\mathbb{C}[t]$-basis of $\mathcal{L}$ is a $\mathbb{C}(t)$-basis for $\mathbb{C}(t)^n$. We say that a free $\mathbb{C}[t]$-module $\mathcal{L}$ *is a lattice in* $\mathbb{C}[t, t^{-1}]^n$ if this $\mathbb{C}[t]$-module has the property that one (and hence every) $\mathbb{C}[t]$-basis of $\mathcal{L}$ is a $\mathbb{C}[t, t^{-1}]$-basis for $\mathbb{C}[t, t^{-1}]^n$.

Exercise 5.2.1 Prove the "*and hence every*" parts of the definition above.

As pointed out in Remark 5.1.1, the set of all ordered bases of $\mathbb{C}[t,t^{-1}]^n$ as a free $\mathbb{C}[t,t^{-1}]$-module can be naturally identified with $GL_n(\mathbb{C}[t,t^{-1}])$, and the set of all ordered bases of $\mathbb{C}[t]^n$ as a free $\mathbb{C}[t]$-module can be naturally identified with $GL_n(\mathbb{C}[t])$.

Let $\mathcal{G}(\mathbb{C}[t,t^{-1}]^n,\mathbb{C}[t])$ be the set of all $\mathbb{C}[t]$-lattices $\mathcal{L}$ in $\mathbb{C}[t,t^{-1}]^n$. So the same reasoning as above in the case of $\mathbb{Z}$-lattices in $\mathbb{Q}^n$ shows that $\mathbb{C}[t]$-lattices $\mathcal{L}$ in $\mathbb{C}[t,t^{-1}]^n$ correspond to matrices in $GL_n(\mathbb{C}[t,t^{-1}])$, so we get a surjective map $GL_n(\mathbb{C}[t,t^{-1}]) \to \mathcal{G}(\mathbb{C}[t,t^{-1}]^n,\mathbb{C}[t])$. And two matrices $g,g' \in GL_n(\mathbb{C}[t,t^{-1}])$ correspond to the same lattice if and only if the columns span the same $\mathbb{C}[t]$-lattice, which is equivalent to say $g^{-1}g'$ is a basis transformation matrix corresponding to two bases of the standard lattice $\mathbb{C}[t]^n = \mathbb{C}[t]e_1 \oplus \cdots \oplus \mathbb{C}[t]e_n$. Or, in other words, $g^{-1}g' \in GL_n(\mathbb{C}[t])$, so we get a bijection

$$GL_n(\mathbb{C}[t,t^{-1}])/GL_n(\mathbb{C}[t]) \xrightarrow{1:1} \mathcal{G}(\mathbb{C}[t,t^{-1}]^n,\mathbb{C}[t]).$$

Together with the Iwasawa decomposition in Theorem 4.5.2 this implies:

Proposition 5.2.1 *We have natural bijections*

$$\Omega^{alg}(U_n(\mathbb{C})) \xleftrightarrow{1:1} GL_n(\mathbb{C}[t,t^{-1}])/GL_n(\mathbb{C}[t]) \xleftrightarrow{1:1} \mathcal{G}(\mathbb{C}[t,t^{-1}]^n,\mathbb{C}[t]).$$

We want to use these bijections to construct a new parameterization of $\Omega^{alg}(U_n(\mathbb{C}))$ and endow the loop group with the structure of a projective Ind-variety.

To do so, we want know to find a characterization of the lattices in $\mathcal{G}(\mathbb{C}[t,t^{-1}]^n,\mathbb{C}[t])$. Let $\mathcal{L}_0$ be the standard lattice $\mathbb{C}[t]^n = \mathbb{C}[t]e_1 \oplus \cdots \oplus \mathbb{C}[t]e_n$ and let $\mathcal{L}$ be a $\mathbb{C}[t]$-lattice in $\mathbb{C}(t)^n$. To say that $\mathcal{L}$ is a lattice in $\mathbb{C}[t,t^{-1}]^n$ is by the bijection in Proposition 5.2.1 and Theorem 4.5.1 equivalent to say that the lattice can be represented by a matrix in $GL(\mathbb{C}[t,t^{-1}])$ of the form $g\underline{t}^{\underline{\lambda}}$ where $g \in GL(\mathbb{C}[t])$ and $\lambda_1 \leq \ldots \leq \lambda_n$.

Now let $\mathcal{L}_{\underline{\lambda}}$ be the $\mathbb{C}[t]$-lattice having as basis $t^{\lambda_1}e_1,\ldots,t^{\lambda_n}e_n$. If $N \in \mathbb{N}$ is such that $N \geq \max\{|\lambda_1|,\ldots,|\lambda_n|\}$, then

$$t^N\mathcal{L}_0 \subseteq \mathcal{L}_{\underline{\lambda}} \subseteq t^{-N}\mathcal{L}_0. \tag{5.1}$$

The $\mathbb{C}$-vector space $t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0$ has dimension $2nN$, having $\mathcal{L}_{\underline{\lambda}}/t^N\mathcal{L}_0$ as a $\mathbb{C}$-subspace of dimension

$$(\lambda_1+N)+(\lambda_2+N)+(\lambda_3+N)+\ldots+(\lambda_n+N) = \Sigma\underline{\lambda}+nN, \tag{5.2}$$

where $\Sigma\underline{\lambda} \in \mathbb{Z}$ is an abbreviation for $\lambda_1 + \ldots + \lambda_n$.

Since left multiplication by $g \in GL_n(\mathbb{C}[t])$ commutes with multiplication by powers of t and $g\mathcal{L}_0 = \mathcal{L}_0$ for $g \in GL_n(\mathbb{C}[t])$, we see that the lattice $\mathcal{L}$ above satisfies (5.1) and (5.2) too.

Lemma 5.2.1 *If $\mathcal{L}$ is a $\mathbb{C}[t]$-lattice in $\mathbb{C}(t)^n$, then $\mathcal{L}$ is a lattice in $\mathbb{C}[t,t^{-1}]^n$ if and only there exists an $N \in \mathbb{N}$ such that*

$$t^N\mathcal{L}_0 \subseteq \mathcal{L} \subseteq t^{-N}\mathcal{L}_0.$$

Moreover, there exists a $\underline{\lambda} = (\lambda_1 \leq \ldots \leq \lambda_n) \in \mathbb{Z}^n$ such that

$$\dim_{\mathbb{C}} \mathcal{L}/t^{N'}\mathcal{L}_0 = nN' + \Sigma\underline{\lambda} \quad \forall\, N' \geq N.$$

Proof. It remains to prove the "$\Leftarrow$" direction of the first part of the lemma. So let $\mathcal{L}$ be a $\mathbb{C}[t]$-lattice in $\mathbb{C}(t)^n$ satisfying the condition. The inclusion $\mathcal{L} \subseteq t^{-N}\mathcal{L}_0$ implies $\mathcal{L} \subseteq \mathbb{C}[t,t^{-1}]^n$, so by choosing a $\mathbb{C}[t]$-basis for $\mathcal{L}$, the lattice $\mathcal{L}$ can be represented by a matrix $L \in M_n(\mathbb{C}[t,t^{-1}])$, invertible over $\mathbb{C}(t)$. Multiplying L from the right by an element in $GL_n(\mathbb{C}[t])$ amounts to a base change and hence does not change the lattice. By multiplying L from the left by an element in $GL_n(\mathbb{C}[t])$ we may replace $\mathcal{L}$ by a new lattice still satisfying the condition. So up to left and right multiplication by elements in $GL_n(\mathbb{C}[t])$ we can assume that $\mathcal{L}$ is represented by a matrix of the form (see (4.5)),

$$\mathcal{L} \widehat{=} \begin{pmatrix} f_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & f_n \end{pmatrix}, \tag{5.3}$$

where f_1 is a divisor of f_2, f_2 is a divisor of f_3 etc. Now the condition $t^N\mathcal{L}_0 \subseteq \mathcal{L}$ implies that the f_i have to be just powers of t. Indeed, the special form in (5.3) reduces the proof to the case $n = 1$. Assume $f = f_1 = a_it^i + \ldots + a_jt^j$ such that $i < j$ and $a_i, a_j \neq 0$. In fact, without loss of generality we may assume $a_i = 1$. Viewed as a $\mathbb{C}$-vector space, $\mathcal{L}$ has as basis $f, tf, t^2f, \ldots$. It is now easy to see that it is impossible to write for any choice of N the monomial t^N as a $\mathbb{C}$-linear combination of these basis elements. So we have necessarily $i = j$, which finishes the proof of the lemma. •

So starting with an algebraic loop $\gamma \in \Omega^{alg}(U_n(\mathbb{C}))$, we can attach to the loop a $\mathbb{C}[t]$-lattice $\mathcal{L}$ in $\mathbb{C}[t,t^{-1}]^n$, which is associated to a matrix of the form

$gt_{\underline{\lambda}}$ for some $g \in GL_n(\mathbb{C}[t])$. By abuse of notation we define in this case

$$\det \mathcal{L} := \deg(\det(gt_{\underline{\lambda}})).$$

Of course, the matrix $gt_{\underline{\lambda}}$ associated to $\mathcal{L}$ is not unique, but note that for all $g, h \in GL_n(\mathbb{C}[t])$ we have $\det g, \det h \in \mathbb{C}^*$ and hence

$$\deg(\det(gt_{\underline{\lambda}})) = \deg(\det t_{\underline{\lambda}}) = \Sigma\underline{\lambda}.$$

Since $\underline{\lambda}$ is uniquely determined by $\mathcal{L}$, $\det \mathcal{L}$ is hence well defined. For $q \in \mathbb{Z}$ denote by

$$\begin{aligned}
\mathcal{G}_q &:= \{\mathcal{L} \in \mathcal{G}(\mathbb{C}[t,t^{-1}]^n, \mathbb{C}[t]) \mid \det \mathcal{L} = q\} \\
&= \{gGL_n(\mathbb{C}[t]) \in GL_n(\mathbb{C}[t,t^{-1}])/GL_n(\mathbb{C}[t]) \mid \deg(\det g) = q\} \\
&= \{\gamma \in \Omega^{alg}U_n(\mathbb{C}) \mid \deg(\det \gamma) = q\}
\end{aligned}$$

Let $\tilde{g}$ be a representative of $gGL_n(\mathbb{C}[t]) \in \mathcal{G}_p$ and let $\tilde{h}$ be a representative of $hGL_n(\mathbb{C}[t]) \in \mathcal{G}_p$. Similarly, let $\gamma, \gamma' \in \Omega^{alg}U_n(\mathbb{C})$ be such that $\gamma \in \mathcal{G}_p$ and $\gamma' \in \mathcal{G}_q$. The multiplicative property of the determinant implies that the class of $\tilde{g} \cdot \tilde{g}$ in $GL_n(\mathbb{C}[t,t^{-1}])/GL_n(\mathbb{C}[t])$ lies in $\mathcal{G}_{p+q}$, similarly, the product $\gamma \cdot \gamma' \in \mathcal{G}_{p+q}$. So the multiplication by $\tilde{g}$ (respectively its inverse), or, in the loop picture, the multiplication by γ, induces bijections

$$\mathcal{G}_q \quad \underset{\cdot\gamma^{-1}}{\overset{\cdot\gamma}{\rightleftarrows}} \quad \mathcal{G}_{p+q}$$

To simplify the analysis, we consider throughout the following only $\mathcal{G}_0$.

5.3 Lattices in $\mathcal{G}_0$ and subspaces

Fix $N \in \mathbb{N}$ and let $V_N \subseteq \mathbb{C}[t,t^{-1}]$ be the complex subspace spanned by the basis vectors

$$V_N = \langle t^{-N}e_1, \dots t^{-N}e_n, \dots, e_1, \dots, e_n, \dots, t^{N-1}e_1, \dots, t^{N-1}e_n \rangle \subseteq \mathbb{C}[t,t^{-1}].$$

We have $\dim V_N = 2nN$, and we have a sequence of inclusions

$$V_1 \subset V_2 \subset V_3 \subset \dots \subset V_{N-1} \subset V_N \subset V_{N+1} \subset \dots \tag{5.4}$$

Denote by $G_{nN,2nN}$ the set of all subspaces of V_N of dimension nN. The set $G_{nN,2nN}$ is called the *Grassmann variety* of nN-dimensional subspaces of V_N.

We would like to have an increasing sequence for the Grassmann varieties $G_{nN,2nN}$ similar to the one in (5.4) for the vector spaces V_N. So given a subspace $W_N \in G_{nN,2nN}$, let $W_{N+1} \in G_{n(N+1),2n(N+1)}$ be the subspace of V_{N+1} obtained from $W_N \subset V_N \subset V_{N+1}$ by taking the span

$$\langle W_N, t^{N-1}e_1, \ldots, t^{N-1}e_n \rangle \subset V_{N+1}.$$

One checks directly that the induced map

$$G_{nN,2nN} \to G_{n(N+1),2n(N+1)}, \quad W_N \mapsto W_{N+1} \tag{5.5}$$

is injective. So we get the desired sequence of inclusions of Grassmann varieties:

$$G_{n,2n} \subset G_{2n,4n} \subset \ldots \subset G_{nN,2nN} \subset G_{n(N+1),2n(N+1)} \subset \ldots \tag{5.6}$$

and we set:

$$G_\infty := \bigcup_{N \geq 1} G_{nN,2nN}. \tag{5.7}$$

Let $\mathcal{L} \in \mathcal{G}_0$ be a lattice, so there exists an $N \in \mathbb{N}$ such that

$$t^N \mathcal{L}_0 \subseteq \mathcal{L} \subseteq t^{-N} \mathcal{L}_0, \tag{5.8}$$

and, by Lemma 5.2.1, the quotient $\mathcal{L}/t^N\mathcal{L}_0 \subset t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0$ is a subspace of dimension nN in the $2nN$-dimensional vector space $\mathcal{L}_0/t^N\mathcal{L}_0$.

After identifying the quotient $t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0$ with V_N:

$$t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0 \mathrel{\widehat{=}} \langle t^{-N}e_1, \ldots, e_1, \ldots, e_n, \ldots, t^{N-1}e_n \rangle = V_N, \tag{5.9}$$

we can hence associate to the lattice (or the associated algebraic loop in $\Omega^{alg}U_n(\mathbb{C})$) a point in $G_{nN,2nN}$. Now if (5.8) holds, then

$$t^{N+1}\mathcal{L}_0 \subseteq \mathcal{L} \subseteq t^{-N-1}\mathcal{L}_0$$

holds too, and the map in (5.5) is exactly the map

$$\begin{array}{ccc} \mathcal{L}/t^N\mathcal{L}_0 & \mapsto & \mathcal{L}/t^{N+1}\mathcal{L}_0 \\ \pitchfork & & \pitchfork \\ G_{nN,2nN} & \hookrightarrow & G_{n(N+1),2n(N+1)} \end{array}$$

which sends the subspace $\mathcal{L}/t^N\mathcal{L}_0$ of $t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0$ to the subspace $\mathcal{L}/t^{N+1}\mathcal{L}_0$ of $t^{-N-1}\mathcal{L}_0/t^{N+1}\mathcal{L}_0$. So the map

$$\mathcal{G}_0 \longrightarrow G_\infty = \bigcup_{N\geq 1} G_{nN,2nN}, \quad \mathcal{L} \mapsto (\mathcal{L}/t^N\mathcal{L}_0 \subset t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0)_{N\gg 0}$$

is well defined. The map is injective because if $t^N\mathcal{L}_0 \subset \mathcal{L}$ and $t^N\mathcal{L}_0 \subset \mathcal{L}'$, then $\mathcal{L}/t^N\mathcal{L}_0 = \mathcal{L}'/t^N\mathcal{L}_0$ implies $\mathcal{L} = \mathcal{L}'$.

It remains to describe the image of the map.

5.4 Subspaces invariant under multiplication with t

By definition, a $\mathbb{C}[t]$-lattice $\mathcal{L}$ is stable under multiplication with the variable t. Now multiplication by t induces a nilpotent endomorphism on $t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0$. To not confuse the variable t with the multiplication by t, let us write $\tilde{t}$ for the multiplication map. Then $\phi := \mathbb{1} + \tilde{t}$ induces a unipotent endomorphism of $t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0$. Since $\mathcal{L}$ is stable under multiplication with t, the subspace $\mathcal{L}/t^N\mathcal{L}_0 \subset t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0$ is stable under $\phi = \mathbb{1} + \tilde{t}$, which is equivalent to say that the point in $G_{nN,2nN}$ corresponding to $\mathcal{L}$ is a fixed point with respect to the action of ϕ on $G_{nN,2nN}$.

Lemma 5.4.1 *Let $W \in G_{nN,2nN}$ be an nN dimensional subspace of V_N. There exists a lattice $\mathcal{L} \in \mathcal{G}_0$ such that*

$$t^N\mathcal{L}_0 \subseteq \mathcal{L} \subseteq t^{-N}\mathcal{L}_0, \ \dim \mathcal{L}/t^N\mathcal{L}_0 = nN$$

and $W = \mathcal{L}/t^N\mathcal{L}_0$ with respect to the identification in (5.9) if and only if W is a fixed point with respect to the action of ϕ.

Denote by $\Omega_0^{alg}(U_n(\mathbb{C}))$ the subgroup of based loops such that $\det\gamma \in \mathbb{C}^*$. The proposition above implies:

Corollary 5.4.1 *We have natural bijections between the following loops, lattices and subspaces:*

$$\Omega_0^{alg}(U_n(\mathbb{C})) \stackrel{1:1}{\longleftrightarrow} \mathcal{G}_0 \stackrel{1:1}{\longleftrightarrow} \bigcup_{N\geq 1} G^{\phi}_{nN,2nN}.$$

Proof. So let $W \in G_{nN,2nN}$ be a fixed point with respect to the action of ϕ, and let $\mathcal{L}$ be the preimage of W with respect to the canonical map (of $\mathbb{C}$-vector spaces)

$$t^{-N}\mathcal{L}_0 \to t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0.$$

Since W is a $\phi = \mathbb{1} + \tilde{t}$-fixed point and $t^N\mathcal{L}_0$ is stable under multiplication by t, the subspace $\mathcal{L}$ is in fact a $\mathbb{C}[t]$-module. Now $\mathbb{C}[t]$ is a Euclidean ring, the module $\mathcal{L}$ is embedded as a submodule in $\mathbb{C}(t)^n$, and hence free of rank at most n. Since $t^N\mathcal{L}_0 \subset L$, it follows the module is free of rank n. Now the same arguments as in the proof of Lemma 5.2.1 show that, after possibly replacing $\mathcal{L}$ by $g\mathcal{L}$ for some $g \in GL(\mathbb{C}[t])$, the lattice can be represented by a matrix of the form

$$g\mathcal{L} \mathrel{\widehat{=}} \begin{pmatrix} t^{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t^{\lambda_n} \end{pmatrix}, \qquad (5.10)$$

where $\lambda_1 \leq \ldots \leq \lambda_n$. It remains to inspect what happens with the subspace W while changing from $\mathcal{L}$ to $g\mathcal{L}$. To replace $\mathcal{L}$ by $g\mathcal{L}$ implies we have to replace W by gW. Note that the lattices $t^N\mathcal{L}_0$ and $t^{-N}\mathcal{L}_0$ are $GL_n(\mathbb{C}[t])$-stable, so we have a well defined action of $GL_n(\mathbb{C}[t])$ on the finite dimensional complex vector space $t^{-N}\mathcal{L}_0/t^N\mathcal{L}_0$ by linear automorphisms. It follows that gW is again a subspace of dimension nN, and since the multiplication by t on $(\mathbb{C}[t,t^{-1}])^n$ commutes with the action of $GL_n(\mathbb{C}[t])$, the subspace $gW \in G_{nN,2nN}$ is again a fixed point with respect to the action of ϕ.

The dimension formula in Lemma 5.2.1 implies $\dim gW = nN$ if and only if $\Sigma\underline{\lambda} = 0$, and hence $g\mathcal{L} \in \mathcal{G}_0$. Now $\det \mathcal{L} = \det g\mathcal{L}$ implies $\mathcal{L} \in \mathcal{G}_0$ too, which finishes the proof of the lemma. •

5.5 The Grassmann variety

Let us start with the most simple example of a Graßmann variety, the projective space $\mathbb{P}^{n-1}$. Recall that the projective space $\mathbb{P}^{n-1}$ is defined as the set of all lines in $V = \mathbb{C}^n$. Another way to formulate the definition is to say that the projective space is the quotient $(V \setminus \{0\})/\sim$, where the equivalence relation is defined by: $v \sim v'$ if there exists an element $r \in \mathbb{C}^*$ such that $rv = v'$.

The definition of a Graßmann variety is a straight forward generalization of the above, only one has to replace lines, i.e. 1-dimensional subspaces, by d-dimensional subspaces.

Definition 5.5.1 *Let $1 \leq d < n$. The Graßmann variety $G_{d,n}$ is defined as the set of all d-dimensional subspaces in V.*

In particular, $G_{1,n} = \mathbb{P}^{n-1}$. To get a description of $G_{d,n}$ as a quotient similar to the description of the projective space above, let $U \in G_{d,n}$ be a d-dimensional subspace of $\mathbb{C}^n$. Fix a basis $\{v_1, \dots, v_d\}$ of U, then we can associate to U an $n \times d$ matrix $A = (a_{i,j})$ of rank d such that the j-th column consists of the coefficients of v_j with respect to the standard basis $\{e_1, \dots, e_n\}$ of V, *i.e.* $v_j = \sum_{i=1}^n a_{ij} e_i$.

Vice versa, to an $n \times d$ matrix $A \in M_{n,d}(\mathbb{C})$ of rank d one associates naturally the d-dimensional subspace U of V obtained as the span of the column vectors. In this language we can give a description of $G_{d,n}$ similar to that of the projective space above: let Z be the set of $n \times d$ matrices of rank strictly less than d, then $G_{d,n} = (M_{n,d}(\mathbb{C}) \setminus Z)/\sim$, where the equivalence relation is defined by: $A \sim A'$ if the column vectors span the same subspace of V.

Above we defined the relation "$\sim$" on $V \setminus \{0\}$ in terms of the group action of $\mathbb{C}^*$ on V. Here we can do the same by using the fact that $GL_d(\mathbb{C})$ acts transitively on the set of bases of a d-dimensional subspace:

$$G_{d,n} = (M_{n,d}(\mathbb{C}) \setminus Z)/\sim, \text{ where } A \sim A' \Leftrightarrow \begin{array}{c} A' = AC \text{ for some} \\ C \in GL_d(\mathbb{C}) \end{array}$$

For $d = 1$, this is exactly the description of the projective space $\mathbb{P}^{n-1} = G_{1,n}$ given above.

$G_{d,n}$ as homogeneous space.

Another very useful description of the Graßmann variety is that of $G_{d,n}$ as a homogeneous space. If $W \subset V$ is a d-dimensional subspace and $g \in SL_n(\mathbb{C})$, then $gW = \{gu \mid u \in W\}$ is again a d-dimensional subspace. In fact, given $W, W' \in G_{d,n}$, there exists always a $g \in SL_n(\mathbb{C})$ such that $gW = W'$.

Denote by $F_j \subset V$ the j-dimensional subspace $F_j = \langle e_1, e_2 \dots, e_j \rangle$ spanned by the first j elements of the standard basis. Then we can identify $G_{d,n}$

with the coset space $SL_n(\mathbb{C})/P_d$, where P_d is the isotropy group of the d-dimensional subspace F_d. Now $g \in SL_n(\mathbb{C})$ is an element of P_d if and only if $ge_j \in F_d$ for $1 \leq j \leq d$, and hence:

$$G_{d,n} = SL_n(\mathbb{C})/P_d\,, \text{ where } P_d = \left\{ A \in SL_n(\mathbb{C}) \;\middle|\; A = \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \right\}.$$

Plücker coordinates.

To endow the Graßmann variety with the structure of an algebraic variety, we will identify $G_{d,n}$ with a subset of the projective space $\mathbb{P}(\Lambda^d V)$. A first step in this direction is the introduction of Plücker coordinates, which can be viewed as linear functions on $\Lambda^d V$ as well as multilinear alternating functions on $M_{n,d}(\mathbb{C})$.

Remark 5.5.1 There are several ways to introduce the vector space $\Lambda^d V$, the following uses a universal property: The d-th exterior power of a finite dimensional vector space is a pair $(\Lambda^d V, \iota)$ consisting of a vector space $\Lambda^d V$ and a a the unique (up to unique isomorphism) multilinear alternating map

$$\iota : \underbrace{V \times \cdots \times V}_{d} \to \Lambda^d V$$

such that for any multilinear and alternation map $\phi : \underbrace{V \times \cdots \times V}_{d} \to W$ in some vector space W, there exists a unique linear map $\tilde{\phi} : \Lambda^d V \to W$, such that the diagram

$$\begin{array}{ccc} \underbrace{V \times \cdots \times V}_{d} & \to^{\phi} & W \\ \downarrow^{\iota} & \nearrow^{\tilde{\phi}} & \\ \Lambda^d V & & \end{array}$$

Or, for short: studying multilinear, alternating maps on $\underbrace{V \times \cdots \times V}_{d}$ is the same a studying linear maps on $\Lambda^d V$.

Let us assume such a vector space $\Lambda^d V$ and the map ι exists. The universal property has some immediate consequences: let $\{e_1, \ldots, e_n\}$ be the standard basis of $V = \mathbb{C}^n$. The multilinear map $\iota : V \times \cdots \times V_d \to \Lambda^d V$ is completely determined by the images $\iota(e_{i_1}, \ldots e_{i_d})$ of the tuples of basis elements, and hence $\Lambda^d V$ has to be spanned by the images $\iota(e_{i_1}, \ldots, e_{i_d})$,

$1 \leq i_1, \ldots, i_d \leq n$. Now above we ask in fact more: the map ι has the additional property of being alternating, so whenever $\sigma \in \mathfrak{S}_d$, then

$$\iota(e_{i_{\sigma(1)}}, \ldots, e_{i_{\sigma(d)}}) = \mathrm{sgn}(\sigma)(e_{i_1}, \ldots, e_{i_d}).$$

It is now easy to see that then the images of the ordered tuples $\iota(e_{i_1}, \ldots, e_{i_d})$, $1 \leq i_1 < \ldots < i_d \leq n$ form indeed a basis of $\Lambda^d V$. The usual notation one uses is the wedge product:

$$e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_d} := \iota(e_{i_1}, \ldots, e_{i_d}), \quad \text{where } 1 \leq i_1 < \ldots < i_d \leq n.$$

and this basis is called the standard basis of $\Lambda^d V$.

More generally, for a d-tuple of vectors we write $v_1 \wedge v_2 \wedge \ldots \wedge v_d := \iota(v_1, \ldots, v_d)$, and using again the properties that ι is multilinear and alternating, one gets the following explicit formula for ι:

$$v_1 \wedge v_2 \wedge \ldots \wedge v_d = \sum_{1 \leq i_1 < \cdots < i_d \leq n} p_{i_1,\ldots,i_d}(v_1 | \ldots | v_d) e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_d},$$

where $(v_1 | \ldots | v_d)$ is the $n \times d$-matrix having the vectors $v_1, \ldots, v_d$ as columns, and $p_{i_1,\ldots,i_d}(v_1 | \ldots | v_d)$ is the minor of the matrix formed by the determinant of the submatrix formed by rows $i_1, i_2, \ldots, i_d$.

Definition 5.5.2 Let $I_{d,n} := \{\underline{i} = (i_1, \ldots, i_d) | 1 \leq i_1 < \cdots < i_d \leq n\}$ be the set of all strictly increasing sequences of length d between 1 and n. For $\underline{i} = (i_1, \ldots, i_d) \in I_{d,n}$ we write $e_{\underline{i}} = e_{i_1} \wedge \cdots \wedge e_{i_d}$. We define a partial order "$\geq$" on $I_{d,n}$ as follows: $\underline{i} \geq \underline{j} \Leftrightarrow i_t \geq j_t$ for all $t = 1, \ldots, d$.

So the standard basis of $\Lambda^d V$ can be written as $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$. Denote by $\{p_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ the dual basis of $(\Lambda^d V)^*$, i.e., $p_{\underline{i}}(e_{\underline{j}}) = \delta_{\underline{i},\underline{j}}$.

Definition 5.5.3 *The linear functions $p_{\underline{i}}$, $\underline{i} \in I_{d,n}$, on $\Lambda^d V$ are called Plücker coordinates.*

By the definition of the d-fold wedge product the space of linear functions on $\Lambda^d V$ can be naturally identified with the space of multilinear alternating functions on d-copies of V, i.e., on $M_{d,n}(\mathbb{C}) = \underbrace{V \times \ldots \times V}_{d \text{ times}}$.

Remark 5.5.2 We use the same name *Plücker coordinates* and the same symbol $p_{\underline{i}}$ for the *linear functions on $\Lambda^d V$* as well as the corresponding *multilinear alternating function on the space $M_{n,d}(\mathbb{C})$*.

To make this relationship more explicit, recall that we have a natural map, the exterior product map:

$$\begin{array}{rcc} \pi_d : M_{n,d}(\mathbb{C}) & \to & \Lambda^d V \\ A = (v_1, \ldots, v_d) & \mapsto & v_1 \wedge \cdots \wedge v_d \end{array}. \tag{5.11}$$

Here $v_1, \ldots, v_d$ are the column vectors of the matrix A. If we express the product $v_1 \wedge \cdots \wedge v_d$ as a linear combination of the elements of the canonical basis, then, by the definition of the dual basis, we have

$$v_1 \wedge \cdots \wedge v_d = \sum_{\underline{i} \in I_{d,n}} p_{\underline{i}}(A) e_{\underline{i}}.$$

The alternating multilinear function on $M_{n,d}(\mathbb{C})$ associated to $p_{\underline{i}}$ is just the $\underline{i}$-th coordinate of the linear combination above, i.e., it is the composition $p_{\underline{i}} \circ \pi_d$. So by abuse of notation we write just $p_{\underline{i}}(A)$ instead of $p_{\underline{i}}(\pi_d(A))$.

$GL_n(\mathbb{C})$-action on $\Lambda^d V$

Given an element $g \in GL_n(\mathbb{C})$, we define a map which is linear in each factor:

$$\begin{array}{ccc} \underbrace{V \times \cdots \times V}_{d} & \to^{g\cdot} & \underbrace{V \times \cdots \times V}_{d} \\ (v_1, \ldots, v_d) & \mapsto & (gv_1, \ldots, gv_d). \end{array}$$

Combining the map with ι, we get a multilinear and alternating map

$$\begin{array}{ccc} \underbrace{V \times \cdots \times V}_{d} & \to^{\iota \circ (g\cdot)} & \Lambda^d V \\ (v_1, \ldots, v_d) & \mapsto & (gv_1) \wedge \ldots \wedge (gv_d). \end{array}$$

Now the universal property of $\Lambda^d V$, we get hence a commutative diagram

$$\begin{array}{ccc} \underbrace{V \times \cdots \times V}_{d} & \to^{\iota \circ (g\cdot)} & \Lambda^d V \\ \downarrow^{\iota} & \nearrow^{\widetilde{\iota \circ (g\cdot)}} & \\ \Lambda^d V & & \end{array}$$

One often writes just $\wedge g$ for the homomorphism $\widetilde{\iota \circ (g\cdot)} : \Lambda^d V \to \Lambda^d V$. So having the group action of $GL_n(\mathbb{C})$ on $V = \mathbb{C}^n$, the universal property of $\Lambda^d V$

has as a consequence that we get for every isomorphism $g : V \to V,\ v \mapsto gv$, an induced isomorphism

$$\wedge g : \Lambda^d V \to \Lambda^d V, \quad v_1 \wedge \ldots \wedge v_d \mapsto (gv_1) \wedge \ldots \wedge (gv_d).$$

Now by definition we have

$$\begin{array}{rcl} \wedge(gh)(v_1 \wedge \ldots \wedge v_d) & = & ((gh)v_1) \wedge \ldots \wedge ((gh)v_d) \\ & = & \wedge g\big((hv_1) \wedge \ldots \wedge (hv_d)\big) \\ & = & (\wedge g \circ \wedge h)(v_1 \wedge \ldots \wedge v_d) \end{array}$$

Since $\Lambda^d V$ is spanned by the "*pure*" wedges $v_1 \wedge \ldots \wedge v_d$, this implies $\wedge(gh) = \wedge g \circ \wedge h$, or, in other words,

$$\wedge : GL_n(\mathbb{C}) \longrightarrow GL(\Lambda^d V), \quad g \mapsto \wedge g,$$

is a group homomorphism. Here is another name for this: it is a *representation.*

$GL_n(\mathbb{C})$-action on $\mathbb{P}(\Lambda^d V)$

To go from a vector space U to the associated projective space $\mathbb{P}(U)$ means to pass from a non-zero vector $u \in U$ to the equivalence class of the vector: $[u] = \{\lambda u \mid \lambda \in \mathbb{C}^*\}$. The linearity of a vector space isomorphism $\phi : U \to U$ implies that we get an induced map $[\phi] : \mathbb{P}(U) \to \mathbb{P}(U)$, $[u] \mapsto [\phi(u)]$.

Given $g \in GL_n(\mathbb{C})$, it follows that the isomorphism $v \mapsto gv$ on V induces an isomorphism $\wedge g : v_1 \wedge \ldots \wedge v_d \mapsto gv_1 \wedge \ldots \wedge gv_d$, which in turn induces a map $[\wedge g] : [v_1 \wedge \ldots \wedge v_d] \mapsto [gv_1 \wedge \ldots \wedge gv_d]$ on $\mathbb{P}(\Lambda^d V)$.

Since $\wedge : GL_n(\mathbb{C}) \longrightarrow GL(\Lambda^d V)$ is a representation, it is easy to see that

$$GL_n(\mathbb{C}) \times \mathbb{P}(\Lambda^d V) \to \mathbb{P}(\Lambda^d V), [\sum_{j=1}^r v_{j,1} \wedge \ldots \wedge v_{j,d}] \mapsto [\sum_{j=1}^r gv_{j,1} \wedge \ldots \wedge gv_{j,d}].$$

defines a group action.

A *pure wedge* in $\Lambda^d V$ is an element which can be written as $v_1 \wedge \ldots \wedge v_d$. Note that not all elements in $\Lambda^d V$ can be written in this way (Exercise!). The action of $GL_n(\mathbb{C})$ stabilizes this set because

$$(\wedge g)(v_1 \wedge \ldots \wedge v_d) = (gv_1) \wedge \ldots \wedge (gv_d)$$

is again a pure wedge. Actually, the set of pure wedges decomposes into two orbits: one consists just of one element: $\{0\}$, the other set is the orbit $\wedge(GL_n(\mathbb{C}))e_1 \wedge \ldots \wedge e_d$. Indeed, the pure wedge $v_1 \wedge \ldots \wedge v_d$ is not equal to zero if and only if the vectors $v_1, \ldots, v_d$ are linearly independent. So the later can be extended to a basis of V, the corresponding matrix g has the property $g(e_1 \wedge \ldots \wedge e_d) = v_1 \wedge \ldots \wedge v_d$.

A *pure wedge* in $\mathbb{P}(\Lambda^d V)$ is an element which can be written as $[v_1 \wedge \ldots \wedge v_d]$. The considerations above show that the set of all pure wedges is stable with respect to the $GL_n(\mathbb{C})$-action on $\mathbb{P}(\Lambda^d V)$. Indeed, the set is just one orbit:

$$GL_n(\mathbb{C}) \cdot [e_1 \wedge \ldots \wedge e_d] = \{[v_1 \wedge \ldots \wedge v_d] \in \mathbb{P}(\Lambda^d V) \mid v_1, \ldots, v_d \in V \ {}_{\text{independent}}^{\text{linearly}}\}$$

Plücker embedding.

Our next step is to identify the Graßmann variety with the subset of pure wedges in the projective space $\mathbb{P}(\Lambda^d V)$.

For $A \in M_{n,d}(\mathbb{C})$ of rang d let $v_1, \ldots, v_d \in k^n$ be the column vectors, let $W \subset V$ be the span of these column vectors and let $u_1, \ldots, u_d \in W$. Denote by $C = (c_{i,j})$ the $d \times d$-matrix expressing the u_j as linear combinations of the v_i. i.e., $u_j = \sum_{i=1}^{d} c_{i,j} v_i$. The exterior product is alternating, so we get $v_1 \wedge \ldots \wedge v_d = (\det C) u_1 \wedge \ldots \wedge u_d$. As a consequence we see that the exterior product map induces a well defined map:

$$\pi : G_{d,n} = ((M_{n,d}(\mathbb{C}) \setminus Z)/\sim) \longrightarrow \mathbb{P}(\Lambda^d V)$$

called the *Plücker embedding*. We have a left action of $GL_n(\mathbb{C})$ on $M_{n,d}(\mathbb{C})$ defined by $g(v_1, \ldots, v_d) = (gv_1, \ldots, gv_d)$, and we have a natural action of $GL_n(\mathbb{C})$ on $\Lambda^d V$ given by $(\wedge g)(v_1 \wedge \cdots \wedge v_d) = (gv_1) \wedge \cdots \wedge (gv_d)$. It follows that the exterior product map $\pi_d : M_{n,d}(\mathbb{C}) \to \Lambda^d V$ is equivariant with respect to these $GL_n(\mathbb{C})$-actions, and hence so is the Plücker embedding. The term *embedding* is justified because:

Proposition 5.5.1 *The Plücker map $\pi : G_{d,n} \to \mathbb{P}(\Lambda^d V)$ is injective.*

Proof. Let F_d be the d-dimensional subspace of V spanned by $e_1, \ldots, e_d$. By the homogeneity of the $GL_n(\mathbb{C})$-action on $G_{d,n}$, it is sufficient to show if $\pi(W) = \pi(F_d)$, then $W = F_d$.

So suppose $\pi(W) = \pi(F_d)$ and let $\{v_1, \ldots, v_d\}$ be a basis of W. Denote by $A \in M_{n,d}(\mathbb{C})$ the corresponding matrix. Since $[\pi_d(A)] = [e_1 \wedge \ldots \wedge e_d]$, we can choose the basis such that $\pi_d(A) = e_1 \wedge \ldots \wedge e_d$. It follows that the submatrix $A_{1,\ldots,d}$ consisting of the first d-rows of A has determinant one, so by replacing A by $A \cdot A^{-1}_{1,\ldots,d}$ if necessary we can (and will) assume that the submatrix of A consisting of the first d rows is the $d \times d$ identity matrix.

Now all $d \times d$ minors except $p_{1,2,\ldots,d}(A)$ vanish. In particular, for $i > d$ we have $\pm a_{i,j} = p_{1,\ldots,j-1,j+1,\ldots,d,i}(A) = 0$ and hence $W = F_d$. •

Projective varieties

The first definition of an affine variety was the following: an affine variety X in a finite dimensional vector space V is a subset such that there exists a set of polynomials $I \subset \mathbb{C}[V]$ and $X = \{v \in V \mid f(v) = 0 \forall\, f \in I\}$. Such a definition does not make sense for the projective space $\mathbb{P}(V) = (V \setminus \{0\})/\sim$ because a point in $\mathbb{P}(V)$ is line in V. We write $[a_1 : \ldots : a_n]$ for the line spanned by the vector $\sum_{i=1}^n a_i e_i$. Note that $[\lambda a_1 : \ldots : \lambda a_n] = [a_1 : \ldots : a_n]$ as points in the projective space but in general one has $f(a_1, \ldots, a_n) \neq f(\lambda a_1, \ldots, \lambda a_n)$ for a polynomial $f \in \mathbb{C}[V]$. So there is no way of seeing polynomials as functions on $\mathbb{P}(V)$.

But if a polynomial f is homogeneous, say of degree m, then

$$f(\lambda a_1, \ldots, \lambda a_n) = \lambda^m f(a_1, \ldots, a_n).$$

In particular, $f(a_1, \ldots, a_n) = 0$ if and only if $f(\lambda a_1, \ldots, \lambda a_n)$, and one just writes $f([a_1 : \ldots : a_n]) = 0$. So the following definition still makes sense:

Definition 5.5.4 A *closed set* $X \subset \mathbb{P}(V)$ for a finite dimensional vector space V is a subset such that there exists a set of homogeneous polynomials $I \subset \mathbb{C}[V]$ such that

$$X = \{[v] \in \mathbb{P}(V) \mid f([v]) = 0 \,\forall\, f \in I,\ f \textit{ homogeneous}\} \tag{5.12}$$

As in the affine case, one shows that a finite union and an arbitrary intersection of closed sets are closed sets, and the empty set as well as $\mathbb{P}(V)$ are closed sets. So it makes sense to define a topology on $\mathbb{P}(V)$ having as closed sets exactly the sets as in Definition 5.5.5 and as open sets exactly the complements of the closed sets. This topology is called the Zariski topology on $\mathbb{P}(V)$.

Definition 5.5.5 A *projective variety* $X \subset \mathbb{P}(V)$ for a finite dimensional vector space V is a closed subset of $\mathbb{P}(V)$. The variety is endowed with the induced Zariski topology.

Theorem 5.5.1 *The Grassmann variety $G_{d,n} \subset \mathbb{P}(\Lambda^d V)$ is a projective variety.*

A proof can be found in the Appendix.

Corollary 5.5.1 $G_{d,n} = GL_n(\mathbb{C}) \cdot [e_1 \wedge e_2 \wedge \ldots \wedge e_d] \subset \mathbb{P}(\Lambda^d V)$.

5.6 $\Omega_0^{alg} U_n(\mathbb{C})$ as Ind-variety

5.6.1 Ind-varieties

By an *ind-variety* we mean a set X together with a filtration

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$$

such that

i) $\bigcup_{j \geq 0} X_n = X$,

ii) each X_n is a finite dimensional complex variety (affine or projective) such that the inclusion $X_n \hookrightarrow X_{n+1}$ is a closed embedding.

Remark 5.6.1 In the following we omit some technicalities, for example we omit the proof that the maps we consider are closed embeddings.

We define the Zariski topology on an ind-variety X by declaring a set $U \subset X$ as open if and only if $U \cap X_n$ is Zariski-open in X_n for all n. A subset $Z \subseteq X$ is closed if and only if $Z \cap X_n$ is closed in X_n for each n (Exercise).

We have seen a first example of such a construction in section 4.2, where we have constructed $L^{alg} GL_n(\mathbb{C})$ as an affine ind-variety. The filtration described on $L^{alg} GL_n(\mathbb{C})$ in (4.1) is exactly a filtration with the properties above.

Next recall that the inclusions described in (5.5):

$$\begin{array}{ccc} \mathbb{P}(\Lambda^{nN} V_N) & & \mathbb{P}(\Lambda^{n(N+1)} V_{N+1}) \\ \cup & & \cup \\ G_{nN,2nN} & \to & G_{n(N+1),2n(N+1)} \\ \Cup & & \Cup \\ W_N & \mapsto & W_{N+1}; \end{array}$$

they induce a sequence of inclusions (see (5.6)):

$$\begin{array}{ccccccccc}
\mathbb{P}(\Lambda^n V_1) & & \mathbb{P}(\Lambda^{2n} V_2) & \dots & & \mathbb{P}(\Lambda^{nN} V_N) & & \dots \\
\cup & & \cup & & & \cup & & \cup \\
G_{n,2n} & \subset & G_{2n,4n} & \dots & \subset & G_{nN,2nN} & \subset & \dots \\
\cup & & \cup & & & \cup & & \cup \\
G^{\phi}_{n,2n} & \subset & G^{\phi}_{2n,4n} & \dots & \subset & G^{\phi}_{nN,2nN} & \subset & \dots
\end{array}$$

The condition of being a fixed point is a closed condition (Exercise: $[\phi]$ is continuous in the Zariski topology), so we see that (see Corollary 5.4.1)

Proposition 5.6.1

$$\Omega_0^{alg} U_n(\mathbb{C}) = \bigcup_{N \geq 1} G^{\phi}_{nN,2nN}$$

is an Ind-projective variety.

Remark 5.6.2 Of course, it remains to prove that the inclusions are closed embeddings. We leave this as an exercise to the reader, or suggest as an alternative to have a look at the book *Kac-Moody groups, their flag varieties and representation theory* by Shrawan Kumar.

5.6.2 An approach towards G_∞ using $GL_\infty(\mathbb{C})$

Consider again the infinite dimensional vector space $V = (\mathbb{C}[t, t^{-1}])^n$, endowed with the standard basis given by the vectors $\{e_i t^j \mid 1 \leq i \leq n,\ j \in \mathbb{Z}\}$.

Subspaces

For a finite dimensional vector space U, the wedge product is a tool to think of a d-dimensional sub-vector space $W \subset U$ as a point in the projective space $\mathbb{P}(\Lambda^d U)$. To achieve something similar in the infinite dimensional case, recall that we have been looking at $\mathbb{C}[t]$-lattices $\mathcal{L}$ in $(\mathbb{C}[t, t^{-1}])^n$ with the property:

$$\exists N \in \mathbb{N} : t^N \mathcal{L}_0 \subset \mathcal{L} \subset t^{-N} \mathcal{L}_0.$$

and we investigated the set of all subspaces $G_{nN,2nN}$ of $t^{-N}\mathcal{L}_0 / t^N \mathcal{L}_0$ of dimension nN. Now a subspace $U \subset t^{-N}\mathcal{L}_0 / t^N \mathcal{L}_0$ of dimension nN is the same as a subspace $\mathcal{U} \subset (\mathbb{C}[t, t^{-1}])^n$ with the property

$$t^N \mathcal{L}_0 \subset \mathcal{U} \subset t^{-N} \mathcal{L}_0, \quad \dim \mathcal{U} / t^N \mathcal{L}_0 = nN.$$

So if one wants to fix a basis $\mathbb{B}_{\mathcal{U}}$ of $\mathcal{U}$ as $\mathbb{C}$-vectorspace, then we can divide such a basis into two parts: one is the infinite part consisting of the standard basis of $t^N\mathcal{L}_0$:

$$\mathbb{B}^2_{\mathcal{U},N} := \{t^N e_1, t^N e_2, \ldots, t^N e_n, t^{N+1}e_1, t^{N+1}e_2, \ldots, t^{N+1}e_n, \ldots\},$$

the remaining finite part $\mathbb{B}^1_{\mathcal{U},N} = \{v_1, \ldots, v_{nN}\}$ is obtained by completing $\mathbb{B}^2_{\mathcal{U},N}$ to a $\mathbb{C}$-basis $\mathbb{B}^1_{\mathcal{U}} \cup \mathbb{B}^2_{\mathcal{U}}$ for $\mathcal{U}$. We associate to this basis the semi-infinite wedge product

$$v_1 \wedge \ldots \wedge v_{nN} \wedge t^N e_1 \wedge t^N e_2 \wedge \ldots \wedge t^N e_n \wedge t^{N+1}e_1 \wedge \ldots \wedge t^{N+1}e_n \wedge \ldots \quad (5.13)$$

More on semi-infinite wedge products

The notion of a semi-infinite wedge product we introduce now is neither really formal nor really standard, we use an ad hoc approach adapted to our needs. In the following we need sometimes an enumeration of the elements of the standard basis of V:

$$b_{i+nj} := e_i t^j, \quad 1 \le i \le n,\ j \in \mathbb{Z}.$$

Definition 5.6.1 The *semi-infinite wedge product* $\Lambda^{\frac{\infty}{2}}V$ of V is the (infinite dimensional) vector space having as basis the vectors

$$\underbrace{b_{k_1} \wedge b_{k_2} \wedge \ldots \wedge b_{k_\ell}}_{head: k_1<k_2<\ldots<k_\ell<1+np} \wedge \underbrace{b_{1+np} \wedge b_{2+np} \wedge \ldots \wedge b_{n+np} \wedge b_{1+n(p+1)} \wedge \ldots}_{tail=stable\, part}$$

i.e., a basis vector has a *head*, which is a finite wedge product, and a *tail*, also called the *stable part*. The tail is an infinite wedge product, it is called the stable part because it is the wedge product of consecutive elements with respect to the enumeration above of the elements of the standard basis of V. The length of the head can be chosen so that the stable part starts with an basis element of the form $b_{1+np} = e_1 t^p$. So by translating the b_{i+nj} back into $e_i t^j$, a tail looks like:

$$e_1 t^p \wedge e_2 t^p \wedge \ldots \wedge e_n t^p \wedge e_1 t^{p+1} \wedge e_2 t^{p+1} \wedge \ldots \wedge e_n t^{p+1} \wedge \ldots$$

The head looks like

$$e_{i_1} t^{j_1} \wedge e_{i_2} t^{j_2} \wedge \ldots \wedge e_{i_\ell} t^{j_\ell}, \quad i_1 + nj_1 < i_2 + nj_2 < \ldots < i_\ell + nj_\ell < 1 + np$$

Example 5.6.1 Suppose $n = 2$, below we give as an example two basis elements. The first is an example emphasizing on the fact that a tail is not unambiguously defined:

$$\underbrace{e_1 \wedge e_2 \wedge te_1 \wedge te_2 \wedge t^2e_1 \wedge t^2e_2 \wedge t^3e_1 \wedge t^3e_2 \wedge t^4e_1 \wedge t^4e_2 \wedge \ldots}_{tail}$$

$$= e_1 \wedge e_2 \wedge \underbrace{te_1 \wedge te_2 \wedge t^2e_1 \wedge t^2e_2 \wedge t^3e_1 \wedge t^3e_2 \wedge t^4e_1 \wedge t^4e_2 \wedge \ldots}_{tail}$$

$$= e_1 \wedge e_2 \wedge te_1 \wedge te_2 \wedge \underbrace{t^2e_1 \wedge t^2e_2 \wedge t^3e_1 \wedge t^3e_2 \wedge t^4e_1 \wedge t^4e_2 \wedge \ldots}_{tail}$$

and

$$\underbrace{t^{-3}e_1 \wedge t^{-2}e_1 \wedge t^{-2}e_2 \wedge t^{-1}e_1}_{head} \wedge \underbrace{t^2e_1 \wedge t^2e_2 \wedge t^3e_1 \wedge t^3e_2 \wedge t^4e_1 \wedge t^4e_2 \wedge \ldots}_{tail}.$$

To be able to talk also about semi-infinite wedge products of finite linear combinations of the e_it^j, we add the following rules, which are of course inspired by what happens in the finite dimensional case:

1) we allow a finite number of the wedge factors to be arbitrary vectors in $V = (\mathbb{C}[t, t^{-1}])^n$;

2) the wedge product is alternating: switching two consecutive wedge factors changes the sign of the vector;

3) multilinearity: if $v = \sum a_{i,j}e_it^j$, then

$$\ldots \wedge v \wedge \ldots = \sum a_{i,j}(\ldots \wedge e_it^j \wedge \ldots).$$

The elements of the form $v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge$ (*stable part*) are called *pure semi-infinite wedge products*.

More on subspaces

Let us again consider the semi-infinite wedge product in (5.13). If we choose a different N' such that $t^{N'}\mathcal{L}_0 \subset \mathcal{U} \subset t^{-N'}\mathcal{L}_0$, then we may assume without loss of generality that $N' \geq N$. Let $\mathbb{B}^2_{\mathcal{U},N'}$ be defined as above and let $\mathbb{B}^1_{\mathcal{U},N'} = \{u_1, \ldots, u_{nN'}\}$ be such that $\mathbb{B}^1_{\mathcal{U},N'} \cup \mathbb{B}^2_{\mathcal{U},N'}$ is a $\mathbb{C}$-basis for $\mathcal{U}$, and we associate to this basis the semi-infinite wedge product:

$$u_1 \wedge \ldots \wedge u_{nN'} \wedge t^{N'}e_1 \wedge t^{N'}e_2 \wedge \ldots \wedge t^{N'}e_n \wedge t^{N'+1}e_1 \wedge \ldots \wedge t^{N'+1}e_n \wedge \ldots$$

Note that both are of the form

$$\underbrace{\ldots\wedge\ldots\wedge\ldots}_{head}\wedge\underbrace{t^{N'}e_1\wedge t^{N'}e_2\wedge\ldots\wedge t^{N'}e_n\wedge t^{N'+1}e_1\wedge\ldots\wedge t^{N'+1}e_n\wedge\ldots}_{tail}$$

so both have a head of length nN', where they may differ, but both have the same tail. The head may hence be seen as the wedge product of two bases of the subspace $U=\mathcal{U}/t^{N'}\mathcal{L}\subseteq t^{-N'}\mathcal{L}_0/t^{N'}\mathcal{L}_0$. In particular, the two heads differ only by a non-zero constant plus a sum of terms involving basis elements of the form e_it^j for some $j\geq N'$. Having in mind the rules mentioned above, one sees that these extra summands will vanish after taking the wedge product on both sides with the tail above. It follows immediately that

$$\begin{aligned}&v_1\wedge\ldots\wedge v_{nN}\wedge t^Ne_1\wedge\ldots\wedge t^Ne_n\wedge t^{N+1}e_1\wedge\ldots\wedge t^{N+1}e_n\wedge\ldots\\&=cu_1\wedge\ldots\wedge u_{nN'}\wedge t^{N'}e_1\wedge\ldots\wedge t^{N'}e_n\wedge t^{N'+1}e_1\wedge\ldots\wedge t^{N'+1}e_n\wedge\ldots\end{aligned}$$

for some non-zero complex number $c\in\mathbb{C}$. So in $\mathbb{P}(\Lambda^{\frac{\infty}{2}}V)$ we have:

$$\begin{aligned}&[v_1\wedge\ldots\wedge v_{nN}\wedge t^Ne_1\wedge\ldots\wedge t^Ne_n\wedge t^{N+1}e_1\wedge\ldots\wedge t^{N+1}e_n\wedge\ldots]\\&=[u_1\wedge\ldots\wedge u_{nN'}\wedge t^{N'}e_1\wedge\ldots\wedge t^{N'}e_n\wedge t^{N'+1}e_1\wedge\ldots\wedge t^{N'+1}e_n\wedge\ldots].\end{aligned}$$

So we can associate to a subspace $\mathcal{U}\subset(\mathbb{C}[t,t^{-1}])^n$ with the property $\exists N\in\mathbb{N}$ such that $t^N\mathcal{L}_0\subset\mathcal{U}\subset t^{-N}\mathcal{L}_0$ and $\dim\mathcal{U}/t^N\mathcal{L}_0=nN$ a point in $\mathbb{P}(\Lambda^{\frac{\infty}{2}}V)$.

The degree zero part

To get a connection between pure semi-infinite wedge products and subspaces having the property above, we introduce the notion of a degree of a pure semi-infinite wedge product. In the following let

$$v_0=e_1\wedge\ldots\wedge e_n\wedge t^1e_1\wedge\ldots\wedge t^1e_n\wedge t^2e_1\wedge\ldots\wedge t^{j+2}e_n\wedge\ldots$$

and, more generally, for $1\leq i\leq n$ and $j\in\mathbb{Z}$, set $q=[(j+1)n+1]-[i+1]$ and denote by v_q the vector

$$v_q=\underbrace{t^je_1\wedge\ldots\wedge t^je_i}_{head}\wedge\underbrace{t^{j+1}e_1\wedge\ldots\wedge t^{j+1}e_n\wedge t^{j+2}e_1\wedge\ldots\wedge t^{j+2}e_n\wedge\ldots}_{tail}.$$

The vector v_q is sometimes called a *vacuum vector*. These are the semi-infinite wedge products of reference for the notion of a degree. In the following let

$\Lambda_q^{\frac{\infty}{2}} V$ be the linear span in $\Lambda^{\frac{\infty}{2}} V$ of all pure semi-infinite wedge products that coincide with v_q after finitely many factors. Or, in other words, for w to be an element in $\Lambda_q^{\frac{\infty}{2}} V$ is equivalent to the existence of a decomposition for both: $v_1 = v_{q,head} \wedge v_{q,tail}, w = w_{head} \wedge w_{tail}$ such that the heads have the same length and the tails coincide. In this case we say that w has degree q.

Degree 0 wedges

By using the calculation rules for semi-infinite wedge products, we may think of $\Lambda_0^{\frac{\infty}{2}} V$ as the union

$$\bigcup_{N \in \mathbb{N}} \underbrace{(\Lambda^{nN} V_N)}_{head} \wedge \underbrace{e_1 t^{N+1} \wedge e_2 t^{N+1} \wedge \ldots \wedge e_n t^{N+1} \wedge e_1 t^{N+2} \wedge \ldots}_{tail}$$

Note that this point of view coincides with our subspace / lattice construction. Recall the inclusion of Grassmann varieties in (5.5): given a subspace $W_N \in G_{nN,2nN}$, let $W_{N+1} \in G_{n(N+1),2n(N+1)}$ be the subspace of V_{N+1} obtained from $W_N \subset V_N \subset V_{N+1}$ by taking the span

$$\langle W_N, t^{N+1} e_1, \ldots, t^{N+1} e_n \rangle \subset V_{N+1}.$$

So the corresponding point in $\mathbb{P}(\Lambda^{n(N+1)} V_{N+1})$ is

$$[(\Lambda^{nN} W_N) \wedge t^{N+1} e_1 \wedge \ldots \wedge t^{N+1} e_n].$$

Now in $\mathbb{P}(\Lambda_0^{\frac{\infty}{2}} V)$ both give rise to the same point:

$$\begin{aligned} &[(\Lambda^{nN} W_N) \wedge t^{N+1} e_1 \wedge \ldots \wedge t^{N+1} e_n \wedge t^{N+2} e_1 \wedge \ldots \wedge t^{N+2} e_n \wedge \ldots] \\ &= [(\Lambda^{n(N+1)} W_{N+1} \wedge t^{N+2} e_1 \wedge \ldots \wedge t^{N+2} e_n \wedge \ldots] \end{aligned}$$

It follows:

Lemma 5.6.1

$$G_\infty = \{[v] \in \mathbb{P}(\Lambda_0^{\frac{\infty}{2}} V) \mid v \text{ is a pure semi-infinite wedge product}\}$$

The group $GL_\infty(\mathbb{C})$

Let $GL_\infty(\mathbb{C})$ be the set of complex $\mathbb{Z}\times\mathbb{Z}$ matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where A is an invertible $\ell\times\ell$-matrix for some $\ell\in\mathbb{N}$. These matrices are invertible and the product of two of them is again of the same form, so these matrices form in a natural way a group. Another way of describing this group is to say that it consists of $\mathbb{Z}\times\mathbb{Z}$-matrices which have only a finite number of off-diagonal entries, all but a finite number of the diagonal entries are equal to one, and the matrix is invertible.

Since $ge_it^j = e_it^j$ for $g\in GL_\infty(\mathbb{C})$ except for a finite number of basis elements, it makes sense to define an action on a pure semi-infinite wedge product by setting:

$$gv = \underbrace{(gv_1)\wedge\ldots\wedge(gv_\ell)}_{head}\wedge\underbrace{(ge_1t^N)\wedge\ldots\wedge(ge_nt^N)\wedge\ldots}_{tail}, \tag{5.14}$$

More precisely, given a pure semi-infinite wedge product $v\in\Lambda_0^{\frac{\infty}{2}}V$, then there exists an $N\in\mathbb{N}$ such that

$$v\in(\Lambda^{nN}V_N)\wedge e_1t^{N+1}\wedge\ldots\wedge e_nt^{N+1}\wedge e_1t^{N+2}\wedge\ldots\subset\Lambda_0^{\frac{\infty}{2}}V,$$

a finite dimensional subspace which can be identified with $\Lambda^{nN}V_N$. Moreover, by replacing N by a larger positive integer if necessary, one can assume that $ge_it^j = e_it^j$ for all $|j|>N$ and $g(V_N)\subseteq V_N$, so g induces an automorphism of $V_N\subset V$:

$$g|_{V_N}: V_N\to V_N, \quad , u\mapsto gu.$$

and hence an automorphism of $\Lambda^{nN}V_N$. Because of the assumption $ge_it^j = e_it^j$ for all $|j|>N$, this gives an induced automorphism of the subspace:

$$(\Lambda^{nN}V_N)\wedge e_1t^{N+1}\wedge\ldots\wedge e_nt^{N+1}\wedge e_1t^{N+2}\wedge\ldots\subset\Lambda_0^{\frac{\infty}{2}}V,$$

such that the linear action coincides with the action on pure semi-infinite wedge products in (5.14). Vice versa, any automorphism $g \in GL(V_N)$ can be extended trivially to an automorphism of V by setting $ge_it^j = e_it^j$ for all $|j| > N$, so we may view g as an element in $GL_\infty(\mathbb{C})$. We conclude:

Lemma 5.6.2 *The action of $GL_\infty(\mathbb{C})$ on pure semi-infinite wedge products defined in (5.14) extends to an action of $GL_\infty(\mathbb{C})$ on $\Lambda_0^{\frac{\infty}{2}} V$ by vector space automorphisms. The induced $GL_\infty(\mathbb{C})$-action on the projective space $\mathbb{P}(\Lambda_0^{\frac{\infty}{2}} V)$ makes G_∞ into a homogeneous space:*

$$G_\infty = GL_\infty(\mathbb{C}) \cdot [e_1 \wedge \ldots \wedge e_n \wedge te_1 \wedge \ldots \wedge te_n \wedge t^2 e_1 \wedge \ldots \wedge t^2 e_n \wedge \ldots]$$

5.6.3 An approach towards $\mathcal{G}_0$ using $GL_n(\mathbb{C}[t, t^{-1}])$

We will leave out many details. We know that $GL_n(\mathbb{C}[t, t^{-1}])$ acts on $V = (\mathbb{C}[t, t^{-1}])^n$ by automorphisms. The naive approach to define an action of $GL_n(\mathbb{C}[t, t^{-1}])$ on $\Lambda^{\frac{\infty}{2}} V$ by

$$\begin{aligned} g \quad \cdot \quad & (v_1 \wedge \ldots \wedge v_\ell \wedge t^N e_1 \wedge t^N e_2 \wedge \ldots \wedge t^N e_n \wedge t^{N+1} e_1 \wedge \ldots) \\ = \quad & (gv_1) \wedge \ldots \wedge (gv_\ell) \wedge (gt^N e_1) \wedge (gt^N e_2) \wedge \ldots \wedge (gt^N e_n) \wedge (gt^{N+1} e_1) \wedge \ldots \end{aligned}$$

does not work because we would get infinite sums. But the point of view of subspaces helps in this case. For simplicity let us stick to pure wedge products of degree 0. A pure semi-infinite wedge product of degree 0 corresponds (in the projective space) to a subspace $\mathcal{U} \subset (\mathbb{C}[t, t^{-1}])^n$ with the property

$$\exists\, N \in \mathbb{N}: \quad t^N \mathcal{L}_0 \subset \mathcal{U} \subset t^{-N} \mathcal{L}_0, \quad \dim \mathcal{U}/t^N \mathcal{L}_0 = nN.$$

Now for $g \in GL_n(\mathbb{C}[t, t^{-1}])$ we know that $g\mathcal{L}_0$ is a $\mathbb{C}[t]$-lattice in $\mathbb{C}[t, t^{-1}]$, so there exists some $N' \geq N$ such that $t^{N'} \mathcal{L}_0 \subseteq g\mathcal{L}_0 \subseteq t^{-N'} \mathcal{L}_0$, and hence

$$t^{N'} \mathcal{L}_0 \subseteq g\mathcal{U} \subseteq t^{-N'} \mathcal{L}_0.$$

So the subspace $g\mathcal{U}$ has again similar properties as the subspaces consider in section 5.6.2, but note that the dimension condition in section 5.6.2 may not hold because $GL_n(\mathbb{C}[t, t^{-1}])$ does not necessarily preserve the decomposition of $\Lambda^{\frac{\infty}{2}} V$ into the direct sum of the $\Lambda_q^{\frac{\infty}{2}} V$. But if we consider only

$$GL_n(\mathbb{C}[t, t^{-1}])_0 = \{g \in GL_n(\mathbb{C}[t, t^{-1}]) \mid \det g \in \mathbb{C}^*\},$$

then i.e. $g\mathcal{U} \in G_\infty$.

Exercise 5.6.1 Prove that $g\mathcal{U} \in G_\infty$ for $g \in GL_n(\mathbb{C}[t,t^{-1}])_0$. Hint: use the decomposition $g = h_1 \underline{t}^{\lambda} h_2$, where $h_1, h_2 \in GL_n(\mathbb{C}[t])$.

Theorem 5.6.1 *It is possible to define a linear action of $GL_n(\mathbb{C}[t,t^{-1}])_0$ on $\mathbb{P}(\Lambda_0^{\frac{\infty}{2}} V)$ such that the induced action on $G_\infty \subset \mathbb{P}(\Lambda_0^{\frac{\infty}{2}} V)$ is exactly the one above:*

$$GL_n(\mathbb{C}[t,t^{-1}]) \times G_\infty \to G_\infty, \quad (g, \mathcal{U}) \mapsto g\mathcal{U}.$$

More details about the construction can be found, for example, in the lecture notes of Pavel Etingof's lecture.

Now the vector $v_0 = e_1 \wedge \ldots \wedge e_n \wedge te_1 \wedge \ldots \in \Lambda_0^{\frac{\infty}{2}} V$ corresponds exactly to the lattice $\mathcal{L}_0$, and we have for $g \in GL_n(\mathbb{C}[t,t^{-1}])$:

$$g[v_0] = [v_0] \Leftrightarrow g \in GL_n(\mathbb{C}[t])$$

So by Proposition 5.2.1, Corollary 5.4.1 and Theorem 4.5.2 we see:

Theorem 5.6.2 *i) The orbit $GL_n(\mathbb{C}[t,t^{-1}])_0 \cdot [v_0]$ has a natural structure as a projective Ind-variety and can be identified with $\mathcal{G}_0$.*

ii) $\mathcal{G}_0 = \Omega_0^{alg} U_n(\mathbb{C}) \cdot [v_0] \leftrightarrow^{1:1} \Omega_0^{alg} U_n(\mathbb{C})$.

5.7 $L^{alg} SU_n(\mathbb{C})$ is dense in $L^\infty SU_n(\mathbb{C})$

The next question we want to address is: how big is the difference between $L^{alg} U_n(\mathbb{C})$ and $L^\infty U_n(\mathbb{C})$?

Consider first the simplest case $n = 1$. We know that $L^{alg} U_1(\mathbb{C})$ consists of 1×1 matrices, the only entry is a Laurent polynomial, invertible in $\mathbb{C}[t,t^{-1}]$, and it defines a map

$$S^1 \to U_1(\mathbb{C}) = \{(a) \in M_1(\mathbb{C}) \mid a\bar{a} = 1\} = S^1.$$

It follows that all elements in $L^{alg} U_n(\mathbb{C})$ are of the form (at^m), where $|a| = 1$ and $m \in \mathbb{Z}$. So there is a big difference between $L^{alg} U_1(\mathbb{C})$ and $L^\infty U_1(\mathbb{C})$ for $n = 1$, for example let f be any real valued smooth function on S^1, then $\exp(2\pi i f) : S^1 \to S^1$ defines a smooth loop. If we consider only the based loops, then the bijection

$$\begin{aligned} \Omega^\infty U_1(\mathbb{C}) \longleftrightarrow GL_1(\mathbb{C}[t,t^{-1}])/GL_1(\mathbb{C}[t]) &= \{at^m \mid a \in \mathbb{C}^*, m \in \mathbb{Z}\}/\mathbb{C}^* \\ &= \{t^m \mid m \in \mathbb{Z}\} \end{aligned}$$

yields yet something discrete.

So let us replace now $U_n(\mathbb{C})$ by $SU_n(\mathbb{C}) = U_n(\mathbb{C}) \cap SL_n(\mathbb{C})$. This looks like only a slight difference, but this change has important consequences. In the following we use the topology introduced in section 3.2. One has to replace in this section $GL_n(\mathbb{C})$ by $SL_n(\mathbb{C})$ and $U_n(\mathbb{C})$ by $SU_n(\mathbb{C})$, and for the Lie algebras one has to replace $M_n(\mathbb{C})$ by $\mathfrak{sl}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid tr(A) = 0\}$ and $\mathfrak{u}_n(\mathbb{C})$ by $\mathfrak{su}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) \cap \mathfrak{sl}_n(\mathbb{C})$.

Theorem 5.7.1 *$L^{alg}SU_n(\mathbb{C})$ is dense in $L^\infty SU_n(\mathbb{C})$.*

Proof. Let $H := \overline{L^{alg}SU_n(\mathbb{C})} \subset L^\infty SU_n(\mathbb{C})$ be the closure. Note that H is a subgroup: by definition, the inversion is a C^∞-map, so

$$\overline{L^{alg}SU_n(\mathbb{C})} = \overline{\{g^{-1} \mid g \in L^{alg}SU_n(\mathbb{C})\}} = \{g^{-1} \mid g \in \overline{L^{alg}SU_n(\mathbb{C})}\}.$$

and, for the same reason, for $\gamma \in L^{alg}SU_n(\mathbb{C})$ one has

$$\gamma \cdot (\overline{L^{alg}SU_n(\mathbb{C})}) = \overline{\gamma \cdot L^{alg}SU_n(\mathbb{C})} = \overline{L^{alg}SU_n(\mathbb{C})}.$$

The last equality implies $\gamma\eta \in \overline{L^{alg}SU_n(\mathbb{C})}$ for all $\gamma \in L^{alg}SU_n(\mathbb{C})$ and $\eta \in \overline{L^{alg}SU_n(\mathbb{C})}$. Using the inversion, we get also $\eta\gamma \in \overline{L^{alg}SU_n(\mathbb{C})}$, and hence for the same reason as above for $\eta \in \overline{L^{alg}SU_n(\mathbb{C})}$:

$$\eta \cdot (\overline{L^{alg}SU_n(\mathbb{C})}) = \overline{\eta \cdot L^{alg}SU_n(\mathbb{C})} = \overline{L^{alg}SU_n(\mathbb{C})}.$$

We want to invest the Lie algebra of this subgroup. The Lie algebra of $L^\infty SU_n(\mathbb{C})$ is $L^\infty(S^1, \mathfrak{su})$. Note that if $\xi \in L^\infty(S^1, \mathfrak{su})$, then so is $t\xi$ for any $t \in \mathbb{R}$. Denote by $\gamma(t) := \exp(t\xi)$ the corresponding one parameter subgroup $\gamma : \mathbb{R} \to L^\infty SU_n(\mathbb{C})$.

Let $V \subset L^\infty(S^1, \mathfrak{su})$ be the subset of elements in the Lie algebra such that the corresponding one parameter subgroup is contained in H. Note that V is a vector space: the set is obviously stable under multiplication with scalars, it remains to prove that V is stable under addition. So let $\xi, \eta \in V$ and consider the sequence $\big(\gamma_\xi(t/n)\gamma_\eta(t/n)\big)^n$ for $n \in \mathbb{N}$. To explain better what this precisely means, choose an appropriate open neighborhood $U_0 \subseteq \mathfrak{su}$ of the origin such that the exponential map defines a diffeomorphism onto an open neighborhood $U_{\mathbb{I}} \subset SU_n(\mathbb{C})$ of the identity. One can now define a sequence of maps

$$f_n : \; U_{\mathbb{I}} \times U_{\mathbb{I}} \to G, \quad (\exp(A), \exp(B)) \mapsto \big(\exp(\frac{A}{n})\exp(\frac{B}{n})\big)^n.$$

Now Lie's product formula implies that this sequence converges for $n \to \infty$ in the C^∞-topology towards the map

$$f:\ U_{\mathbb{1}} \times U_{\mathbb{1}} \to G, \quad (\exp(A), \exp(B)) \mapsto \exp(A+B).$$

For our one parameter subgroups we get

$$\lim_{n\to\infty} \big(\gamma_\xi(t/n)\gamma_\eta(t/n)\big)^n = \gamma_{\xi+\eta}(t).$$

So if the one parameter subgroups γ_ξ, γ_η are contained in H, then so is $\gamma_{\xi+\eta}$. It follows that V is a vector space, and, since H is closed, so is V.

Since the exponential map is locally a diffeomorphism, to prove that $H = L^\infty SU_n(\mathbb{C})$ it suffices to prove that $V = L^\infty(S^1, \mathfrak{su}_n)$. Let us start with the simplest case $n = 2$. The Lie algebra loops

$$\xi_n(t) := \begin{pmatrix} 0 & t^n \\ -t^{-n} & 0 \end{pmatrix}, \quad \eta_n(t) \begin{pmatrix} 0 & it^n \\ it^{-n} & 0 \end{pmatrix}$$

are elements in V. Indeed, the corresponding one parameter subgroups lie in $L^{alg}(S^1, \mathfrak{su}_2)$:

$$\exp(s\xi(t)) = (\sum_{i=1}^{\infty} (-1)^i \frac{s^{2i}}{2i!})\mathbb{1} + st \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} +$$

By linearity and the fact that V is closed, it follows that

$$f \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + g \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

are elements in V for all smooth, real valued functions f, g on the circle. ETC

TO BE CONTINUED.........

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5.8 Appendix: $G_{d,n}$ as a projective variety

One has to show that there exists a homogeneous ideal $I \subset \mathbb{C}[\Lambda^d \mathbb{C}^n]$ such that the zero set $\mathcal{V}(I) \subseteq \mathbb{P}(\Lambda^d \mathbb{C}^n)$ is exactly $G_{d,n}$. This will be proved in Theorem 5.8.1.

Again Plücker coordinates.

In section 5.5 we introduced the name Plücker coordinate for the dual basis $p_{\underline{i}}$ of the standard basis of $\Lambda^d V$. To simplify the notation we use $p_{\underline{i}}$ in the following for arbitrary d-tuples and not only for elements $\underline{i} \in I_{d,n}$.

We give a description of the functions as alternating multilinear functions on the columns of $M_{n,d}(\mathbb{C})$ (instead of describing them as linear functions on $\Lambda^d V$).

For $1 \leq i_1, \ldots, i_d \leq n$ (not necessarily distinct nor in increasing order) set $\underline{i} = (i_1, \ldots, i_d)$. For an $n \times d$ matrix A let $A_{\underline{i}}$ be the $d \times d$ matrix having as first row the i_1-th row of A, as second row the i_2-th row of A and so on. We set $p_{\underline{i}}(A) = \det A_{\underline{i}}$.

Clearly, $p_{\underline{i}} = 0$ if the i_j's are not distinct, and if they are all distinct, then

$$p_{i_1,\ldots,i_d} = \operatorname{sgn}(\sigma) p_{\sigma(i_1),\ldots,\sigma(i_d)} \tag{5.15}$$

where $\sigma \in \mathfrak{S}_d$ is such that $(\sigma(i_1), \ldots, \sigma(i_d)) \in I_{d,n}$.

Alternating functions.

In view of Proposition 5.5.1, we can identify $G_{d,n}$ with $\operatorname{Im} \pi$. In general the image will not be all of $\mathbb{P}(\Lambda^d V)$, so the Plücker coordinates restricted to $G_{d,n}$ must satisfy some relations.

By definition, the Plücker coordinates are *i)* linear functions on $\Lambda^d V$ as well as *ii)* multilinear alternating functions on the columns of $M_{n,d}(\mathbb{C})$ (the latter being identified with d-copies of V).

These functions are defined as determinants of maximal submatrices, so they have a third property: *iii)* the Plücker coordinate $p_{\underline{i}}$ is a *multilinear and alternating function in the i_1-th, i_2-th etc. row of* $M_{n,d}(\mathbb{C})$.

Suppose now $\underline{i} \cap \underline{j} = \emptyset$, then the product $p_{\underline{i}} p_{\underline{j}}$ is a quadratic function on $\Lambda^d V$ which is definitely not anymore multilinear in the columns of $M_{n,d}(\mathbb{C})$. But this function is still multilinear in the i_1-th, i_2-th, ..., j_1-th, j_2-th etc. row of $M_{n,d}(\mathbb{C})$, and alternating separately in the i_k and j_ℓ. So if we alternate this function so that it becomes alternating in the rows say $i_1, \ldots, i_d, j_1$, then we have an alternating function on $d+1$-copies of a d-dimensional vector space (the space of row vectors of $M_{n,d}(\mathbb{C})$). Hence this function is zero on $M_{n,d}(\mathbb{C})$. Or, in other words, viewed as a quadratic function on $\Lambda^d V$, we have a function such that the restriction to $\operatorname{Im} \pi_d$ vanishes.

Example 5.8.1 Before starting with the formal approach consider the example $G_{2,4}$ and the product of Plücker coordinates $p_{1,2}p_{3,4} \in k[\Lambda^2 k^4]$. The composition with $\pi_2 : M_{4,2} \to \Lambda^2 k^4$ gives a function which is of course not anymore multilinear in the columns of $M_{4,2}(\mathbb{C})$, but which is still multilinear in the rows of this space of matrices. We will "formally alternate" this function. For example (we will see below why this is the alternated function)

$$p_{1,2}p_{3,4} + p_{2,3}p_{1,4} - p_{2,4}p_{1,3} \tag{5.16}$$

is a quadratic polynomial on $\Lambda^2 k^4$. The restriction to $\operatorname{Im} \pi_2$ is a multilinear function on $M_{4,2}(\mathbb{C})$ which is alternating in the first, the third and the fourth row of $M_{4,2}(\mathbb{C})$. The only function with this property (i.e. being alternating on 3 copies of a 2-dimensional space) is the zero function, so the function above vanishes identically on $\operatorname{Im} \pi_2$. But this means that the restriction of the quadratic polynomial in (5.16) to $G_{2,4}$ is identically zero, and hence the Plücker coordinates satisfy on $G_{2,4}$ a quadratic relation.

To formalize this idea, let us start with some generalities. We work inside the ring $k[x_{i,j}]$ of polynomial functions on $M_{n,d}(\mathbb{C})$ and we write just $x_1, \ldots, x_n$ for the vector variables corresponding to the **rows** of $M_{n,d}(\mathbb{C})$. Let $f(x_1, \ldots, x_n)$ be a multilinear function, then we can alternate it by setting:

$$Alt(f) := \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)({}^{\sigma}f)(x_1, \ldots, x_n),$$

where ${}^{\sigma}f(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$.

Suppose $n \geq d+1$. Instead of assuming that the function is multilinear in all vector variables, fix a subset $M = \{k_1, \ldots, k_{d+1}\}$, $1 \leq k_1 \leq \ldots \leq k_{d+1} \leq n$, of pairwise different indices, and assume the function is multilinear in the rows corresponding to the indices $k_1, \ldots, k_{d+1}$. The function

$$Alt_M(f) := \sum_{\sigma \in \mathfrak{S}_{d+1}} \operatorname{sgn}(\sigma) f(\ldots, x_{k_{\sigma^{-1}(1)}}, \ldots, x_{k_{\sigma^{-1}(d+1)}}, \ldots)$$

(i.e. all vector variables different from $x_{k_1}, \ldots, x_{k_{d+1}}$ are not changed) is alternating and multilinear in $x_{k_1}, \ldots, x_{k_{d+1}}$.

For $1 \leq t < d+1$ let $M = M_1 \cup M_2$ be a disjoint decomposition such that $\sharp M_1 = t$. If f is alternating separately in the variables $\{x_k \mid k \in M_1\}$ and $\{x_\ell \mid \ell \in M_2\}$, then

$$\begin{aligned} \operatorname{sgn}(\sigma) f(\ldots, x_{k_{\sigma^{-1}(1)}}, \quad \ldots \quad , x_{k_{\sigma^{-1}(d+1)}}, \ldots) \\ = \quad \operatorname{sgn}(\sigma') f(\ldots, x_{k_{\sigma'^{-1}(1)}}, \ldots, x_{k_{\sigma'^{-1}(d+1)}}, \ldots) \end{aligned}$$

whenever σ and σ' are in the same coset in $\mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$. Here we identify the subgroup $\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$ with the subgroup of permutations in $\mathfrak{S}_{d+1}$ which separately permute only the elements in M_1 and M_2 among themselves.

So to get an alternating function one has to take the sum

$$Alt_{M_1,M_2}(f) := \sum_{\sigma \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}} \operatorname{sgn}(\sigma) f(\ldots, x_{k_{\sigma^{-1}(1)}}, \ldots, x_{k_{\sigma^{-1}(d+1)}}, \ldots)$$

only over a system of representatives of the cosets.

Example 5.8.2 Suppose $n = 4$ and $d = 2$. Let $f(x_1, x_2, x_3, x_4) = p_{1,2}p_{3,4}$ be the product of these two Plücker coordinates, then f is a multilinear function on $M_{4,2}(\mathbb{C})$, alternating separately in the 1st and 2nd and the 3rd and 4th row. Set $M_1 = \{1\}$, $M_2 = \{3,4\}$ and $M = M_1 \cup M_2$, and denote by $\mathfrak{S}_M$ respectively $\mathfrak{S}_{M_i}$ the permutation groups of the sets. Then

$$id = \begin{pmatrix} 134 \\ 134 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 134 \\ 314 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 134 \\ 341 \end{pmatrix},$$

is a set of representatives of $\mathfrak{S}_M/\mathfrak{S}_{M_1} \times \mathfrak{S}_{M_2}$ (see section 5.8 for a procedure to get the representatives) and

$$\begin{aligned} Alt_{M_1,M_2}(f) &= f + \operatorname{sgn}(\sigma_1)({}^{\sigma_1}f) + \operatorname{sgn}(\sigma_2)({}^{\sigma_2}f) \\ &= f(x_1, x_2, x_3, x_4) - f(x_3, x_2, x_1, x_4) + f(x_4, x_2, x_1, x_3) \\ &= p_{1,2}p_{3,4} + p_{2,3}p_{1,4} - p_{2,4}p_{1,3} \end{aligned}$$

is the function on $M_{4,2}(\mathbb{C})$ in equation 5.16, which is alternating in the 1st, 3rd and 4th row.

Quadratic relations.

A product $f = p_{\underline{i}} p_{\underline{j}}$ of Plücker coordinates is a quadratic polynomial on $\Lambda^d V$. Suppose now all indices i_k, j_ℓ are different. The product is a function on $M_{n,d}(\mathbb{C})$ which is multilinear with respect to the rows of this space of matrices. Fix $1 \leq t < d$, then f is, by construction, alternating separately in the (row) vector variables $x_{i_1}, \ldots, x_{i_t}$ and $x_{j_t}, \ldots, x_{j_d}$.

Given $\sigma \in \mathfrak{S}_{d+1}$, note that σ shuffles the indicees $i_1, \ldots, i_t$ and $j_t, \ldots, j_d$. Denote by $\underline{i}^\sigma$ and $\underline{j}^\sigma$ the d-tuples

$$(\sigma^{-1}(i_1), \ldots, \sigma^{-1}(i_t), i_{t+1}, \ldots, i_d)$$

and

$$(j_1, \ldots, j_{t-1}, \sigma^{-1}(j_t), \ldots, \sigma^{-1}(j_d)).$$

Recall that the function $\mathrm{sgn}(\sigma)({}^\sigma f)$, $\sigma \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$, is independent of the choice of a representative for σ. The function we get by alternating $f = p_{\underline{i}} p_{\underline{j}}$ is:

$$Alt_{\{i_1,\ldots,i_t\},\{j_t,\ldots,j_d\}}(p_{\underline{i}} p_{\underline{j}}) = \sum_{\sigma \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}} \mathrm{sgn}(\sigma) p_{\underline{i}^\sigma} p_{\underline{j}^\sigma}.$$

Lemma 5.8.1 *Suppose $n \geq 2d$. Let $\underline{i}, \underline{j}$ be two d-tuples, $1 \leq i_k, j_l \leq n$, such that the entries are all distinct. Fix $1 \leq t < d$, the homogeneous polynomial $Alt_{\{i_1,\ldots,i_t\},\{j_t,\ldots,j_d\}}(p_{\underline{i}} p_{\underline{j}}) \in k[\Lambda^d V]$ vanishes on $G_{d,n} \subset \mathbb{P}[\Lambda^d V]$.*

Proof. By composing the function with the exterior product map, we see that the quadratic polynomial vanishes on $G_{d,n}$ if and only if, viewed as a sum of products of minors, the function vanishes on $M_{n,d}(\mathbb{C})$. But this function is multilinear and alternating in the $d+1$ row vector variables $x_{i_1}, \ldots, x_{i_t}$, $x_{j_t}, \ldots, x_{j_d}$. The space of the row vectors is of dimension d, so this function vanishes on $M_{n,d}(\mathbb{C})$. •

To weaken the condition that all indices have to be different, consider two arbitrary d-tuples $\underline{i}$ and $\underline{j}$, $1 \leq i_k, j_l \leq n$. We will now define a new pair $\underline{i}', \underline{j}'$ such that all entries are different. Set

$$\begin{aligned} i'_k &= i_k + mn \quad \text{where} \quad m = \sharp\{\ell \mid \ell < k, i_k = i_\ell\} \\ j'_k &= j_k + mn \quad \text{where} \quad m = \sharp\{\ell \mid j_k = i_\ell\} + \sharp\{\ell \mid \ell < k, j_k = j_\ell\} \end{aligned}$$

For example, suppose i_1, i_2, j_1, j_2 are pairwise different, then this procedure applied to the pair

$$\begin{array}{c} \underline{i} = (i_1, i_2, i_1, i_1, i_2)\ \underline{j} = (j_1, i_2, i_1, j_1, j_2) \\ \downarrow \\ \underline{i}' = (i_1, i_2, i_1 + n, i_1 + 2n, i_2 + n)\ \underline{j}' = (j_1, i_2 + 2n, i_1 + 3n, j_1 + n, j_2) \end{array}$$

provides a new pair $(\underline{i}', \underline{j}')$ such that all entries are different. So we can formally define the quadratic polynomial (in a larger ring with more vector variables)

$$Alt_{\{i'_1,\dots,i'_t\},\{j'_t,\dots,j'_d\}}(p_{\underline{i}'}p_{\underline{j}'}). \tag{5.17}$$

We define the polynomial (which is either zero or a quadratic polynomial)

$$Alt_{(i_1,\dots,i_t),(j_t,\dots,j_d)}(p_{\underline{i}}p_{\underline{j}})$$

now as the function obtained from (5.17) by replacing in the Plücker coordinates all indices i'_k, j'_ℓ by the original indices, i.e. all indices $i'_k, j'_\ell > n$ are replaced by $i'_k \pmod n$ respectively $j'_\ell \pmod n$.

Example 5.8.3 Suppose $n = 5$, $d = 3$ and $\underline{i} = (2, 1, 5)$ and $\underline{j} = (1, 3, 4)$. Then $\underline{i}' = \underline{i}$ and $\underline{j}' = (6, 3, 4)$. For $t = 1$ we have $M_1 = \{2\}$, $M_2 = \{6, 3, 4\}$ and $M = M_1 \cup M_2$. For the permutation groups we have $\mathfrak{S}_{M_1} \simeq \mathfrak{S}_1$, $\mathfrak{S}_{M_2} \simeq \mathfrak{S}_3$, $\mathfrak{S}_M \simeq \mathfrak{S}_4$ and $\mathfrak{S}_M/(\mathfrak{S}_{M_1} \times \mathfrak{S}_{M_2}) \simeq \mathfrak{S}_4/(\mathfrak{S}_1 \times \mathfrak{S}_3)$. Denote by s_1, s_2, s_3 the simple transpositions of $\mathfrak{S}_4$. The elements $id, s_1, s_2s_1, s_3s_2s_1$ form a system of representatives for the cosets in $\mathfrak{S}_4/\mathfrak{S}_1 \times \mathfrak{S}_3$ and we get

$$Alt_{\{2\},\{6,3,4\}}(p_{\underline{i}'}p_{\underline{j}'}) = p_{2,1,5}p_{6,3,4} - p_{6,1,5}p_{2,3,4} + p_{3,1,5}p_{2,6,4} - p_{4,1,5}p_{2,6,3}$$

After specializing (i.e. replacing 6 back by 1) we get

$$\begin{aligned} Alt_{(2),(1,3,4)}(p_{\underline{i}}p_{\underline{j}}) &= p_{2,1,5}p_{1,3,4} - p_{1,1,5}p_{2,3,4} + p_{3,1,5}p_{2,1,4} - p_{4,1,5}p_{2,1,3} \\ &= -p_{1,2,5}p_{1,3,4} + p_{1,3,5}p_{1,2,4} - p_{1,4,5}p_{1,2,3} \end{aligned}$$

Theorem 5.8.1 *Let $\underline{i}$ and $\underline{j}$, $1 \le i_k, j_l \le n$, be two arbitrary d-tuples. For all $1 \le t < d$, the polynomial $Alt_{(i_1,\dots,i_t),(j_t\dots,j_d)}(p_{\underline{i}}p_{\underline{j}})$ vanishes on the Graßmann variety $G_{d,n}$.*

Proof. Suppose the polynomial is different from zero. As above, by composing the function with the exterior product map, one sees that this quadratic polynomial vanishes on $G_{d,n}$ if and only if, viewed as a sum of products of minors, the function vanishes on $M_{n,d}(\mathbb{C})$. If the entries in $\underline{i}$ and $\underline{j}$ are all different, then this is Lemma 5.8.1. Otherwise consider first the multilinear function $Alt_{(i'_1,\dots,i'_t),(j'_t\dots,j'_d)}(p_{\underline{i}'}p_{\underline{j}'})$ defined in (5.17), this function is defined on the space $M_{2dn,d}(\mathbb{C})$ of $2dn \times d$ matrices, and vanishes identically since it is multilinear and alternating in $d+1$ of the vector variables.

The original space $M_{n,d}(\mathbb{C})$ can be seen as a subspace of $M_{2dn,d}(\mathbb{C})$ by identifying a $n \times d$-matrix A with the $2dn \times d$-matrix obtained by putting $2d$ copies of A on the top of each other. By construction we have then

$$Alt_{(i_1,\dots,i_t),(j_t\dots,j_d)}(p_{\underline{i}}p_{\underline{j}}) = Alt_{(i'_1,\dots,i'_t),(j'_t\dots,j'_d)}(p_{\underline{i}'}p_{\underline{j}'})|_{M_{n,d}(\mathbb{C})} \equiv 0.$$

•

Definition 5.8.1 *If $Alt_{(i_1,\dots,i_t),(j_t\dots,j_d)}(p_{\underline{i}}p_{\underline{j}})$ is not the zero polynomial in $k[\Lambda^d V]$, then this quadratic polynomial is called a shuffle relation or a Plücker relation.*

Shuffles.

We will describe how to obtain shuffles or coset representatives. Fix $1 \leq t \leq d$, we want to describe a special set of coset representatives of $\mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$. Let $\mathfrak{S}_{d+1}$ act on the set $\{1,\dots,d+1\}$. Then a coset $\overline{\sigma} \in \mathfrak{S}_{d+1}/\mathfrak{S}_t \times \mathfrak{S}_{d+1-t}$ is identified by the relative position of the first t and the second $d+1-t$ elements. Expressed in a pictorial way: suppose we are given a configuration of t-balls and $d+1-t$ triangles:

$$\bigcirc \bigcirc \triangle \bigcirc \triangle\triangle\triangle \bigcirc \dots$$

If we fill the balls with any permutation of $\{1,\dots,t\}$ and the triangles with any permutation of $\{t+1,\dots,d+1\}$, we always get a permutation which is an element of the same coset.

A canonical representative of such a coset is hence obtained by putting $1,2,\dots,t$ in order in the balls and $t+1,\dots,d+1$ in order in the triangles. Such a representative is called a *t-shuffle*.

Example 5.8.4 To determine the set of all 2-shuffles in $\mathfrak{S}_4$ consider first the set of all configuration of 2-balls and 2 triangles:

$$\bigcirc\bigcirc\triangle\triangle,\ \bigcirc\triangle\bigcirc\triangle,\ \triangle\bigcirc\bigcirc\triangle,\ \bigcirc\triangle\triangle\bigcirc,\ \triangle\bigcirc\triangle\bigcirc,\ \triangle\triangle\bigcirc\bigcirc.$$

The 2-shuffles and the decomposition of the inverse are given by:

$$\begin{array}{rcllllll} \sigma & = & \begin{pmatrix}1234\\1234\end{pmatrix}, & \begin{pmatrix}1234\\1324\end{pmatrix}, & \begin{pmatrix}1234\\3124\end{pmatrix}, & \begin{pmatrix}1234\\1342\end{pmatrix}, & \begin{pmatrix}1234\\3142\end{pmatrix}, & \begin{pmatrix}1234\\3412\end{pmatrix} \\ \sigma^{-1} & = & id, & s_2, & s_2s_1, & s_2s_3, & s_2s_1s_3, & s_2s_1s_3s_2 \end{array}$$

Example 5.8.5 Let $n=5$, $d=3$, $\underline{i}=(2,3,4), \underline{j}=(1,4,5)$, $t=2$. By the example above we have

$$\begin{aligned} Alt_{(2,3)(4,5)}p_{\underline{i}}p_{\underline{j}} &= p_{2,3,4}p_{1,4,5} - p_{2,4,4}p_{1,3,5} + p_{3,4,4}p_{1,2,5} \\ &\quad + p_{2,5,4}p_{1,3,4} - p_{3,5,4}p_{1,2,4} + p_{4,5,4}p_{1,2,3} \\ &= p_{2,3,4}p_{1,4,5} - p_{2,4,5}p_{1,3,4} + p_{3,4,5}p_{1,2,4} \end{aligned}$$

Closed embedding.

Next we will see that one can identify $G_{d,n}$ with is a closed subset of $\mathbb{P}(\Lambda^d V)$, i.e. the Graßmann variety is naturally endowed with the structure of a projective variety.

Theorem 5.8.2 *The Graßmann variety $G_{d,n} \subset \mathbb{P}(\Lambda^d V)$ is the zero set of the homogeneous ideal generated by the following polynomials:*

$$\sum_{l=1}^{d+1} (-1)^l p_{i_1,\ldots,\widehat{i_l},\ldots,i_{d+1}} p_{j_1,\ldots,j_{d-1},i_l}, \tag{5.18}$$

where $i_1,\ldots,i_{d+1}$ and $j_1,\ldots,j_{d-1}$ are any numbers between 1 and n.

Proof. The relation in (5.18) is a special case of the shuffle relations (see Theorem 5.8.1, $t=d$), so $G_{d,n}$ is contained in the zero set of the homogeneous ideal generated by these equations.

Conversely, let $y = [\sum_{\underline{i}\in I_{d,n}} y_{\underline{i}} e_{\underline{i}}]$ satisfy the equations in (5.18). Suppose $y_{l_1,\ldots,l_d} \neq 0$ for some $\underline{\ell} = (l_1,\ldots,l_d) \in I_{d,n}$, without loss of generality we may (and will) assume $y_{l_1,\ldots,l_d} = 1$. For $1 \le i \le n$, $1 \le j \le d$, set

$$a_{ij} = y_{l_1,\ldots,l_{j-1},i,l_{j+1},\ldots,l_d}.$$

We apply the usual rules as in (5.15): $y_{l_1,\ldots,l_{j-1},i,l_{j+1},\ldots,l_d}$ is zero if two indices are equal etc. Let A be the $n \times d$ matrix $A = (a_{ij})$. By construction $A_{l_1,\ldots,l_d} = \mathrm{I}_d$ because $a_{l_j,j} = y_{l_1,\ldots,l_d} = 1$ for $j = 1,\ldots,d$ and for $i \neq j$ we have $a_{l_j,i} = y_{l_1,\ldots,l_{i-1},l_j,l_{i+1},\ldots,l_d} = 0$. Clearly $\operatorname{rank} A = d$, let U be the d-dimensional subspace spanned by the columns of A. We have to show that $\pi(U) = [\sum_{\underline{i}\in I_{d,n}} p_{\underline{i}}(A)e_{\underline{i}}] = [\sum_{\underline{i}\in I_{d,n}} y_{\underline{i}}e_{\underline{i}}] = y$ and hence $y \in G_{d,n}$.

For two d-tuples $\underline{\kappa}, \underline{\kappa}'$ denote by $\sharp\{\underline{\kappa}\cap\underline{\kappa}'\}$ the number of common entries. We will show $p_{\underline{j}}(A) = y_{\underline{j}}$ by decreasing induction on $\sharp\{\underline{\ell} \cap \underline{j}\}$. We know already that $p_{\underline{\ell}}(A) = 1 = y_{\underline{\ell}}$. For $\underline{j} = (l_1, \ldots, l_{j-1}, i, l_{j+1}, \ldots, l_d)$ we have $p_{\underline{j}}(A) = a_{i,j} = y_{\underline{j}}$ by the definition of A, so this proves the claim if $\sharp\{\underline{\ell}\cap\underline{j}\} \geq d-1$.

Let $\underline{j}$ be arbitrary such that $\sharp\{\underline{\ell}\cap\underline{j}\} < d-1$. There exists an entry in $\underline{j}$ which is not an entry in $\underline{\ell}$. Without loss of generality (i.e., after permuting the entries if necessary) we assume that j_d has this property. Now y satisfies all the relations in (5.18), so the coordinates $y_{\underline{\ell}}$ and $y_{\underline{j}}$ satisfy a relation of the form above: $y_{\underline{\ell}}y_{\underline{j}} + \sum \pm y_{\underline{\ell}'}y_{\underline{j}'} = 0$, where $\underline{\ell}'$ differs from $\underline{\ell}$ in just one place. Further, if $y_{\underline{\ell}'}y_{\underline{j}'} \neq 0$, then $\sharp\{\underline{j}'\cap\underline{\ell}\} > \sharp\{\underline{j}\cap\underline{\ell}\}$ since j_d has been replaced by an element in $\underline{\ell}$. Thus we know by induction $y_{\underline{\ell}'} = p_{\underline{\ell}'}(A)$, $y_{\underline{j}'} = p_{\underline{j}'}(A)$.

By Theorem 5.8.1, the d-minors of A satisfy the relations in (5.18), so $p_{\underline{\ell}}(A)p_{\underline{j}}(A) + \sum \pm p_{\underline{\ell}'}(A)p_{\underline{j}'}(A) = 0$. Now $p_{\underline{\ell}}(A) = y_{\underline{\ell}} = 1$, so $p_{\underline{j}}(A) = -\sum \pm p_{\underline{\ell}'}(A)p_{\underline{j}'}(A) = -\sum \pm y_{\underline{\ell}'}y_{\underline{j}'} = y_{\underline{j}}$. •