An introduction to pluripotential theory

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Introduction

The note is organized as follows. Chapter 1 and part of Chapter 2 are standard material in pluripotential theory about subharmonic functions in complex plane and psh functions. The presentation for these parts is based on [13, 9, 24, 28, 31]. Chapter 3 deals with a main object in the intersection of $(1,1)$-currents: Monge-Ampère operators and its continuity. ....????? Most of results presented in the note already appear in published papers. Nevertheless there are seemingly some new ones. The note is incomplete. More references need to be added.

Chapter 1

Subharmonic functions on the complex plane

1.1 Harmonic functions on the complex plane

Let (x, y) be the standard coordinates on $\mathbb{C}$ and $z := x + iy$. Let Ω be a connected open subset of $\mathbb{C}$. Such a subset Ω is called a domain. Let $u : \Omega \to \mathbb{R}$ be a function. We say that u is *harmonic* if $u \in \mathscr{C}^2(\Omega)$ and $\Delta u = 0$, where $\Delta := \partial^2/\partial^2 x + \partial^2/\partial^2 y$ is the Laplacian operator.

Theorem 1.1.1. (i) *Let f be a holomorphic function on Ω. Then $\operatorname{Re} f$ is a harmonic function.*

(ii) *Let u be a harmonic function on Ω. If Ω is simple connected (a disk for example), then $u = \operatorname{Re} f$ for some holomorphic function f on Ω; moreover such f is unique up to a constant. In particular every harmonic function is smooth.*

Proof. We prove (i). Write $f = u + iv$. The Cauchy-Riemann equations give $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$. Thus

$$\Delta u = \partial_x^2 u + \partial_y^2 u = \partial_x \partial_y v - \partial_y \partial_x v = 0.$$

As to (ii), we first check the uniqueness of f. Suppose that f is a holomorphic function such that $\operatorname{Re} f = u$. Compute

$$\partial_x f = \partial_x u + i\partial_x v = \partial_x u - i\partial_y u, \quad \partial_y f = \partial_y u + i\partial_y v = \partial_y u + i\partial_x u.$$

Thus,

$$\partial_z f = 1/2(\partial_x f - i\partial_y f) = \partial_x u - i\partial_y u$$

Hence the derivative of f depends only on u. It follows that f is unique up to a constant. We now prove the existence of f. Let $g := \partial_x u - i\partial_y u$. Observe $g \in \mathscr{C}^1(\Omega)$ and g satisfies the Cauchy-Riemann equations. Consequently, g is holomorphic. Let $z_0 \in \Omega$. Put

$$f(z) = u(z_0) + \int_{z_0}^{z} g(z)dz,$$

where the integral is taken over any smooth path joining z_0 and z. The value $f(z)$ is independent of the chosen path because g is holomorphic and Ω is simple connected. We can check that $\operatorname{Re} f = u$. This finishes the proof. □

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. We denote by $\mathbb{D}(w, r)$ the disk centered at w and of radius r in $\mathbb{C}$.

Corollary 1.1.2. (i) *(mean-value property) For every harmonic function u on Ω and every disk $\mathbb{D}(w, r) \Subset \Omega$, we have*

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta$$

(ii) *(maximum principle) Let Ω be a bounded domain and u be a harmonic function on Ω. Then if u attains a local maximum then it is constant. Consequently, if $\limsup_{x\to\partial\Omega} u(x) \leq 0$, then $u \leq 0$ on Ω.*

Proof. By Theorem 1.1.1, we can write $u = \operatorname{Re} f$ on an open neighborhood of $\overline{D}(w, r)$, where f is holomorphic. Recall

$$f(w) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}(w,r)} \frac{f(z)}{w - z} dz = \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{i\theta}) d\theta.$$

Taking the real parts of both sides gives the mean-valued property (i). The desired assertion in (ii) is a direct consequence of (i). □

Theorem 1.1.3. *(Poisson's formula)* (i) *Let f be a continuous function on $\partial\mathbb{D}$. Then*

$$P(f) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} f(e^{i\theta}) d\theta$$

is a harmonic function on $\mathbb{D}$ which is continuous on $\overline{D}$ and $P(f) = f$ on $\partial\mathbb{D}$.

(ii) *Let u be a harmonic function on an open neighborhood of $\overline{\mathbb{D}}$. Then, we have*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta$$

for every $z \in \mathbb{D}$.

Proof. First observe that for $\xi \in \partial\mathbb{D}$, we have

$$\frac{1 - |z|^2}{|\xi - z|^2} = \operatorname{Re}\left(\frac{\xi + z}{\xi - z}\right)$$

which is the real part of a holomorphic function in z (for ξ fixed). Hence $\tilde{u}$ is harmonic by Theorem 1.1.1. We leave an exercise to verify that $P(f)$ is continuous up to boundary and equal to f on $\partial\mathbb{D}$ (see [31, Theorem 1.2.4]).

To get (ii), just observe that for $f := u|_{\partial\mathbb{D}}$, we have $u = P(f)$ on $\partial\mathbb{D}$. This together with the maximum principle gives $u = P(f)$ on $\mathbb{D}$. □

1.2 Upper semi-continuity

Denote $[-\infty,\infty) := \mathbb{R}\cup\{-\infty\}$. We recall rules to work with $-\infty$. For every $a\in\mathbb{R}$, one has

$$-\infty < a, \quad -\infty + a = -\infty, \quad -\infty + -\infty = -\infty,$$

and

$$(-\infty)\cdot a = \begin{cases} -\infty & \text{if } a>0 \\ 0 & \text{if } a=0 \\ \infty & \text{if } a<0 \end{cases}$$

and $(-\infty)\cdot(-\infty) = \infty$. We don't define the quotient $(-\infty/(-\infty)$. The rules have been made so that for every sequence $(b_j)_j \subset \mathbb{R}$ converging to $-\infty$, there holds

$$-\infty + a = \lim_{n\to\infty}(b_j + a), \quad -\infty\cdot a = \lim_{j\to\infty} b_j\cdot a.$$

Let $m\geq 2$ be an integer. We denote $\mathbb{R}_{\geq 0} := \{x\in\mathbb{R} : x\geq 0\}$, and $\mathbb{R}_{>0} := \{x\in\mathbb{R} : x > 0\}$. The notations $\mathbb{R}_{\leq 0}$ and $\mathbb{R}_{<0}$ are defined similarly. For every $x\in\mathbb{R}^m$ and $r\in\mathbb{R}_{\leq 0}$, let $\mathbb{B}(x,r)$ be the ball centered at x of radius r (with respect to the Euclidean distance).

Let Ω be an open subset of $\mathbb{R}^m$. Let $u:\Omega\to[-\infty,\infty)$. We say the u is *upper semi-continuous* if for every $a\in\mathbb{R}$, the set $\{x\in\Omega : u(x)<a\}$ is open in Ω. Observe that u is upper semi-continuous if and only if for every $x\in\Omega$, we have

$$\limsup_{y\to x} u(y) = u(x),$$

where

$$\limsup_{y\to x} u(y) := \lim_{\epsilon\to 0}\sup_{y\in\mathbb{B}(x,\epsilon)} u(y)$$

Every continuous function is upper semi-continuous.

Lemma 1.2.1. *Let $(u_\alpha)_{\alpha\in A}$ be a sequence of upper semi-continuous functions. Then $u := \inf_{\alpha\in A} u_\alpha$ is upper semi-continuous.*

Proof. Observe

$$\{x : u(x) < a\} = \cap_{\alpha\in A}\{x : u_\alpha(x) < a\}$$

which is open. This finishes the proof. □

Lemma 1.2.2. *Let u be an upper semi-continuous function on Ω. Let K be a compact subset of Ω. Then there exists $x_0\in K$ such that $u(x_0) = \sup_{x\in K} u(x)$.*

This means that every upper semi-continuous u is locally bounded from above on Ω, *e.g.* for every $x\in\Omega$, there exists a small open ball U containing x such that $\sup_{x\in U} u(x) < \infty$.

Proof. Put $b := \sup_{x\in K} u(x)$. Let $(x_j)_j$ be a sequence of points in K such that $u(x_j)\to b$ as $j\to\infty$. □

Lemma 1.2.3. *Let $u : \Omega \to [-\infty, \infty)$ be upper semi-continuous. Then there exists a decreasing sequence $(u_j)_j$ of continuous functions (u_j continuous and $u_j \geq u_{j+1}$ for every j) such that $u(x) = \lim_{j\to\infty} u_j(x)$ for every $x \in \Omega$.*

Proof. If $u = -\infty$, then it is clear: just take $u_j := -j$. Assume $u \not\equiv -\infty$. Put

$$u_j(x) := \sup_{y\in\Omega} \big(u(y) - j|x - y|\big).$$

We can check that u_j decreases to u and satisfies

$$|u_j(x) - u_j(x')| \leq j|x - x'|.$$

Hence u_j is continuous. □

Let $u : \Omega \to [-\infty, \infty)$ be a function which is locally bounded from above. *The upper semi-continuous regularization u^* of u* is given by

$$u^*(x) := \limsup_{y\to x} u(y) = \lim_{\epsilon\to 0} \sup_{y\in\mathbb{B}(x,\epsilon)} u(y).$$

Note that if u is upper semi-continuous, then $u^* = u$.

Lemma 1.2.4. *The function u^* is upper semi-continuous and $u^* \geq u$.*

Proof. The inequality $u^* \geq u$ is clear. We prove the first desired assertion. By the definition of u^*, one sees that $u^*(x) < a$ if there exists an $\epsilon > 0$ such that $u(y) < a$ for every $y \in \mathbb{B}(x,\epsilon)$. Let $a \in \mathbb{R}$ and $x \in \Omega$ such that $u^*(x) < a$. We need to check that $u^*(y) < a$ for y closed enough to x. By definition, we get

$$\sup_{y\in\mathbb{B}(x,\epsilon')} u(y) < a$$

for $\epsilon' > 0$ small enough. Hence by the above observation, $u^*(y) < a$ for every $y \in \mathbb{B}(x,\epsilon')$. This finishes the proof. □

Lemma 1.2.5. *(Choquet's lemma) Let $(u_\alpha)_{\alpha\in A}$ be a family of functions from $\Omega \to [-\infty, \infty)$. Assume that the family $(u_\alpha)_\alpha$ is locally bounded from above (*i.e, *for every $x \in \Omega$, there exists a ball U in Ω containing x such that $\sup_{\alpha\in A} \sup_{x\in U} u < \infty$). Then there exists a countable subset B of A such that*

$$(\sup_{\alpha\in A} u_\alpha)^* = (\sup_{\alpha\in B} u_\alpha)^*.$$

Proof. See [13, Page 38]. Idea: there exists a countable basis for the metric topology in Ω. □

1.3 Subharmonic functions on the complex plane

Definition 1.3.1. *A function $u : \Omega \to [-\infty, \infty)$ is said to be subharmonic if $u \not\equiv -\infty$ on every connected component of Ω, and u is upper semi-continuous and for every $z \in \Omega$, there exists a constant $r_z > 0$ small enough such that $\{z' : |z' - z| < r_z\} \Subset \Omega$ and the submean inequality*

$$u(z) \le \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$$

holds for every $0 < r \le r'$.

Note that by definition, being subharmonic is a local property. One can define u to be subharmonic on an arbitrary open subset Ω in $\mathbb{C}$ by asking that u is so on every connected open component of Ω. From now on, we only consider the setting where Ω is a domain.

Lemma 1.3.2. *Every subharmonic function is locally integrable.*

Proof. Let u be a subharmonic function on Ω. Let A be the set of $z \in \Omega$ such that u is locally integrable in an open neighborhood of z. Since $u \not\equiv -\infty$ on Ω, there exists $z_0 \in \Omega$ such that $u(z_0) > -\infty$. This combined with the submean inequality gives that u is integrable in a small disk centered at z_0. Thus, A is non-empty. Moreover A is open by its definition. We check that A is closed. Let $z_1 \in \partial A$. Let $z_1' \in A$ be close enough to z_1 such that there is a disk $\mathbb{D}(z_1', r) \Subset \Omega$ such that $z_1 \in \mathbb{D}(z_1', r/10)$. Since u is locally integrable around z_1', there exists $z_1'' \in \mathbb{D}(z_1', r/10)$ such that $u(z_1'') > -\infty$. By previous arguments, we know that u is integrable on $\mathbb{D}(z_1'', r/2)$ which contains z_1. Hence $z_1 \in A$. In other words, A is closed and open. The connectedness of Ω yields $A = \Omega$. This ends the proof. □

Theorem 1.3.3. *(maximum principle) Let Ω be a bounded domain and u be a subharmonic function on Ω. Then if u attains a local maximum then it is constant. Moreover for every harmonic function h on an open neighborhood of $\overline{\Omega}$ such that $\limsup_{x \to x_0 \in \partial\Omega} u(x) \le h(x_0)$ for every $x_0 \in \partial\Omega$, then $u \le h$ on Ω.*

Proof. The proof is similar to the case of harmonic function. □

Corollary 1.3.4. *For every disk $\mathbb{D}(w, r) \Subset \Omega$, we have*

$$u(z) \le \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z - w|^2}{|re^{i\theta} - (z - w)|^2} u(w + re^{i\theta}) d\theta \tag{1.3.1}$$

(in particular the submean inequality holds for every relatively compact disk inside Ω). Hence the function

$$M_u(w, r) := \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta$$

is increasing in r (r small).

Proof. Let $(u_j)_j$ be a sequence of continuous function decreasing to u; see Lemma 1.2.3. Combining Theorems 1.3.3 and 1.1.3 gives

$$u(w) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|(z-w)/r|^2}{|e^{i\theta}-(z-w)/r|^2} u_j(w+re^{i\theta})d\theta$$

because the right-hand side is a subharmonic function which is equal to $u_j \geq u$ on $\partial\mathbb{D}(w,r)$. Letting $j \to \infty$ gives the first desired inequality. Let $0 < r' < r$ be a constant. Integrating the just-obtained submean inequality over $z \in \partial\mathbb{D}(w,r')$ gives

$$\begin{aligned} M_u(w,r') &\leq \frac{1}{2\pi}\int_0^{2\pi} d\theta' \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-r'^2}{|re^{i\theta}-r'e^{i\theta'}|^2} u(w+re^{i\theta})d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} u(w+re^{i\theta})d\theta \frac{1}{2\pi}\int_0^{2\pi} d\theta' \frac{r^2-r'^2}{|re^{i\theta}-r'e^{i\theta'}|^2} \end{aligned} \tag{1.3.2}$$

By using a change of variable, one see that the number

$$\int_0^{2\pi} d\theta' \frac{r^2-r'^2}{|re^{i\theta}-r'e^{i\theta'}|^2}$$

is independent of θ, and must be equal to 1 (otherwise we get a contradiction in (1.3.2) by putting $u := \pm 1$). Hence $M_u(w,r)$ is increasing in r. This finishes the proof. □

Let $\chi \geq 0$ be a smooth radial function with support in $\mathbb{D}$ such that $\int_{\mathbb{C}} \chi d\,\mathrm{Leb} = 1$, where Leb is the Lebesgue measure on $\mathbb{C}$. Here being radial means $\chi(z) = \chi(|z|)$ for every $z \in \mathbb{C}$. For every constant $\epsilon > 0$, put

$$\chi_\epsilon(z) := \epsilon^{-2}\chi(z/\epsilon), \quad u_\epsilon(z) := \int_{\mathbb{C}} u(z-w)\chi_\epsilon(w)d\,\mathrm{Leb}\,.$$

Note that the function u_ϵ is well-defined on the set Ω_ϵ which consists of $z \in \Omega$ of distance at least ϵ to Ω.

Theorem 1.3.5. *(regularisation of subharmonic functions) The function u_ϵ is a smooth subharmonic function and u_ϵ decreases to u as $\epsilon \to 0$.*

Proof. The smoothness is clear because u_ϵ is a convolution:

$$u_\epsilon(z) = \int_{\mathbb{C}} u(w)\chi_\epsilon(z-w)d\,\mathrm{Leb}\,.$$

Using the submean inequality for u gives

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} u_\epsilon(z+re^{i\theta})d\theta &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_\Omega u(z+re^{i\theta}-w)\chi_\epsilon(w)d\,\mathrm{Leb}(w) \\ &= \int_\Omega \left(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta}-w)d\theta\right)\chi_\epsilon(w)d\,\mathrm{Leb}(w) \\ &\geq \int_\Omega u(z-w)\chi_\epsilon(w)d\,\mathrm{Leb}(w) = u_\epsilon(r). \end{aligned}$$

Hence u_ϵ is smooth and subharmonic. Let $0 < \epsilon' < \epsilon$ be a constant. Now using the fact that χ is radial and the polar coordinates gives

$$\begin{aligned} u_\epsilon(z) &= \int_0^\infty r\chi_\epsilon(r)dr \int_0^{2\pi} u(z - re^{i\theta})d\theta \\ &= \int_0^\infty r\chi(r)dr \int_0^{2\pi} u(z - \epsilon re^{i\theta})d\theta \\ &\geq \int_0^\infty r\chi(r)dr \int_0^{2\pi} u(z - \epsilon' re^{i\theta})d\theta u(z) \geq 2\pi u(z) \int_0^\infty r\chi(r)dr = u(z) \end{aligned}$$

by Corollary 1.3.4 and the submean inequality. By the last inequality and the fact that $\mathrm{Supp}\chi_\epsilon \in \mathbb{D}(\epsilon)$, we also get

$$u_\epsilon(z) \leq \int_0^\infty \big(\sup_{\mathbb{D}(z,\epsilon)} u(w)\big) r\chi_\epsilon(r)dr.$$

Letting $\epsilon \to 0$ and using the upper semi-continuity of u yield that $\limsup_{\epsilon\to 0} u_\epsilon(z) \leq u(z)$. This combined with the fact that u_ϵ decreases as $\epsilon \to 0$ gives the desired assertion. This finishes the proof. □

We call u_ϵ *standard regularisation of* u.

Lemma 1.3.6. *Let $u \in \mathscr{C}^2(\Omega)$. Then u is subharmonic if and only if $\Delta u \geq 0$.*

Proof. We assume u is subharmonic. Without loss of generality, we can suppose $0 \in \Omega$. It suffices to check $\Delta u(0) \geq 0$. By Taylor's expansion, we have

$$u(x+iy) = u(0)+x\partial_x u(0)+y\partial_y u(0)+\frac{1}{2}\big(x^2\partial_x^2 u(0)+y^2\partial_y^2 u(0)\big)+xy\partial_x\partial_y u(0)+o(|x|^2+|y|^2).$$

Integrating the last inequality over $\partial\mathbb{D}(r)$ gives

$$M_u(r) = u(0) + cr^2\Delta u(0) + o(|r|^2),$$

for some constant $c > 0$. Hence if $\Delta u(0) < 0$, then we would get $M_u(r) < u(0)$ for r small enough. This is a contradiction. Hence $\Delta u(0) \geq 0$.

Now consider $u \in \mathscr{C}^2(\Omega)$ with $\Delta u \geq 0$. In order to get the desired assertion, it suffices to check the sub-mean inequality. Let $\mathbb{D}(w, r)$ be a small disk in Ω. Let h be a harmonic function on $\mathbb{D}(w, r)$ which is continuous up to the boundary such that $u \leq h$ on $\partial\mathbb{D}(r)$. We choose h later. We will verify that $u \leq h$ on $\mathbb{D}(w, r)$. Let $\epsilon > 0$ be a constant. Put $v := u - h + \epsilon|z - w|^2$. Since $\Delta v = \Delta u + \epsilon > 0$, we see that v cannot have local maximum in $\mathbb{D}(w, r)$ (by Taylor's expansion as above). Hence

$$v \leq \limsup_{z\to\partial\mathbb{D}(w,r)} v(z) \leq \epsilon r^2.$$

It follows that $u \leq h + \epsilon r^2$ for every constant $\epsilon > 0$. Hence $u \leq h$ on $\mathbb{D}(w, r)$. Now we choose h to be the function in the right-hand side of (1.3.1). The desired submean inequality hence follows. This finishes the proof. □

Lemma 1.3.7. *Let $\Omega = U + iV$, where U, V are open subsets in $\mathbb{R}$. Let u be a subharmonic function on Ω such that $u(z)$ depends only on $\operatorname{Re} z$. Then the function $u(x)$ with $x \in U$ is convex.*

Proof. By regularisation of u (which depends also only on $\operatorname{Re} z$), we can assume $u \in \mathscr{C}^2$. In this case, we have $0 \le \Delta u(x) = \partial_x^2 u(x)$. Hence u is convex. □

Lemma 1.3.8. *Let $f : \Omega' \to \Omega$ be a holomorphic function. Let u be subharmonic on Ω. Then $u \circ f$ is subharmonic.*

Proof. By regularisation, it suffices to check the desired assertion for u smooth. Put $\partial_z := 1/2(\partial_x - i\partial_y)$ and $\partial_{\bar z} := 1/2(\partial_x + i\partial_y)$. We have $\partial_z \partial_{\bar z} = 1/4\Delta$. Using this formula and the fact that $\partial_{\bar z} f = 0$, one can check that

$$\Delta(u \circ f) = |f'|^2 (\Delta u \circ f) \ge 0.$$

This finishes the proof. □

Theorem 1.3.9. *Let $w \in \Omega$. The function*

$$M_u(r) := \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta$$

is an increasing convex function in $\log r$ (r small).

Proof. We have already known that $M_u(r)$ is increasing. Consider the function

$$M_u(z) := \frac{1}{2\pi} \int_0^{2\pi} u(w + e^z e^{i\theta}) d\theta$$

which is subharmonic by Lemma 1.3.8. This function depends only on $\operatorname{Re} z$. Hence applying Lemma 1.3.7 to $M_u(z)$ implies that $M_u(r)$ is convex. □

1.4 Construction of subharmonic functions

Lemma 1.4.1. *Let $\chi : \mathbb{R}^m \to \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_m)$ is increasing in each variable t_j, and χ can be extended continuously to be a function from $[-\infty, \infty)^m$ to $[-\infty, \infty)$. Let $u_1, \ldots, u_m$ be subharmonic functions on Ω. Then $\chi(u_1, \ldots, u_m)$ is also subharmonic. In particular, the functions $u_1 + \cdots + u_m$, $\max\{u_1, \ldots, u_m\}$, and $\log(e^{u_1} + \ldots + e^{u_m})$ are subharmonic.*

Proof. Let $u_{j,\epsilon}$ be a sequence of subharmonic function decreasing to u_j as $\epsilon \to 0$. By continuity of χ, we see that the continuous function $\chi(u_{1,\epsilon}, \ldots, u_{m,\epsilon})$ decreases to $u := \chi(u_1, \ldots, u_m)$. Hence u is upper semi-continuous. We need to check the submean inequality. To this end, by regularisation, we can assume that u_j is smooth for every j. Let

$w \in \Omega$ and $\mathbb{D}(w,r) \Subset \Omega$. For every $k \in \mathbb{N}$, let $(t_1, \ldots, t_k)$ be points equidistributed in $\partial\mathbb{D}(w,r)$. Since u_j is continuous, we obtain that

$$a_{jk} := \frac{1}{k}\sum_{s=1}^{k} u_j(t_s) \to M_{u_j}(w,r)$$

as $k \to \infty$. Put $b_s := \big(u_1(t_s), \ldots, u_m(t_s)\big)$ for $1 \le s \le m$. Observe that

$$\chi(a_{1k}, \ldots, a_{mk}) = \chi\big(\frac{1}{k}b_1 + \cdots + \frac{1}{k}b_k\big) \le \frac{1}{k}\sum_{s=1}^{k}\chi(b_j)$$

which converges to

$$\frac{1}{2\pi}\int_0^{2\pi} \chi\big(u_1(w+re^{i\theta}), \ldots, u_m(w+re^{i\theta})\big)d\theta = M_u(w,r)$$

as $k \to \infty$ because of the continuity of χ and the choice of $t_1, \ldots, t_k$. On the other hand, since χ is increasing in each variable, we have

$$u(w) \le \chi\big(M_{u_1}(w,r), \ldots, M_{u_m}(w,r)\big) = \lim_{k\to\infty}\chi(a_{1k}, \ldots, a_m k) \le M_u(w,r).$$

This finishes the proof. □

Lemma 1.4.2. *Let f be a holomorphic function on Ω. Then $\log|f|$ is subharmonic on Ω.*

Proof. Firstly $u := \log|f|$ is upper semi-continuous. Observe that u is smooth on $\Omega\backslash\{f = 0\}$, and on the last open set we have

$$\Delta u = 2\partial_z\partial_{\bar z}\log|f|^2 = \partial_z\big(\bar f^{-1}\partial_{\bar z}\bar f\big) = 0.$$

Hence u is harmonic on $\Omega' := \Omega\backslash\{f = 0\}$. In particular the submean inequality holds for every $z \in \Omega'$ and for every small enough disk centered at z. Consider $z_0 \in \{f = 0\}$. Since $u(z_0) = -\infty$, it is clear that the submean inequality holds for z_0. This finishes the proof. □

Corollary 1.4.3. *Let $f_1, \ldots, f_m$ be holomorphic functions. Then for every positive constant $a_1, \ldots, a_m$, we have that $\log(|f_1|^{a_1} + \ldots + |f_m|^{a_m})$ is subharmonic.*

Lemma 1.4.4. *Let $(u_j)_{j\in J}$ be a family of subharmonic function which is locally bounded from above uniformly. Then $(\sup_{j\in J} u_j)^*$ is also subharmonic.*

Proof. The first desired assertion is direct from the definition of subharmonic functions. □

Lemma 1.4.5. *The limit of a decreasing sequence of subharmonic functions is either identically equal to $-\infty$ or a subharmonic function.*

Proof. It is clear from the definition. □

Theorem 1.4.6. *Let u be a subharmonic function on Ω. Let U be an open subset of Ω and v be a subharmonic function on U. Assume that $\limsup_{z' \to z} v(z') \le u(z)$ for every $z \in \partial U \cap \Omega$. Then the function*

$$w = \begin{cases} \max\{u, v\} \text{ on } U, \\ u \text{ on } \Omega \backslash U \end{cases}$$

is subharmonic on Ω.

Proof. We can check easily that w is upper semi-continuous. The submean inequality is also immediate from the hypothesis. □

Lemma 1.4.7. *(strong upper semi-continuity) Let u be a subharmonic function on Ω. Let B be a set of zero Lebesgue measure on Ω. Then for every $z \in \Omega$, we have*

$$\limsup_{z' \notin B \to z} u(z') = u(z).$$

Proof. This is a direct consequence of the upper semi-continuity property and submean inequality. Indeed, by the submean inequality and the polar coordinates, we get

$$u(z) \le \int_{\mathbb{D}(z,\epsilon)} u(z) d\,\text{Leb} = \int_{\mathbb{D}(z,\epsilon)\backslash \mathbb{B}} u(z) d\,\text{Leb} \le \sup_{z' \in \mathbb{D}(z,\epsilon)\backslash B} u(z').$$

Letting $\epsilon \to 0$ gives $\limsup_{z' \notin B \to z} u(z') \ge u(z)$. The converse inequality follows from the upper semi-continuity. □

Theorem 1.4.8. *Let A be a closed subset in $\mathbb{C}$ such that $A = \{v = -\infty\}$ for some subharmonic function v on Ω. Let u be a subharmonic function on $\Omega \backslash A$ such that for every compact subset K on Ω, the function u is bounded from above on $K \backslash A$. Then u can be extended uniquely to be a subharmonic function $\tilde{u}$ on Ω.*

Proof. We check the uniqueness of $\tilde{u}$. Now observe that A is of zero Lebesgue measure in Ω because v is locally integrable. Hence, using Lemma 1.4.7, for $z \in A$, we have

$$\tilde{u}(z) = \limsup_{z' \notin A \to z} \tilde{u}(z') = \limsup_{z' \notin A \to z} u(z').$$

In other words, $\tilde{u}$ is unique, if it exists.

We now prove the existence. Since the problem is local, we can assume $v < 0$. Define $u_\epsilon := u + \epsilon v$ on $\Omega \backslash A$ and $u_\epsilon := -\infty$ on A. One can see that u_ϵ is upper semi-continuous, and satisfies the submean inequality. Hence, u_ϵ is subharmonic. Hence by Lemma 1.4.4, the function $(\sup_{\epsilon > 0} u_\epsilon)^*$ is a subharmonic function on Ω which is equal to u on $\Omega \backslash A$. This finishes the proof. □

1.5 Laplacian of subharmonic functions

We know that if a $\mathscr{C}^2$ function is subharmonic, then its Laplacian is positive. We explain in this section how to extend the last property to all subharmonic functions.

Recall that *a distribution* T on Ω is a continuous linear functional from $\mathscr{C}^\infty_c(\Omega)$ to $\mathbb{C}$. Here $\mathscr{C}^\infty_c(\Omega)$ denotes the set of smooth functions with compact support in Ω, and by continuity we mean that for every sequence $(f_j)_{j\in\mathbb{N}} \subset \mathscr{C}^\infty_c(\Omega)$ such that there exists a compact $K \subset \Omega$ satisfying $\mathrm{Supp} f_j \subset K$ for every j and f_j converges to some $f_\infty \in \mathscr{C}^\infty_c(\Omega)$ in $\mathscr{C}^\infty$ topology, we have $\langle T, f_j\rangle \to \langle T, f_\infty\rangle$ as $j \to \infty$.

Lemma 1.5.1. *A linear functional $T : \mathscr{C}^\infty_c(\Omega) \to \mathbb{C}$ is continuous if and only if for every compact $K \subset \Omega$, there exist an integer $k \in \mathbb{N}$ and a constant $C > 0$ such that*

$$\langle T, f\rangle \le C\|f\|_{\mathscr{C}^k(\Omega)},$$

for every smooth f with compact support in K.

Proof. Straightforward. □

Every locally integrable function g on $\mathbb{C}$ can be viewed as a distribution T_g by putting

$$\langle T_g, f\rangle := \int_{\mathbb{C}} gf \, d\,\mathrm{Leb}\,.$$

One can check that T_g is linear and continuous in the above sense. In practice we usually identify T_g with g, and use the same notation g to denote T_g. Every (positive) Radon measure is also a distribution by integration functions against it. Here we recall that a Radon measure on a topological space X is a (Borel) measure μ on X such that $\mu(K) < \infty$ for every compact subset K in X.

Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of distribution on Ω. We say that T_j *converges weakly to* T if

$$\langle T_j, f\rangle \to \langle T, f\rangle$$

as $j \to \infty$ for every $f \in \mathscr{C}^\infty_c(\Omega)$. If $(u_j)_{k\in\mathbb{N}}$ is a sequence of locally integrable functions converges in L^1_{loc} to a function u, then $u_j \to u$ as $j \to \infty$ in the sense of distributions.

Let T be a distribution on Ω. We can define partial derivatives $\partial_x T$ and $\partial_y T$ by the following formula

$$\langle \partial_x T, f\rangle := -\langle T, \partial_x f\rangle$$

for every $f \in \mathscr{C}^\infty_c(\Omega)$, and similarly for $\partial_y T$. By integration by parts formula, these operators extend the usual partial derivatives of $\mathscr{C}^1$ functions. We say that T is positive and write $T \ge 0$ if $\langle T, f\rangle \ge 0$ for every $f \in \mathscr{C}^\infty_c(\Omega)$ with $f \ge 0$. For distributions T_1, T_2, we write $T_1 \ge T_2$ of $T_1 - T_2 \ge 0$. Recall the following fundamental fact.

Theorem 1.5.2. *([32, Theorem 2.14]) Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on the space $\mathscr{C}_c(X)$ of continuous functions with compact support in X. Then there exists a Radon measure μ on X representing Λ,* i.e,

$$\langle \Lambda, f\rangle = \int_X f d\mu$$

for every $f \in \mathscr{C}_c(X)$.

When X is a locally compact Hausdorff space in which every open subset can be written as a countable union of compact subsets, every Radon measure is regular (see [32, Theorem 2.18]), *i.e* for every Borel set E in X, we have

$$\mu(E) = \inf\{\mu(V) : E \subset V,\, V\,\text{open}\} = \sup\{\mu(K) : K \subset E,\, K\,\text{compact}\}.$$

In particular every Radon measure on $\mathbb{R}^m$ is regular.

Corollary 1.5.3. *Every positive distribution is a positive Radon measure.*

By the last result, for every positive distribution T, we can define $\langle T, f\rangle$ for every continuous function f with compact support, or more generally for every bounded (Borel) measurable function f on X. Note by Lebesgue's dominate convergence theorem, for every sequence $(f_j)_j$ of uniformly bounded smooth functions supported in a fixed compact subset in Ω such that f_j converges pointwise to some function f as $j \to \infty$, then we have $\langle T, f_j\rangle \to \langle T, f\rangle$ as $j \to \infty$. The following is fundamental in the theory of distributions.

Theorem 1.5.4. *Let $(T_j)_j$ is a sequence of distributions such that the sequence $\langle T_j, f\rangle$ is bounded for every $f \in \mathscr{C}^\infty_c(\Omega)$. Then for every compact K in Ω, there exist an integer $k \geq 0$ and a constant $C > 0$ such that*

$$|\langle T_j, f\rangle| \leq C\|f\|_{\mathscr{C}^k(\Omega)}, \tag{1.5.1}$$

for every j and every smooth f with compact support in K. Consequently, if the limit $\langle T, f\rangle := \lim_{j\to\infty}\langle T_j, f\rangle$ exists for every $f \in \mathscr{C}^\infty_c(\Omega)$, then T is a distribution and T_j converges weakly to T as $j \to \infty$.

Proof. The second desired assertion follows directly from the first one. We check the first one. It is a consequence of the Banach-Steinhaus theorem in functional analysis (see [33, Theorem 2.6]). Let K be a compact subset in Ω, and $\mathscr{C}^\infty(K, \Omega)$ be the set of $f \in \mathscr{C}^\infty_c(\Omega)$ such that $\mathrm{Supp} f \subset K$. Observe that $\mathscr{C}^\infty(K, \Omega)$ with the topology induced by that of $\mathscr{C}^\infty_c(\Omega)$ (here $f_j \to f$ in $\mathscr{C}^\infty(K, \Omega)$ if $\|f_j - f\|_{\mathscr{C}^k(\Omega)} \to 0$ as $j \to \infty$ for every $k \in \mathbb{N}$) is naturally endowed with a metric which makes it to be a complete metric space.

By Lemma 1.5.1, every distribution S induces naturally a continuous linear functional S_K on $\mathscr{C}^\infty(K, \Omega)$. Hence we obtain continuous functionals $T_{j,K} : \mathscr{C}^\infty(K, \Omega) \to \mathbb{C}$ induced by T_j. By hypothesis, $\langle T_{j,K}, f\rangle$ is bounded uniformly in j for every $f \in \mathscr{C}^\infty(K, \Omega)$. This combined with the Banach-Steinhaus theorem applied to $(T_{j,K})_j$ gives (1.5.1). This finishes the proof. □

Let T be a distribution and let g be a smooth function with compact support in Ω. The *convolution* of T with g is defined by

$$T * g(x) := \langle T(\cdot), g(x - \cdot)\rangle,$$

for $x \in \Omega$ such that the distance from x to $\partial\Omega$ is greater than the diameter of the support of g.

Lemma 1.5.5. *The following statements are true:*

(i) *the convolution $T * g$ is smooth, and*

$$D^\alpha(T * g) = (D^\alpha T) * g = T * D^\alpha g,$$

(ii) *for every smooth f with compact support in Ω,*

$$\langle T * g, f\rangle = \langle T, f * g_1\rangle,$$

where $g_1(z) := g(-z)$,

(iii) *let χ be a smooth radial function with compact support in Ω such that $\int_\Omega \chi d\,\mathrm{Leb} = 1$ and $\chi_\epsilon(z) := \epsilon^{-2}\chi(z/\epsilon)$, then $T * \chi_\epsilon$ converges weakly to T as $\epsilon \to 0$,*

(iv) *if $(T_j)_j$ is a sequence of distributions converging to a distribution T then $T_j * g \to T * g$ in $\mathscr{C}^\infty$ topology.*

Proof. We check (i). It suffices to do it for $\alpha = 1$. Using linearity of T gives

$$\begin{aligned} D(T * g)(x) &= \lim_{h\to 0} h^{-1}\big(T * g(x+h) - T * g(x)\big) \\ &= \lim_{h\to 0}\langle T, h^{-1}(g(x+h-\cdot) - g(x-\cdot))\rangle. \end{aligned}$$

Put $g_x(y) := g(x-y)$. Observe now that

$$h^{-1}(g(x+h-y) - g(x-y)) = h^{-1}(g_x(y-h) - g_x(y))$$

converges to $-Dg_x(y)$ in $\mathscr{C}^\infty$ topology as $h \to 0$ (x fixed). Hence by continuity of T, we get

$$D(T * g)(x) = \langle T, -Dg_x\rangle = \langle DT, g_x\rangle.$$

Since $Dg_x(y) = -Dg(x-y)$, we also obtain

$$D(T * g)(x) = \langle T, -Dg_x\rangle = \langle T, Dg\rangle.$$

We check (ii). Since $T * g$ is smooth, we can decompose the following integral into Riemann sum:

$$\langle T * g, f\rangle = \int_{\mathbb{C}} T * g(x) f(x) = \pi^{-1}\lim_{\epsilon\to 0}\epsilon^{-2}\sum_{j\in A_\epsilon} T * g(x_j) f(x_j),$$

where we decompose $\mathbb{C}$ into squares of size ϵ indexed by a countable family A_ϵ and choose x_j to be the center of the squares. Observe that

$$D^\alpha\big(\pi^{-1}\epsilon^{-2}\sum_{j\in A_\epsilon} g(x_j - y) f(x_j)\big) = \pi^{-1}\epsilon^{-2}\sum_{j\in A_\epsilon} -D^\alpha g(x_j - y) f(x_j)\big)$$

which converges uniformly in y to

$$\int_{\mathbb{C}} -D^\alpha g(x-y) f(x) d\,\mathrm{Leb}(x)$$

as $\epsilon \to 0$ for every α. This combined with the continuity of T gives

$$\langle T * g, f\rangle = \langle T, f * g_1\rangle.$$

The desired assertion (iii) is a direct consequence of (ii) and the fact that $f * \chi_\epsilon \to f$ as $\epsilon \to 0$ in $\mathscr{C}^\infty$ topology.

It remains to verify (iv). Firstly we have

$$D^\alpha T_j * g(x) = T_j * D^\alpha g(x) \to T * D^\alpha g(x) = D^\alpha T * (x)$$

as $j \to \infty$ for every x by the weak convergence of T_j. The pointwise convergence of $D^\alpha T_j * g$ is actually uniform thanks to Theorem 1.5.4. Thus, we obtain the $\mathscr{C}^\infty$ convergence of $T_j * g$ to $T * g$. This finishes the proof. □

Remark 1.5.6. *All of above properties of distributions have direct analogues for distributions in a domain in* $\mathbb{R}^n$.

We come back to subharmonic functions.

Theorem 1.5.7. *Let u be a subharmonic function on Ω. Then Δu is a positive measure.*

Proof. Let $(u_\epsilon)_\epsilon$ be a regularisation of u. We have $\Delta u_\epsilon \to \Delta u$ as $\epsilon \to 0$ in the sense of distributions. This combined with the fact that $\Delta u_\epsilon \geq 0$ implies that Δu is a positive distribution. By this and Corollary 1.5.3, we obtain the desired assertion. □

Theorem 1.5.8. *Let u be a distribution on Ω such that $\Delta u \geq 0$. Then there exists a subharmonic function u' on Ω such that $u = u'$ (that means u is the distribution induced by u').*

Proof. Let $u_\epsilon := u * \chi_\epsilon$ be standard regularisation of u. We have $\Delta u_\epsilon = (\Delta u) * \chi_\epsilon \geq 0$. Hence u_ϵ is subharmonic. We check that u_ϵ decreases as $\epsilon \to 0$. To this end, we consider

$$u_{\epsilon,\delta} := u_\epsilon * \chi_\delta$$

which converges weakly to u_δ as $\epsilon \to 0$ because $u_\epsilon \to u$ weakly. Since $u_{\epsilon,\delta}$ is decreasing in δ because it is a standard convolution of u_ϵ which is a subharmonic function. Letting $\epsilon \to 0$ and using the above observation implies that u_δ is decreasing in δ as distributions. Hence u_δ is decreasing in δ as functions because they are smooth. By this and Lemma 1.4.5, the pointwise limit of $(u_\epsilon)_\epsilon$ is either identically equal to $-\infty$ (in this case $u_\epsilon \to -\infty$ in L^1_{loc} by Lebesgue's monotone convergence theorem) or a subharmonic function. The former case cannot happen because $u_\epsilon \to u$ as distributions. We infer that u_ϵ decreases to a subharmonic function u' which is equal to u almost everywhere. This finishes the proof. □

Corollary 1.5.9. *Let $(u_j)_{j\in J}$ be a family of subharmonic function which is locally bounded from above uniformly. Then the set of $z \in \Omega$ such that $(\sup_{j\in J} u_j)^*(z) > (\sup_{j\in J} u_j)(z)$ is of zero Lebesgue measure.*

Proof. Let $u := (\sup_{j\in J} u_j)$. We know that $\Delta u \geq 0$ as distributions. Let u_ϵ be standard regularisation of u. By the proof of Theorem 1.5.8, the sequence $(u_\epsilon)_\epsilon$ is decreasing. Hence u_ϵ decreases to a subharmonic function u'. We get $u' = u$ as distribution. Since both functions are locally integrable, we see that they are equal almost everywhere. On the other hand, by the upper semi-continuity, we have $u \leq u^* \leq u_\epsilon$. It follows that $u^* \leq u'$. Since $u' = u$ almost everywhere, we deduce that $u' = u^*$ almost everywhere. □

The set of x such that $(\sup_{j\in J} u_j)^*(z) > (\sup_{j\in J} u_j)(z)$ is called *a negligible set*. By the above result, every negligible set is of zero Lebesgue measure. We will see later that a much deeper property holds: every negligible set is (pluri)polar.

Corollary 1.5.10. *Let u be a distribution on Ω such that $\Delta u = 0$. Then there exists a harmonic function u' on Ω such that $u = u'$.*

Proof. It suffices to apply Theorem 1.5.8 to u and $-u$. □

Lemma 1.5.11. *We have*

$$\Delta \log|z| = 2\pi\delta_0,$$

where δ_0 is the Dirac mass at 0.

Proof. Let $f \in \mathscr{C}^\infty_c(\mathbb{C})$. We compute

$$\begin{aligned}\langle \Delta \log|z|, f\rangle &= \langle \log|z|, \Delta f\rangle \\ &= \frac{1}{2}\lim_{\epsilon\to 0}\int_{\mathbb{C}} \log(|z|^2+\epsilon)\Delta f\, d\,\mathrm{Leb} = \frac{1}{2}\lim_{\epsilon\to 0}\int_{\mathbb{C}} \Delta\log(|z|^2+\epsilon) f\, d\,\mathrm{Leb}\,.\end{aligned}$$

Note that

$$\Delta \log(|z|^2+\epsilon) = 4\partial_z\partial_{\bar z}\log(|z|^2+\epsilon) = \frac{4\epsilon}{(|z|^2+\epsilon)^2}.$$

Using this and the polar coordinates gives

$$\begin{aligned}\int_{\mathbb{C}} \Delta\log(|z|^2+\epsilon) f\, d\,\mathrm{Leb} &= 2\pi\int_0^\infty \frac{4\epsilon r f(re^{i\theta})d\theta}{(r^2+\epsilon)^2}dr \\ &= 4\pi f(0)\int_0^\infty \frac{2\epsilon r d\theta}{(r^2+\epsilon)^2}dr + 2\pi\int_0^\infty \frac{4\epsilon r O(r))d\theta}{(r^2+\epsilon)^2}dr,\end{aligned}$$

where we decomposed $f(re^{i\theta}) = f(0) + O(r)$ as r small. Direct computations show that the first integral converges to $4\pi f(0)$ as $\epsilon \to 0$, whereas the second one converges to 0 as $\epsilon \to 0$. We infer that $\langle \Delta \log|z|, f\rangle = 2\pi f(0)$. This finishes the proof. □

The following result tells us that every measure with compact support in $\mathbb{C}$ is indeed the Laplacian of some subharmonic function on $\mathbb{C}$.

Theorem 1.5.12. *Let μ be a measure with compact support in $\mathbb{C}$. The function*

$$u_\mu(z) := \int_{w\in\mathbb{C}} \log|z-w| d\mu(w)$$

is subharmonic on $\mathbb{C}$, and $\Delta u_\mu = 2\pi\mu$.

Proof. For every constant $\epsilon > 0$, put

$$u_\epsilon := \int_{w\in\mathbb{C}} \log(|z-w|+\epsilon) d\mu(w).$$

Observe that u_ϵ is continuous. Since $\log(|z-w|+\epsilon)$ is subharmonic and continuous, we see that u_ϵ is an average of subharmonic functions. Hence u_ϵ is subharmonic. Note that u_ϵ decreases to u_μ as $\epsilon \to 0$. Hence u_μ is either $- \equiv -\infty$ or subharmonic. For $z \notin \operatorname{Supp}\mu$, we have $u_\mu(z) > -\infty$. Hence $u \not\equiv -\infty$. We deduce that u_μ is subharmonic.

It remains to check that $\Delta u_\mu = \mu$. Let $f \in \mathscr{C}_c^\infty(\mathbb{C})$. We compute

$$\begin{aligned}
\langle \Delta u, f\rangle = \langle u, \Delta f\rangle &= \int_{z\in\mathbb{C}} \Delta f(z) d\,\mathrm{Leb} \int_{\mathbb{C}} \log|z-w| d\mu(w) \\
&= \int_{w\in\mathbb{C}} d\mu(w) \int_{z\in\mathbb{C}} \Delta f(z) \log|z-w| d\,\mathrm{Leb} \\
&= \int_{w\in\mathbb{C}} d\mu(w) \int_{z\in\mathbb{C}} f(z) \Delta \log|z-w| d\,\mathrm{Leb} = \int_{\mathbb{C}} f d\mu
\end{aligned}$$

by Lemma 1.5.11. Hence $\Delta u_\mu = \mu$. This ends the proof. □

Theorem 1.5.13. *(Riesz's representation formula) Let u be a subharmonic function on an open neighborhood of $\overline{\mathbb{D}}$. Then we have*

$$2\pi u(z) = \int_{\mathbb{D}} \log\left|\frac{z-\xi}{1-z\bar\xi}\right| \Delta u + \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} u(e^{i\theta}) d\theta,$$

where Δu is identified with a measure.

Proof. As the first step, we assume u smooth and prove the desired formula. Let v_1, v_2 be the first and second integral in the sum on the right-hand side of the desired formula. For a fixed $\xi \in \mathbb{D}$ and $z \in \mathbb{D}$, note that $1 - z\bar\xi$ has no zero in $\mathbb{D}$. Hence $\Delta \log|1-z\bar\xi| = 0$ on $\mathbb{D}$. It follows that

$$\Delta w = \Delta \log|z-\xi| = 2\pi\delta_\xi,$$

where $w := \log\left|\frac{z-\xi}{1-z\bar\xi}\right|$. Note also that e^w is continuous. Hence w is subharmonic. Since $w = 0$ on $\partial\mathbb{D}$, we get $w \le 0$ on $\mathbb{D}$ by the maximum principle. Hence v_1 is a subharmonic function whose Laplacian is equal to $2\pi\Delta_u$. On the other hand, for every $z_0 \in \partial\mathbb{D}$, since Δu is smooth, we have $\lim_{z\to z_0} v_1(z) = 0$. In other words, v_1 can be extended continuously up to boundary and equal to 0 on the boundary. On the other hand, v_2 is subharmonic and equal to $2\pi u$ on $\partial\mathbb{D}$. We deduce that $v_1 + v_2$ is subharmonic and equal to $2\pi u$ on $\partial\mathbb{D}$, and $\Delta(v_1+v_2) = 2\pi\Delta u$. Hence $v_1 + v_2 - 2\pi u$ is harmonic on $\mathbb{D}$ and equal to 0 on $\partial\mathbb{D}$. It follows that $v_1 + v_2 - 2\pi u \equiv 0$ by the maximum principle.

Now consider the general case. Since u is defined on an open neighborhood of $\overline{\mathbb{D}}$, we can construct a sequence of smooth subharmonic functions $(u_\epsilon)_\epsilon$ defined on an open neighborhood of $\overline{\mathbb{D}}$ such that u_ϵ decreases to u. By the first part of the proof, we have the

Riesz representation formula for u_ϵ. In order to obtain that for u, we just need to check that

$$v_{1,\epsilon}(z) := \int_{\mathbb{D}} w\Delta u_\epsilon \to v_1(z) = \int_{\mathbb{D}} \log\left|\frac{z-\xi}{1-z\bar{\xi}}\right|\Delta u \tag{1.5.2}$$

as $\epsilon \to 0$ for every $z \in \mathbb{D}$. Let $w' := w$ on $\mathbb{D}$ and $w' := 0$ on $\mathbb{C}\backslash\mathbb{D}$. As observed above, w' is subharmonic on $\mathbb{D}$ and $w' \leq 0$. Let χ be a radial function used to defined the regularisation u_ϵ, and define χ_ϵ as usual. For every We have

$$\langle \Delta u_\epsilon, w'\rangle = \langle \Delta(u * \chi_\epsilon), w'\rangle = \langle \Delta u, w' * \chi_\epsilon\rangle.$$

The function $w' * \chi_\epsilon$ decreases to w on $\overline{\mathbb{D}}$ as $\epsilon \to 0$ (recall $w = 0$ on $\partial\mathbb{D}$), and to 0 on $\mathbb{C}\backslash\overline{\mathbb{D}}$. Hence (1.5.2) follows by Lebesgue's monotone convergence theorem. The proof is finished. □

1.6 Compactness properties

We start with the following result suggesting that the singularity of subharmonic functions in $\mathbb{C}$ is "not much worse" than that of the logarithmic function.

Theorem 1.6.1. *Let u be a subharmonic function defined on an open neighborhood of $2\overline{\mathbb{D}}$ such that $\|u\|_{L^1(2\mathbb{D})} \leq 1$. Let K be a compact subset of $\mathbb{D}$. Then there exist constants $C, \alpha > 0$ independent of u such that*

$$\int_K e^{-\alpha u} d\,\mathrm{Leb} \leq C.$$

In particular the L^p norm of u on K is uniformly bound.

Proof. Using partition of unity, it suffices to prove the desired assertion for $K := 1/4\mathbb{D}$. Suppose that there exists $z_0 \in 3/2\mathbb{D}$ such that $u(z_0) \geq 10$ then by the submean inequality, we get

$$\|u\|_{L^1(2\mathbb{D})} \geq \int_{\mathbb{D}(z_0,1)} |u| d\,\mathrm{Leb} \geq 1/2u(z_0) = 9/8 > 1.$$

This is a contradiction. Hence $u \leq 10$ on $\mathbb{D}_1$. By similar arguments, we also see that there exists a $z_1 \in 1/8\mathbb{D}$ such that $u(z_1) \geq -C_0$ for some constant $C_0 > 0$ independent of u. By considering $1/(10 + C_0)(u(\cdot - z_1) + C_0)$ instead of u, we reduce the question to the following statement: for a subharmonic function u defined on $\overline{\mathbb{D}}$ such that $u(0) \geq 0$ and $u \leq 1$ on $\mathbb{D}$, then there exist constants $\alpha, C > 0$ both independent of u such that

$$\int_{\mathbb{D}_{\frac{1}{2}}} e^{-\alpha u} d\,\mathrm{Leb} \leq C.$$

We check it now. Let α be a small strictly positive constant to be chosen later. By maximum principle, $u \leq 1$ on $\mathbb{D}$. By Riesz representation formula applied to $(u-1)$ in Theorem 1.5.13, we get

$$u(z) - 1 = \frac{1}{2\pi}\int_{\mathbb{D}} \log\left|\frac{z-\xi}{1-z\bar{\xi}}\right|\Delta u + \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}(u(e^{i\theta}) - 1)d\theta.$$

Let $u_1(z), u_2(z)$ be the first term and the second term in the right-hand side of the last equality.

We use the letter C to denote a general positive constant independent of u, although its value can vary line from line. Note that $u_1 \leq 0$ (see the proof of Theorem 1.5.13). We also have $u_2 \leq 0$ because $u \leq 1$ on $\partial\mathbb{D}$. Using the equality $-1 = u_1(0) + u_2(0)$ (let $z = 0$ in the Riesz representation formula) yields that

$$-1 \leq u_1(0) \leq 0, \quad -1 \leq u_2(0) \leq 0.$$

Observe that there exists a constant $C > 0$ such that

$$|u_2(z)| \leq C\frac{1}{2\pi}\int_0^{2\pi} -(u(e^{i\theta}) - 1)d\theta = -Cu_2(0) \leq C$$

for every $|z| \leq 1/2$. We deduce that

$$\int_{\mathbb{D}_{1/2}} e^{-\alpha u} d\,\mathrm{Leb} \leq C_\alpha \int_{\mathbb{D}_{1/2}} e^{-\alpha u_1} d\,\mathrm{Leb}\,.$$

It remains to estimate the last integral for suitable small α.

Since $u_1(0) = \frac{1}{2\pi}\int_{\mathbb{D}} \log|\xi|\Delta u$, we infer that $A := \Delta u(\mathbb{D}_{\frac{1}{2}}) \leq C$ for some constant C independent of u. Since e^t is convex and $\mu(\partial\mathbb{D}) \leq$, by Jensen's inequality, for every constant $\alpha > 0$, one obtains

$$e^{-\alpha u_1(z)} \lesssim \int_{\xi\in\mathbb{D}} \left|\frac{z-\xi}{1-z\bar{\xi}}\right|^{-A\alpha} (\Delta u/A) \lesssim \int_{\xi\in\mathbb{D}} |z-\xi|^{-A\alpha}\big(\Delta u/A\big).$$

Integrating over $z \in \mathbb{D}_{1/2}$ and noting that $\int_{\mathbb{D}_{1/2}} |z-\xi|^{-A\alpha} < \infty$ if $\alpha < 2/A \leq 2/C$, we obtain

$$\int_{\mathbb{D}_{1/2}} e^{-\alpha u_1(z)} \lesssim \int_{\xi\in\mathbb{D}} (\Delta u/A) = 1.$$

This ends the proof. □

We recall the following standard fact.

Lemma 1.6.2. *Let X be a metric space which is a countable union of compact subsets. Let $(\mu_j)_j$ be a sequence of measures on X of mass bounded uniformly on compact subsets on X. Then there exists a subsequence $(\mu_{j'})_{j'}$ of $(\mu_j)_j$ such that $\mu_{j'}$ converges weakly to some measure μ_∞.*

Note here the weak convergence of measures seen as distributions coincides with the usual notion of converges of measures.

Proof. By a diagonal argument, we can assume X is compact. We will only use this result for X is an open subset of $\mathbb{R}^m$. So we give here a proof for this case. The general case is done similarly. The space $\mathscr{C}^0(X)$ of continuous functions with the supnorm on X is separable. This means that it has a countable dense subset. Let $A = \{f_1, f_2, \ldots\}$ be such a dense subset. By a diagonal argument and the fact that $\mu_j(X) \leq M$ for some constant

M independent of j, we can extract a subsequence $(\mu_{j'})_{j'}$ such that $\langle \mu_{j'}, f_k\rangle$ is convergent as $j \to \infty$ for every k. Define μ_∞ as follows. Put

$$\langle \mu_\infty, f_k\rangle := \lim_{j'\to\infty} \langle \mu_{j'}, f_k\rangle.$$

Since A is dense, for every continuous function f on X, we can find a sequence $(f_{s_k})_k$ converging to f in the supnorm. Since

$$|\langle \mu_\infty, f_{s_k} - f_{s_{k'}}\rangle| = \lim_{j'\to\infty} |\langle \mu_{j'}, f_{s_k} - f_{s_{k'}}\rangle| \le M\|f_{s_k} - f_{s_{k'}}\|,$$

we infer that the sequence $\langle \mu_\infty, f_{s_k}\rangle$ is convergent. Hence we can put

$$\langle \mu_\infty, f\rangle := \lim_{k\to\infty} \langle \mu_\infty, f_{s_k}\rangle.$$

By similar reasoning, one can check that this definition is independent of the choice of $(f_{s_k})_k$. Hence we obtain a function $\mu_\infty : \mathscr{C}^0(X) \to \mathbb{R}$ which is linear and positive. Hence by Theorem 1.5.2, μ_∞ is a measure. We leave the readers to check that $\mu_j \to \mu_\infty$. □

Lemma 1.6.3. *Let $(u_j)_j$ be a sequence of functions on Ω which are of L^1-norm locally bounded uniformly in j,* i.e*, for every compact K in Ω, there exists a constant M_K such that $\|u_j\|_{L^1(K)} \le M_K$ (L^1-norm is computed with respect to Lebesgue measure on $\mathbb{C}$). Then there exists a sequence $(j_j)_k \subset \mathbb{N}$ such that u_{j_k} (considered as a distribution) converges weakly to some distribution on Ω as $j \to \infty$.*

Proof. Let $u_j^+ := \max\{u_j, 0\}$ and $u_j^- := -\min\{u_j, 0\}$. We have $u_j^\pm \ge 0$ and $u_j = u_j^+ - u_j^-$. Since

$$\int_K |u_j| d\,\mathrm{Leb} = \int_K |u_j^+| d\,\mathrm{Leb} + \int_K |u_j^-| d\,\mathrm{Leb},$$

we infer that $u_j^\pm$ is of L^1-norm locally bounded uniformly in j. Since $u_j^\pm$ is non-negative, we can view them as positive distribution (hence measures). Now one just applies Lemma 1.6.2 to $u_j^\pm$ to obtain the desired assertion. □

The following is fundamental in pluripotential theory.

Theorem 1.6.4. (i) *Let $(u_j)_j$ be a sequence of subharmonic functions converging weakly to some distribution u_∞ on Ω. Then u_∞ is also a subharmonic function and $u_j \to u_\infty$ in L^p_{loc} for every constant $0 < p < \infty$. Furthermore for every compact subset K in Ω and every continuous function f on K, we have*

$$\limsup_{j\to\infty} \sup_K (u_j - f) \le \sup_K (u - f). \tag{1.6.1}$$

(ii) *Let $(u_j)_j$ be a sequence of subharmonic functions uniformly locally bounded from above defined on Ω. Then either u_j converges uniformly on compact subsets in Ω to $-\infty$ as $j \to \infty$ or there exists a subsequence $(u_{j'})_{j'}$ which converges in L^p to a subharmonic function u_∞ for every real number $0 < p < \infty$.*

Proof. Since $\Delta u_\infty = \lim_{j\to\infty} \Delta u_j \geq 0$, by Theorem 1.5.8, u_∞ is a subharmonic function. As the next step, we verify that $u_j \to u_\infty$ in L^1_{loc}.

Let $u_{j,\epsilon}, u_{\infty,\epsilon}$ be standard regularisation of u_j, u_∞ respectively defined using the same cut-off function χ. Let χ_ϵ be the function induced by χ and ϵ as usual. By the first part of the proof, the sequence u_j is of L^1-norm locally bounded uniformly. We have

$$u_{j,\epsilon}(z) = \int_\Omega \chi_\epsilon(z-x)u_j(x)d\,\mathrm{Leb}(x).$$

Since $u_j \to u_\infty$ as distributions, using Lemma 1.5.5 (iv), we see that $u_{j,\epsilon}$ is equicontinuous in j for ϵ fixed. This together with the fact that $u_{j,\epsilon}$ converges pointwise to $u_{\infty,\epsilon}$ yields that the convergence $u_{j,\epsilon} \to u_{\infty,\epsilon}$ is uniformly on compact subsets, for ϵ fixed. By subharmonicity, we have

$$u_j \leq u_{j,\epsilon}, \quad u_\infty \leq u_{\infty,\epsilon}.$$

Let $\delta > 0$ be a constant. We estimate

$$u_j - u_{\infty,\epsilon} - \delta \leq u_j - u_\infty \leq u_{j,\epsilon} - u_\infty + \delta.$$

Since $u_{j,\epsilon} \to u_{\infty,\epsilon} \geq u_\infty$ uniformly, we infer that $u_{j,\epsilon} - u_\infty + \delta \geq 0$ if j is large enough. Likewise

$$u_j - u_{\infty,\epsilon} - \delta \leq u_{j,\epsilon} - u_{\infty,\epsilon} - \delta \leq 0$$

for j big enough. Hence

$$\begin{aligned}\|u_j - u_\infty\|_{L^1(K)} &\leq \int_K \max\{u_{j,\epsilon} - u_\infty + \delta, u_{\infty,\epsilon} - u_j + \delta\} d\,\mathrm{Leb} \\ &\leq \int_\Omega f(u_{j,\epsilon} - u_\infty + \delta + u_{\infty,\epsilon} - u_j + \delta) d\,\mathrm{Leb}\end{aligned}$$

where f is a nonnegative smooth function with compact support which is equal to 1 on K. Letting $j \to \infty$ in the last inequality gives

$$\limsup_{j\to\infty} \|u_j - u_\infty\|_{L^1(K)} \leq \int_\Omega f(2u_{\infty,\epsilon} - 2u_\infty + 2\delta) d\,\mathrm{Leb}\,.$$

Letting $\epsilon, \delta \to 0$ implies that $u_j \to u_\infty$ in L^1_{loc}. We check (1.6.1). Let f be a continuous function ≥ 0 on K. By above arguments (the fact that $u_{j,\epsilon} \to u_{\infty,\epsilon}$ uniformly on compact subsets),

$$\limsup_{j\to\infty} \sup_K (u_j - f) \leq \limsup_{j\to\infty} \sup_K (u_{j,\epsilon} - f) = \sup_K (u_{\infty,\epsilon} - f)$$

for every constant $\epsilon > 0$. Put $M := \sup_K (u - f)$. Observe that the continuous function $\max\{u_{\infty,\epsilon} - f, M\}$ decreases pointwise to the constant function M as $\epsilon 0$. Hence by Dini's theorem, $\max\{u_{\infty,\epsilon} - f, M\}$ converges uniformly to M as $\epsilon \to 0$. We infer that $\sup_K (u_{\infty,\epsilon} - f) \to M$ as $\epsilon \to 0$. Thus (1.6.1) follows.

Since $(u_j)_j$ is uniformly locally bounded from above and the problem is local, we can assume that $u_j \leq 0$ on Ω for every j. Assume that u_j does not converge uniformly on compact subsets in Ω to $-\infty$ as $j \to \infty$. This means that there exists a compact subset $K \subset \Omega$ and $z_j \in K$ such that $u_j(z_j) \geq -C$ for every $j \in \mathbb{N}$ some constant C independent of j. We need to prove that there exists a subsequence of $(u_j)_j$ converging in L^p for every $1 \leq p < \infty$.

By considering a subsequence of $(z_j)_j$, we can assume that z_j converges to $z_\infty \in K$. Since $u_j(z_j) > -C$ and z_j close to z_∞ as j big, using the submean inequality implies that $\|u_j\|_{L^1(B)}$ is uniformly bounded for some small ball B centered at z_∞. Let A be the set of $z \in \Omega$ such that there exists a small ball B containing z and $\|u_j\|_{L^1(B)}$ is uniformly bounded. We have just seen that A is non empty. Moreover A is open because of its definition. By an argument similar to those in the proof of Lemma 1.3.2, we can prove that A is also closed. Hence $A = \Omega$ (we always assume Ω is connected). In other words, the sequence $(u_j)_j$ is of L^1 norm locally bounded uniformly in j. Hence, by extracting a subsequence, we can assume that u_j converges weakly to a distribution u_∞ as $j \to \infty$. By Part (i), u_j converges to u_∞ in L^1_{loc}.

Consider now $p \geq 1$. By Hölder's inequality, we have

$$\begin{aligned} \|u_j - u_\infty\|^p_{L^p(K)} &= \int_K |u_j - u_\infty|^{1/2} |u_j - u_\infty|^{p-1/2} d\,\mathrm{Leb} \\ &\leq \Big(\int_K |u_j - u_\infty| d\,\mathrm{Leb} \Big)^2 \Big(\int_K |u_j - u_\infty|^{2p-1} d\,\mathrm{Leb} \Big)^2. \end{aligned}$$

The second integral in the right-hand side of the last inequality is bounded uniformly in j by Theorem 1.6.1. Whereas the first one converges to 0 by the previous part of the proof. This finishes the proof. □

Corollary 1.6.5. (i) *Let $(u_j)_j$ be a sequence of subharmonic functions converging weakly to some distribution u on Ω. Let $\varphi_j := \sup_{k \geq j} u_k$. Then the upper semi-continuity regularisation φ_j^* of φ_j decreases to u.*

(ii) *Let χ be as in Lemma 1.4.1. For $1 \leq k \leq m$, let u_{jk} be subharmonic function such that u_{jk} converges in L^1_{loc} to u_k as $j \to \infty$. Then we also have $\chi(u_{j1}, \dots, u_{jm})$ converges in L^1_{loc} to $\chi(u_1, \dots, u_m)$ (hence in L^p_{loc} for every $p > 0$).*

Proof. We check (i). By Theorem 1.6.4, $(u_j)_j$ is uniformly locally bounded from above. Hence φ_j^* is a well-defined subharmonic function. Since $(\varphi_j^*)_j$ is decreasing sequence and $\varphi_j^* \geq u_j \to u$ as distributions, we get that φ_j^* decreases to a subharmonic function u', and $u' \geq u$. By Theorem 1.6.4 and extracting a subsequence if necessary, we can assume that $u_j \to u$ in L^1_{loc} and $u_j(x) \to u(x)$ for almost everywhere $x \in \Omega$. Hence $\varphi_j(x) \to u(x)$ for almost everywhere x. This combined with the fact that

$$\{x : \varphi_j^*(x) > \varphi_j(x)\}$$

is of zero Lebesgue measure (Corollary 1.5.9) implies that $\varphi_j^*(x) \to u(x)$ for almost everywhere x. We infer that $u'(x) = u(x)$ for a.e. x. Hence $u' = u$ by the strong upper semi-continuity of u' and u.

It remains to check (ii). It suffices to prove the desired assertion for a subsequence of $(\chi(u_{j1}, \dots, u_{jm}))_j$. By extracting a subsequence if necessary, using Theorem 1.6.4 yields that $\chi(u_{j1}, \ldots, u_{jm})$ is locally bounded uniformly in j and $\chi(u_{j1}, \ldots, u_{jm})$ converges in L^1_{loc} to some harmonic function v as $j \to \infty$. On the other hand, Theorem 1.6.4 again, u_{jk} converges pointwise almost everywhere to u_k as $j \to \infty$, for $1 \leq k \leq m$ (after again extracting a subsequence). Hence $\chi(u_{j1}, \ldots, u_{jm})$ converges pointwise almost everywhere to $\chi(u_1, \ldots, u_m)$. It follows that $\chi(u_1, \ldots, u_m) = v$. This finishes the proof. □

Corollary 1.6.6. *Let $(u_j)_j$ be a sequence of harmonic functions converging weakly to a harmonic function u on Ω. Then u_j converges to u in $\mathscr{C}^\infty$ topology in Ω.*

Proof. By Theorem 1.6.4, we have $u_j \to u$ in L^1_{loc}. Let $\mathbb{D}(w, r_1) \Subset \mathbb{D}(w, r_2) \Subset \Omega$ be two disks. Using the Poisson formula, we obtain that

$$u(z) = \int_{\mathbb{D}(w,r_2)\backslash\mathbb{D}(w,r_1)} K(z, z')u(z')d\,\mathrm{Leb},$$

for $z \in \mathbb{D}(w.r_1/2)$, where $K(z, z')$ is some smooth function on (z, z') in some open neighborhood of the closure of $\mathbb{D}(w, r_1/2) \times \mathbb{D}(w, r_2)\backslash\mathbb{D}(w, r_1)$. Hence

$$D^\alpha u(z) = \int_{\mathbb{D}(w,r_2)\backslash\mathbb{D}(w,r_1)} D^\alpha_z K(z, z')u(z')d\,\mathrm{Leb}\,.$$

We also have similar equality for u_j. Combining this with the L^1_{loc} convergence of $(u_j)_j$ yields that

$$\sup_{\mathbb{D}(w.r_1/2)} |D^\alpha u_j - D^\alpha u| \lesssim \int_{\mathbb{D}(w,r_2)\backslash\mathbb{D}(w,r_1)} |u_j - u| \to 0$$

as $j \to \infty$. Hence the desired assertion follows. □

Remark 1.6.7. *The notions of harmonic functions and subharmonic functions can be extended to the case of $\mathbb{R}^m$ by using the Laplacian in $\mathbb{R}^m$. The (sub)harmonic functions on open subsets in $\mathbb{R}^m$ shares many similar properties as in the case of $\mathbb{R}^2 \approx \mathbb{C}$. However they are not the main object of the course. We will see, in the next chapter, a more refined generalization of subharmonic functions on $\mathbb{C}$ which is the notion of so-called plurisubharmonic functions.*

Notes. All of results presented in this chapter are classical, except possibly the notion of strong upper semi-continuity which was introduced in [17]. The presentation is based on [13, 24, 28, 31].

Chapter 2

Plurisubharmonic functions

2.1 Plurisubharmonic functions

Let Ω be a domain in $\mathbb{C}^n$.

Definition 2.1.1. *A function $u : \Omega \to [-\infty, \infty)$ is said to be plurisubharmonic (psh) if $u \not\equiv -\infty$ on Ω, and u is upper semi-continuous and for $x \in \Omega$ and every complex line L passing through x, the restriction $u|_{L\cap\Omega}$ of u to $L \cap \Omega$ is either $\equiv -\infty$ or a subharmonic function on a small neighborhood of x in $L \cap \Omega$.*

Recall that a complex line is an affine complex vector subspace of dimension 1 in $\mathbb{C}^n$. We identified $L \cap \Omega$ with an open subset in $\mathbb{C}$. Let $z = (z_1, \dots, z_n) \in \Omega$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{\geq 0}$. Denote $\mathbb{D}(z, r) := \mathbb{D}(z_1, r_1) \times \cdots \times \mathbb{D}(z_n, r_n)$, and

$$\partial\mathbb{D}(z, r) := \partial\mathbb{D}(z_1, r_1) \times \cdots \times \partial\mathbb{D}(z_n, r_n)$$

which is a proper subset of the Euclidean topological boundary of $\mathbb{D}(r, z)$. This set can be identified with $[0, 2\pi)^n$.

Lemma 2.1.2. *Let u be a psh function on Ω. Then for every polydisk $\mathbb{D}(z, r) \Subset \Omega$, we have*

$$u(z) \leq \frac{1}{(2\pi)^n} \int_{\partial\mathbb{D}(z,r)} u(z_1 + r_1 e^{i\theta_1}, \dots, z_n + r_n e^{i\theta_n}) d\theta_1 \cdots d\theta_n.$$

Proof. Apply consecutively the submean inequality to $u(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_n)$ on $\mathbb{D}(z_j, r_j)$, where $z_{j'}$ fixed for $j' \neq j$. □

We now present basic properties of psh functions. Some results are given without proofs if they are either deduced directly from the 1-dimensional case or can be proved in a similar ways as their analogue in the 1-dimensional case.

Lemma 2.1.3. *Every psh function is locally integrable.*

Theorem 2.1.4. *(maximum principle) Let Ω be a bounded domain and u be a psh function on Ω. Then if u attains a local maximum then it is constant. Consequently,*

$$\sup_{x\in\Omega} u(x) = \sup_{x_0\in\partial\Omega} \limsup_{x\to x_0\in\partial\Omega} u(x)$$

.

Corollary 2.1.5. *For every polydisk $\mathbb{D}(w,r) \Subset \Omega$, we have*

$$u(z) \leq \frac{1}{(2\pi)^n} \int_{\partial \mathbb{D}(w,r)} \prod_{j=1}^n \frac{r_j^2 - |z_j - w_j|^2}{|re^{i\theta_j} - (z_j - w_j)|^2} u(w_1 + r_1 e^{i\theta_1}, \dots, w_n + r_n e^{i\theta_n}) d\theta_1 \cdots d\theta_n \tag{2.1.1}$$

Hence the function

$$M_u(w, r_1, \dots, r_n) := \frac{1}{2\pi} \int_0^{2\pi} u(w_1 + r_1 e^{i\theta_1}, \dots, w_n + r_n e^{i\theta_n}) d\theta_1 \dots d\theta_n$$

is increasing in each variable r_j for $1 \leq j \leq n$.

Let $\chi_1, \dots, \chi_n \geq 0$ be smooth radial functions with compact support in $\mathbb{D}$ such that $\int_{\mathbb{C}} \chi_j d\,\mathrm{Leb} = 1$ for $1 \leq j \leq n$. Put $\chi(z_1, \dots, z_n) = \chi_1(z_1) \dots \chi_n(z_n)$. For every constant $\epsilon > 0$, put

$$\chi_\epsilon(z) := \epsilon^{-2n} \chi(z/\epsilon), \quad u_\epsilon(z) := \int_{\mathbb{C}^n} u(z-w)\chi_\epsilon(w) d\,\mathrm{Leb}\,.$$

Note that the function u_ϵ is well-defined on the set Ω_ϵ which consists of $z \in \Omega$ of distance at least ϵ to Ω.

Theorem 2.1.6. *(regularisation of psh functions) The function u_ϵ is a smooth psh function and u_ϵ decreasing pointwise to u as $\epsilon \to 0$.*

Proof. The smoothness is clear. Let L be a complex line intersecting Ω. Using the equality $u_\epsilon(z) = \int_\Omega u(z-w)\chi_\epsilon(w) d\,\mathrm{Leb}(w)$, one see that the restriction $u_\epsilon|_L$ of u_ϵ to L is an average of subharmonic function on L. Thus $u_\epsilon|_L$ is itself subharmonic. Hence u_ϵ is subharmonic. Let

$$u(\epsilon_1, \dots, \epsilon_n, z) := \int_\Omega u(z-w)(\chi_1)_{\epsilon_1}(w_1) \cdots (\chi_n)_{\epsilon_n}(w_n) d\,\mathrm{Leb}(w),$$

where ϵ_j is a small positive constant for $1 \leq j \leq n$. Observe that $u_\epsilon(z) = u(\epsilon, \dots, \epsilon, z)$. Put $z' = (z_2, \dots, z_n)$ and $w' = (w_2, \dots, w_n)$. By Fubini's theorem,

$$u(\epsilon_1, \dots, \epsilon_n, z) := \int_{w_2, \dots, w_n} (\chi_2)_{\epsilon_2}(w_2) \cdots (\chi_n)_{\epsilon_n}(w_n) d\,\mathrm{Leb} \times \int_{w_1} u(z_1 - w_1, z' - w')(\chi_1)_{\epsilon_1}(w_1) d\,\mathrm{Leb},$$

here it is not important to specify the open subsets over which the integrals are taken. Since $u(z_1 - w_1, z' - w')$ is either subharmonic or $\equiv -\infty$ on an open subset in $\mathbb{C}$, using Theorem 1.3.5 implies that $u(\epsilon_1, \dots, \epsilon_n, z)$ is increasing in ϵ_1. Similarly, we also obtain that $u(\epsilon_1, \dots, \epsilon_n, z)$ is increasing in every ϵ_j for $1 \leq j \leq n$. On the other hand, by Lemma 2.1.2, $u(\epsilon_1, \dots, \epsilon_n, z) \geq u$. By this, in order to get $u(\epsilon_1, \dots, \epsilon_n, z)$ decreases to u as $(\epsilon_1, \dots, \epsilon_n) \to 0$, one just needs to use the upper semi-continuity of u as in the proof of Theorem 1.3.5. $\square$

We call u_ϵ *standard regularisation of* u. Recall some basic on differential forms on $\mathbb{R}^m$, where m is a positive integer. For every subset $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ (here $i_1 < \ldots < i_k$), denote $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Every differential k-form Φ on $\mathbb{R}^n$ can be written uniquely as

$$\Phi = \sum_I f_I(x) dx_I,$$

where I runs over every subset of $\{1, \ldots, m\}$ of cardinality k. We say $\Phi \in \mathscr{C}^s$ if f_I is so for every I. A k-form is said to be *real* if its coefficients are real. Let $\mathscr{D}^k$ be the set of smooth k-form with compact support in Ω.

Let $f(x_1, \ldots, x_m)$ be a $\mathscr{C}^1$ function on $\mathbb{R}^m$. The exterior differential operator d acting on differential forms is computed as follows. We have

$$df(x_1, \ldots, x_m) = \sum_{j=1}^m \partial_{x_j} f(x_1, \ldots, x_m) dx_j.$$

More generally for every differential k-form $\Phi = \sum_I f_I(x) dx_I$, recall

$$d\Phi = \sum_I df_I \wedge dx_I$$

which is a $(k+1)$-form.

Let $z = (z_1, \ldots, z_n)$ be the standard complex coordinates in $\mathbb{C}^n$. For $j = 1, \ldots, n$, put $z_j = x_j + iy_j$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by sending $(z_1, \ldots, z_n)$ to $(x_1, y_1, \ldots, x_n, y_n)$. Set $dz_j := dx_j + idy_j$ and $d\bar{z}_j := dx_j - idy_j$. Similarly as above, we put

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_k}, \quad d\bar{z}_I = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_k}.$$

Observe $dx_j = 1/2(dz_j + d\bar{z}_j)$ and $dy_j = 1/(2i)(dz_j - d\bar{z}_j)$. Using these formulae, we can decompose Φ uniquely as

$$\Phi = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J,$$

where I, J run overs non-empty subsets of $\{1, \ldots, m\}$. Let $\overline{\Phi} := \sum_{I,J} \overline{f}_{I,J} d\bar{z}_I \wedge dz_J$. We say that Φ is of bi-degree (p, q) (and say Φ is a (p, q)-form or a form of bi-degree (p, q)) if $f_{IJ} = 0$ for every (I, J) such that either $|I| \neq p$ or $|J| \neq q$. Denote by $\mathscr{D}^{p,q}(\Omega)$ the set of smooth (p, q)-forms with compact support in Ω. We have seen that

$$\mathscr{D}^k(\Omega) = \oplus_{p+q=k} \mathscr{D}^{p,q}(\Omega). \tag{2.1.2}$$

Using this decomposition, we see that $d = \partial + \bar{\partial}$, where for every (p, q)-form Φ, we define $\partial\Phi$ to be the $(p+1, q)$-form in the decomposition of $d\Phi$ given by (2.1.2). Analogously, $\bar{\partial}\Phi$ is the $(p, q+1)$-form in the last decomposition of $d\Phi$. The operators $\partial, \bar{\partial}$ act on $\mathscr{D}^k(\Omega)$ by using (2.1.2). Since $d^2 = 0$, using bi-degree decomposition, we get

$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Put $\partial_{z_j} := 1/2(\partial_{x_j} - i\partial_{y_j})$ and $\partial_{\bar{z}_j} := 1/2(\partial_{x_j} + i\partial_{y_j})$ for $1 \leq j \leq n$. For $\Phi = f_{IJ} dz_I \wedge d\bar{z}_J$,

$$\partial\Phi = \partial_{z_j} f_{IJ} dz_j \wedge dz_I \wedge d\bar{z}_J, \quad \overline{\partial}\Phi = \partial_{\bar{z}_j} f_{IJ} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J.$$

Put $d^c := i/(2\pi)(\overline{\partial} - \partial)$. Hence $dd^c = i/\pi \partial\bar{\partial}$. When $n = 1$, the operator $dd^c u = c_0 \Delta u (idz \wedge d\bar{z})$, where $c_0 > 0$ is a constant.

Lemma 2.1.7. *(i) Let $\Phi = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J$ be a form. Then Φ is real if and only if $\Phi = \overline{\Phi}$, or equivalently $f_{IJ} = (-1)^{|I||J|}\overline{f}_{JI}$ for every I, J.*

(ii) Let Φ be a real form, then $dd^c\Phi$ is real as well, in other words, dd^c is a real operator.

Proof. Consider a form Φ formally as a polynomial of real variables dx_j, dy_j for $1 \le j \le n$ with complex coefficients. Then $\overline{\Phi}$ is simply the complex conjugate of the polynomial Φ. Hence (i) follows. The (ii) is deduced by similar reasons. □

Lemma 2.1.8. *Let $g : \Omega' \to \Omega$ be a holomorphic map. Let Φ be a form. Then $g^*\partial\Phi = \partial g^*\Phi$, and a similar equality for $\bar{\partial}$ also holds. Moreover if Φ is of bi-degree (p, q) then $g^*\Phi$ is so.*

Proof. Since f^* and ∂ are linear, it suffices to check the desired equality for Φ of bi-degree (p, q). Since $dg^*\Phi = g^*d\Phi$, and g^* preserves the bi-degree, we get the desired equality. □

Lemma 2.1.9. *Let $u \in \mathscr{C}^2(\Omega)$. Then u is psh if and only if $dd^cu \ge 0$,* i.e, *the matrix of coefficients of $-idd^cu$ is positive semidefinite.*

Proof. Let $z \in \Omega$ and $v = (v_1, \dots, v_n) \in \mathbb{C}^n$. Let $L := z + \mathbb{C}v$ which is a complex line passing through z. Put $u_L(t) := u(x + tv)$ which is the restriction of u to L. We have

$$dd^cu(z) = i/\pi \sum_{1\le j,k\le n} \partial^2_{z_j\bar{z}_k} f(z) dz_j \wedge d\bar{z}_k.$$

Using Lemma 2.1.8, we compute

$$\begin{aligned} c_0\Delta u(idt \wedge d\bar{t}) &= dd^cu_L(t) = (dd^cu)|_L \\ &= i/\pi \partial^2_{z_j\bar{z}_k} f(z + tv) d(z_j + v_jt) \wedge d\overline{(z_k + v_jt)} \\ &= \Bigg(\sum_{1\le j,k\le n} \partial^2_{z_j\bar{z}_k} f(z + tv) v_j\bar{v}_k\Bigg) i/\pi dt \wedge d\bar{t}. \end{aligned}$$

It follows that u is psh if and only if $\sum_{1\le j,k\le n} \partial^2_{z_j\bar{z}_k} f(z)v_j\bar{v}_k \ge 0$ for every $v \in \mathbb{C}^n$. This finishes the proof. □

Lemma 2.1.10. *Let $\Omega = U + iV$, where U, V are open subsets in $\mathbb{R}^n$. Let u be a psh function on Ω such that $u(z)$ depends only on $\operatorname{Re} z$. Then the function $u(x)$ with $x \in U$ is convex.*

Proof. By regularisation of u (which depends also only on $\operatorname{Re} z$), we can assume $u \in \mathscr{C}^2$. In this case, we have $0 \le dd^cu(x) = H_xu(x)$ (Hessian of $u(x)$). Hence u is convex. □

Lemma 2.1.11. *Let $f : \Omega' \to \Omega$ be a holomorphic function. Let u be psh on Ω. Then $u \circ f$ is also psh.*

Proof. By regularisation, it suffices to check the desired assertion for u smooth. To check $u \circ f$ is psh, we need to restrict it to a complex line L' in Ω'. Hence without loss of generality, we can assume that Ω' is in $\mathbb{C}$. Let $t_0 \in \Omega' \subset \mathbb{C}$. We need to check that $dd^c(u\circ f)(t_0) \ge 0$. Locally near t_0, we have $f(t_0+t) = f(z_0)+vt+O(t^2)$. Now we compute $dd^c(u\circ f)(t_0)$ as in the proof of Lemma 2.1.9 to obtain the positivity of $dd^c(u\circ f)(t_0)$. This finishes the proof. □

Theorem 2.1.12. *Let $w \in \Omega$. The function*

$$M_u(w; r_1, \dots, r_n) := \frac{1}{(2\pi)^n} \int_{\partial\mathbb{D}(z,r)} u(w_1 + r_1 e^{i\theta_1}, \dots, w_n + r_n e^{i\theta_n}) d\theta_1 \cdots d\theta_n$$

is a convex function in $(\log r_1, \dots, \log r_n)$ which is increasing in each variable r_j for $1 \le j \le n$.

Proof. We have already known that $M_u(r)$ is increasing in each variable r_j. Consider the function

$$M_u(z) := \frac{1}{(2\pi)^n} \int_{\partial\mathbb{D}(z,r)} u(z_1 + e^{z_1} e^{i\theta_1}, \dots, z_n + e^{z_n} e^{i\theta_n}) d\theta_1 \cdots d\theta_n$$

which is psh by Lemma 2.1.11. This function depends only on $\operatorname{Re} z$. Hence applying Lemma 2.1.10 to $M_u(z)$ implies that $M_u(r)$ is convex in $(\log r_1, \dots, \log r_n)$. □

Corollary 2.1.13. *For every $z \in \Omega$, the limit*

$$\nu(u, z) := \lim_{r \to 0} \frac{M_u(z; r, \dots, r)}{\log r} \ge 0$$

exists and we have

$$u(z') \le \nu(u, z) \log \frac{\max\{|z_1' - z_1|, \dots, |z_n' - z_n|\}}{r} + M_u(z; \log r, \dots, \log r) \qquad (2.1.3)$$

for every $z' \in \mathbb{D}(z_1, r) \times \cdots \times \mathbb{D}(z_n, r)$.

Proof. Let $f(t) := M_u(z; e^t, \dots, e^t)$ for $-\infty < t \le 0$. By Theorem 2.1.12, we infer that $f(t)$ is a convex increasing function in t. It follows that the function

$$\frac{f(t) - f(t_0)}{t - t_0} \ge 0$$

is increasing in t for t_0 fixed. Thus $f(t)/t$ is convergent as $t \to -\infty$. The first desired assertion follows. We check the second one. Since

$$\nu(u, z) = \lim_{t \to -\infty} f(t)/t = \lim_{t \to -\infty} \frac{f(t) - f(t_0)}{t - t_0},$$

using the increasing property of the function in the limit, we get

$$f(t) \le \nu(u, z)(t - t_0) + f(t_0)$$

for every $t \le t_0$. This combined with the fact that $u(z') \le f(t)$ if $z' \in \mathbb{D}(z_1, e^t) \times \cdots \times \mathbb{D}(z_n, e^t)$ gives the second desired assertion. □

The nonnegative number $\nu(u, z)$ is called *the Lelong number of u at z*. Bounded psh functions have zero Lelong number everywhere. However the converse is far from being true. For example, the function $u(z) = -\sqrt{-\log z}$ (for $\|z\| < 1$) is psh by Lemma 2.2.1 below, and it has zero Lelong number everywhere (by (2.1.3)). Observe that for $u \le v$ psh functions, then $\nu(u, z) \ge \nu(v, z)$ for every z because $M_u \le M_v$. The Lelong numbers is a simple and important object measuring the singularity of psh functions.

2.2 Construction of plurisubharmonic functions

Lemma 2.2.1. *Let $\chi : \mathbb{R}^m \to \mathbb{R}$ be a convex function such that $\chi(t_1, \dots, t_m)$ is increasing in each variable t_j, and χ can be extended continuously to be a function from $[-\infty, \infty)^m$ to $[-\infty, \infty)$. Let $u_1, \dots, u_m$ be psh functions on Ω. Then $\chi(u_1, \dots, u_m)$ is also psh. In particular, the functions $u_1 + \cdots + u_m$, $\max\{u_1, \dots, u_m\}$, and $\log(e^{u_1} + \ldots + e^{u_m})$ are psh.*

A function f on Ω is said to be *holomorphic* if for every $z_0 \in \Omega$, there exists a small open neighborhood U of z_0 such that

$$f(z_1, \dots, z_n) = f(z_0) + \sum_{k=1}^{\infty} \sum_{|I|=k} a_I (z - z_0)^I$$

which is an absolutely convergent series for $z \in U$, where $a_I \in \mathbb{C}$, $I = (i_1, \dots, i_n) \subset\in \mathbb{N}^n$, $|I| := i_1 + \cdots + i_n$ and $z^I := z_1^{i_1} \cdots z_n^{i_n}$.

Lemma 2.2.2. *Let f be a holomorphic function on Ω. Then $\log|f|$ is psh on Ω.*

Corollary 2.2.3. *Let $f_1, \dots, f_m$ be holomorphic functions. Then for every positive constant $a_1, \dots, a_m$, we have that $\log(|f_1|^{a_1} + \ldots + |f_m|^{a_m})$ is psh.*

By the last result, the function $\log\max\{|z'_1 - z_1|, \dots, |z'_n - z_n|$ is psh. This combined with (2.1.3) shows that the Lelong number $\nu(u, z)$ of a given psh function u at z is the largest constant λ such that

$$u(z') \le \lambda \log\max\{|z'_1 - z_1|, \dots, |z'_n - z_n|\} + O(1)$$

for z' in a small polydisk around z. Moreover using the fact that

$$(\mathbb{D}(z, r/n))^n \subset \mathbb{B}(z, r) \subset (\mathbb{D}(z, r))^n$$

($\mathbb{B}(z, r)$ is the ball of radius r centered at z), we also see that $\nu(u, z)$ is the largest constant λ such that

$$u(z') \le \lambda \log \|z' - z\| + O(1)$$

for z' in a small ball around z.

Lemma 2.2.4. *Let $(u_j)_{j \in J}$ be a family of psh function which is locally bounded from above uniformly. Then $(\sup_{j \in J} u_j)^*$ is also psh.*

Lemma 2.2.5. *The limit of a decreasing sequence of psh functions is either identically equal to $-\infty$ or a psh function.*

Theorem 2.2.6. *Let u be a psh function on Ω. Let U be an open subset of Ω and v be a psh function on U. Assume that $\limsup_{z' \to z} v(z') \le u(z)$ for every $z \in \partial U \cap \Omega$. Then the function*

$$w = \begin{cases} \max\{u, v\} \text{ on } U, \\ u \text{ on } \Omega \backslash U \end{cases}$$

is psh on Ω.

Lemma 2.2.7. *(strong upper semi-continuity) Let u be a psh function on Ω. Let B be a set of zero Lebesgue measure on Ω. Then for every $z \in \Omega$, we have*

$$\limsup_{z' \notin B \to z} u(z') = u(z).$$

Theorem 2.2.8. *Let A be a closed subset in $\mathbb{C}$ such that $A = \{v = -\infty\}$ for some psh function v on Ω. Let u be a psh function on $\Omega \backslash A$ such that for every compact subset K on Ω, the function u is bounded from above on $K \backslash A$. Then u can be extended uniquely to be a psh function $\tilde{u}$ on Ω.*

A $\mathscr{C}^2$ function u is said to be *pluriharmonic* if $dd^c u = 0$.

Lemma 2.2.9. *Let u be pluriharmonic. For every polydisk $\mathbb{D}(w, r) \Subset \Omega$, we have*

$$u(z) = \frac{1}{(2\pi)^n} \int_{\partial \mathbb{D}(w,r)} \prod_{j=1}^{n} \frac{r_j^2 - |z_j - w_j|^2}{|re^{i\theta_j} - (z_j - w_j)|^2} u(w_1 + r_1 e^{i\theta_1}, \ldots, w_n + r_n e^{i\theta_n}) d\theta_1 \cdots d\theta_n.$$

In particular u is smooth.

A higher dimensional analogue of Theorem 1.1.1 also holds: u is pluriharmonic if and only if it is locally the real part of a holomorphic function. We refer to [13, Page 42] for a proof.

2.3 Complex Hessian of psh functions

Let $(z_1, \ldots, z_n)$ be the standard coordinates on $\mathbb{C}^n$. We orient $\mathbb{C}^n$ by using the standard volume form $vol_n := (i/2 dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (i/2 dz_n \wedge d\bar{z}_n)$.

A k-current on Ω is a continuous linear functional T from $\mathscr{D}^{n-k}(\Omega)$ to $\mathbb{C}$. Here $\mathscr{D}^{2n-k}(\Omega)$ denotes the set of smooth $(2n-k)$-forms with compact support in Ω, and by continuity we mean that for every sequence $(\Phi_j)_{j \in \mathbb{N}} \subset \mathscr{D}^{2n-k}(\Omega)$ such that there exists a compact $K \subset \Omega$ satisfying $\mathrm{Supp}\Phi_j \subset K$ for every j and Φ_j converges to some $\Phi_\infty \in \mathscr{D}^{n-k}(\Omega)$ in $\mathscr{C}^\infty$ topology, we have $\langle T, \Phi_j \rangle \to \langle T, \Phi_\infty \rangle$ as $j \to \infty$. Every k-form with locally integrable coefficients Ψ on $\mathbb{C}$ can be viewed as a distribution T_Ψ by putting

$$\langle T_\Psi, \Phi \rangle := \int_\Omega \Psi \wedge \Phi.$$

In practice we usually identify T_Ψ with Ψ, and use the same notation Ψ to denote T_Ψ. More generally, every k-form whose coefficients are Randon measures is a k-current.

Let $(T_j)_{j \in \mathbb{N}}$ be a sequence of k-current on Ω. Let T be a k-current on Ω. We say that T_j *converges weakly to* T if

$$\langle T_j, \Phi \rangle \to \langle T, \Phi \rangle$$

as $j \to \infty$ for every $\Phi \in \mathscr{C}_c^\infty(\Omega)$. For $I, J \subset \{1, \ldots, n\}$ and $f \in \mathscr{C}_c^\infty(\Omega)$ put

$$\langle T_{IJ}, f \rangle := \langle T, \delta_{IJ} f dz_{I^c} \wedge d\bar{z}_{J^c} \rangle,$$

where δ_{IJ} is defined by the equality

$$dz_I \wedge d\bar{z}_J \wedge dz_{I^c} \wedge d\bar{z}_{J^c} = \delta_{IJ}(i/2dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (i/2dz_n \wedge d\bar{z}_n).$$

We infer that T_{IJ} are distributions on Ω and

$$T = \sum_{I,J} T_{IJ} dz_I \wedge d\bar{z}_J.$$

Every distribution T on Ω can be naturally identified with a $2n$-current by identifying T with $T\,vol_n$.

Lemma 2.3.1. *A linear functional $T : \mathscr{D}^{2n-k}(\Omega) \to \mathbb{C}$ is a current if and only if for every compact $K \subset \Omega$, there exist an integer $s \in \mathbb{N}$ and a constant $C > 0$ such that*

$$\langle T, \Phi \rangle \le C\|\Phi\|_{\mathscr{C}^s(\Omega)}, \tag{2.3.1}$$

for every smooth f with compact support in K.

Proof. Straightforward. □

When (2.3.1) holds for $s = 0$ for every K, we say that T *is of order* 0. The following is a generalization of Theorem 1.5.2.

Theorem 2.3.2. *([32, Theorem 2.14]) Let X be a compact Hausdorff space, and let Λ be a bounded linear functional on the space $\mathscr{C}(X)$ of continuous functions in X. Then there exists a complex Radon measure μ on X representing Λ,* i.e,

$$\langle \Lambda, f \rangle = \int_X f d\mu$$

for every $f \in \mathscr{C}(X)$.

Consequently, we get

Corollary 2.3.3. *Every current of order 0 is a differential form whose coefficients are complex Radon measures.*

Let T be a k-current on Ω. We define *the exterior differential* dT of T to be the $(k+1)$-current given by

$$\langle T, \Phi \rangle = (-1)^{k+1} \langle T, d\Phi \rangle$$

for every $\Phi \in \mathscr{D}^{2n-k}(\Omega)$. We say that T is of bi-degree (p,q) or of bi-dimension $(n-p, n-q)$ if $\langle T, \Phi \rangle = 0$ for every (p', q')-form Φ with $(p'.q') \neq (n-p, n-q)$. By decomposing forms into sums of (p,q)-forms, we see that every k-current can be decomposed uniquely as the sum of (p,q)-currents. For a (p,q)-current T and (p', q')-form Φ, put

$$\langle \partial T, \Phi \rangle := (-1)^{p+q+1} \langle T, \partial \Phi \rangle$$

if $(p', q') = (n-p-1, q)$, and $\langle \partial T, \Phi \rangle = 0$ otherwise. By linearity, we obtain a well-defined $(p+1, q)$-current ∂T. We define $\overline{\partial} T$ similarly. Note that $d = \partial + \overline{\partial}$. When T is smooth,

∂T and $\overline{\partial} T$ coincide with the definition in the smooth case. We say that T is *closed (or d-closed)* if $dT = 0$, we define similarly ∂-closedness, and $\overline{\partial}$-closedness.

A simple positive continuous (p,p)-form is $(i\gamma_1 \wedge \overline{\gamma}_1) \wedge \cdots \wedge (i\gamma_p \wedge \overline{\gamma}_p)$, where $\gamma_1, \ldots, \gamma_p$ are $(1,0)$-form (with complex coefficients). Every simple positive form is real. Positivity of forms are preserved under holomorphic maps.

Lemma 2.3.4. *Every constant (p,p)-form can be written as a linear combination (with functions coefficients) of constant simple positive forms.*

Proof. For the first desired assertion, it suffices to use the formula

$$\begin{aligned} 4dz_j \wedge d\bar{z}_k = (dz_j + dz_k) \wedge \overline{(dz_j + dz_k)} - (dz_j - dz_k) \wedge \overline{(dz_j - dz_k)} \\ + i(dz_j + idz_k) \wedge \overline{(dz_j + idz_k)} - i(dz_j - idz_k) \wedge \overline{(dz_j - idz_k)}. \end{aligned}$$

□

A real $(1,1)$-current T is said to be *positive* (and write $T \geq 0$) if $\langle T, \alpha \rangle \geq 0$ for every simple positive form α with compact support. Since simple positive form are preserved under holomorphic maps, the positivity is independent of the Euclidean coordinates on $\mathbb{C}^n$.

Lemma 2.3.5. *Let $\alpha = i\sum_{j,k} a_{jk} dz_j \wedge d\bar{z}_k$ be a real continuous $(1,1)$-form. Then $\alpha \geq 0$ if and only if the Hermitian matrix $[a_{jk}]_{1 \leq j,k \leq n}$ is positive semidefinite. In particular, for every $\mathscr{C}^2$ function u, then u is psh if and only if $dd^c u$ is a closed positive form.*

Proof. the second desired assertion follows from the first one and Lemma 2.1.9. We check the first one. We assume first that the Hermitian matrix $[a_{jk}]_{1 \leq j,k \leq n}$ is positive semidefinite. Note that for a Hermitian matrix, being positive semidefinite is preserved under an $\mathbb{C}$-linear change of coordinates in $\mathbb{C}^n$. Let β be a simple $(n-1, n-1)$-form. Fix $z_0 \in \Omega$. By a $\mathbb{C}$-linear change of coordinates, we can assume that $\beta(z_0) = c(idz_2 \wedge d\bar{z}_2) \wedge \cdots \wedge (idz_n \wedge d\bar{z}_n)$, where c is a positive constant. Now we compute

$$\alpha(z_0) \wedge \beta(z_0) = a_{11}(z_0)\, vol_n \geq 0$$

because $a_{11} \geq 0$. Thus $\alpha \geq 0$. Conversely, assume $\alpha \geq 0$. Let $t = (t_1, \ldots, t_n) \in \mathbb{C}^n \backslash \{0\}$. Let $(t, z_2', \ldots, z_n')$ be new orthogonal coordinates on $\mathbb{C}^n$. Let vol_n' be the canonical volume form in $\mathbb{C}^n$ induced by these new coordinates. Compute

$$0 \leq \alpha \wedge (idz_2' \wedge d\bar{z}_2') \wedge \cdots \wedge (idz_n' \wedge d\bar{z}_n') = \Bigg(\sum_{1 \leq j,k \leq n} a_{jk} t_j \bar{t}_k \Bigg) vol_n'.$$

Since $vol_n' \geq 0$, we get $\sum_{1 \leq j,k \leq n} a_{jk} t_j \bar{t}_k \geq 0$. The desired assertion follows. □

Lemma 2.3.6. *Every positive $(1,1)$-current T has measures coefficients.*

Proof. Let α be a constant simple positive $(n-1, n-1)$-form. By positivity, $T \wedge \alpha$ is a positive (n,n)-current. Hence it is a measure. By Lemma 2.3.4, every coefficient T_{IJ} can be written as a linear combination of some $T \wedge \alpha$. Hence T_{IJ} is a complex measure. □

Theorem 2.3.7. *Let u be a psh function on Ω. Then $dd^c u$ is a closed positive current.*

Proof. Let $(u_\epsilon)_\epsilon$ be a regularisation of u. We have $dd^c u_\epsilon \to dd^c u$ as $\epsilon \to 0$ in the sense of distributions. On the other hand by Lemma 2.3.5, we get $dd^c u_\epsilon \geq 0$. It follows that $dd^c u \geq 0$. The closedness is clear because $d(dd^c u) = 0$. □

Theorem 2.3.8. *Let u be a distribution on Ω such that $dd^c u \geq 0$. Then there exists a psh function u' on Ω such that $u = u'$.*

Corollary 2.3.9. *Let $(u_j)_{j\in J}$ be a family of psh function which is locally bounded from above uniformly. Then the set of $z \in \Omega$ such that $(\sup_{j\in J} u_j)^*(z) > (\sup_{j\in J} u_j)(z)$ is of zero Lebesgue measure.*

The set of x such that $(\sup_{j\in J} u_j)^*(z) > (\sup_{j\in J} u_j)(z)$ is called *a negligible set*. By the above result, every negligible set is of zero Lebesgue measure. We will see later that a much deeper property holds: every negligible set is pluripolar (*i.e,* contained in $\{u = -\infty\}$ for some psh function u on Ω or even in $\mathbb{C}^n$).

Corollary 2.3.10. *Let u be a distribution on Ω such that $dd^c u = 0$. Then there exists a pluriharmonic function u' on Ω such that $u = u'$.*

Proof. It suffices to apply Theorem 2.3.8 to u and $-u$ and use the mean equality. □

We admit the following important result.

Theorem 2.3.11. *([13, Page 135]) Let T be a closed positive $(1,1)$-current. Then T is locally equal to $dd^c u$ for some psh function u.*

2.4 Compactness properties

Recall Ω is a domain in $\mathbb{C}^n$.

Theorem 2.4.1. *Let u be a negative psh function on Ω such that $\|u\|_{L^1(\Omega)} \leq 1$. Let K be a compact subset of Ω. Then there exist constants $C, \alpha > 0$ independent of u such that*

$$\int_K e^{-\alpha u} d\,\mathrm{Leb} \leq C.$$

In particular the L^p norm of u on K is uniformly bound.

Proof. Without loss of generality we can assume that $u \leq 1$ on Ω. By using a partition of unity, we can assume that $K \subset (\mathbb{D}_{1/2})^n$ and $\Omega = \mathbb{D}^n$. Since $\|u\|_{L^1(\Omega)} \leq 1$, there exists $z_0 \in (\mathbb{D}_{1/2})^n$, such that $u(z_0) \geq -M$, where $M > 0$ is a constant independent of u. We can assume $z_0 = 0$. For $1 \leq j \leq n$, let $F_j : \mathbb{D} \times \mathbb{D}^{n-1} \to \mathbb{C}^n$ be given by

$$F(t, z'_1, \ldots, z'_{n-1}) = t(z'_1, \ldots, z'_{j-1}, 1, z'_{j+1}, \ldots, z'_{n-1}).$$

For $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$, there exists j such that $|z_{j'}| \leq |z_j|$ for every $j' \neq j$. Hence such z belongs to the image of F_j. We deduce that the images of F_j's cover the polydisk

$\mathbb{D}^n$. Moreover since F_j is $1-1$ almost everywhere, using change of variables formula, we obtain

$$\int_K e^{-\alpha u} d\,\mathrm{Leb} \lesssim \sum_{j=1}^{n} \int_{\mathbb{D}^n} e^{-\alpha u \circ F} d\,\mathrm{Leb}\,.$$

It remains to estimate

$$A_j := \int_{\mathbb{D}^n} e^{-\alpha u \circ F} d\,\mathrm{Leb} = \int_{\mathbb{D}^{n-1}} d\,\mathrm{Leb}_{\mathbb{C}^{n-1}} \int_{\mathbb{D}} e^{-\alpha u \circ F} d\,\mathrm{Leb}_{\mathbb{C}}\,.$$

Consider the function $v := u \circ F(\cdot, z')$ for fixed z'. We have $v \leq 1$ on $\mathbb{D}$ and $v(0) \geq -M$ and v is subharmonic on an open neighborhood of $\overline{\mathbb{D}}$. Thus applying Theorem 1.6.1 to v implies that

$$\int_{\mathbb{D}} e^{-\alpha u \circ F} d\,\mathrm{Leb}_{\mathbb{C}} \lesssim 1$$

uniformly in z'. This implies that $A_j \lesssim 1$. The proof is finished. □

The following is fundamental in pluripotential theory.

Theorem 2.4.2. (i) *(Hartogs' lemma) Let $(u_j)_j$ be a sequence of psh functions converging weakly to some distribution u_∞ on Ω. Then u_∞ is also a psh function and $u_j \to u_\infty$ in L^p_{loc} for every constant $0 < p < \infty$. Furthermore for every compact subset K in Ω and every continuous function f on K, we have*

$$\limsup_{j\to\infty} \sup_K (u_j - f) \leq \sup_K (u - f). \tag{2.4.1}$$

(ii) *Let $(u_j)_j$ be a sequence of psh functions uniformly locally bounded from above defined on Ω. Then either u_j converges uniformly on compact subsets in Ω to $-\infty$ as $j \to \infty$ or there exists a subsequence $(u_{j'})_{j'}$ which converges in L^p to a psh function u_∞ for every real number $0 < p < \infty$.*

Proof. We argue verbatim as in the proof of Theorem 1.6.4. □

Corollary 2.4.3. (i) *Let $(u_j)_j$ be a sequence of psh functions converging weakly to some distribution u on Ω. Let $\varphi_j := \sup_{k\geq j} u_k$. Then the upper semi-continuity regularisation φ^*_j of φ_j decreases to u.*

(ii) *Let χ be as in Lemma 1.4.1. For $1 \leq k \leq m$, let u_{jk} be psh function such that u_{jk} converges in L^1_{loc} to u_k as $j \to \infty$. Then we also have $\chi(u_{j1}, \dots, u_{jm})$ converges in L^1_{loc} to $\chi(u_1, \dots, u_m)$ (hence in L^p_{loc} for every $p > 0$).*

Corollary 2.4.4. *Let $(u_j)_j$ be a sequence of pluriharmonic functions converging weakly to a harmonic function u on Ω. Then u_j converges to u in $\mathscr{C}^\infty$ topology in Ω.*

2.5 Quasi-plurisubharmonic functions

Let X be a complex manifold (a differentiable manifold of even real dimension equipped with a $\mathscr{C}^\infty$ atlats whose transition functions are holomorphic). The notion of bidegree (p,q) is independent of local coordinates by Lemma 2.1.8. The operators $\partial, \overline{\partial}$ hence extends globally to X. Likewise the notion of positivity for real $(1,1)$-currents are also naturally extended.

A function u on X is said to be *psh* if it is locally psh, *i.e*, for every $x \in X$, there exists a biholomorphic map f (hence for such every f by Lemma 2.1.11) from an open neighborhood of x to an open subset $\mathbb{C}^n$ such that $u \circ f^{-1}$ is psh on $f(U)$. By the maximum principle, we have the following.

Lemma 2.5.1. *There is no non-constant psh function on compact complex manifold.*

This is the reason to introduce the notion of quasi-plurisubharmonicity for functions on compact complex manifolds.

Let X be a complex manifold of dimension n. A function from X to $[-\infty, \infty)$ is said to be *quasi-plurisubharmonic* (quasi-psh for short) if it can be written locally as the sum of a psh function and a smooth one. Obviously every smooth function on X is quasi-psh. A bit more elaborated example is as follows. Let $\mathbb{B}_r$ be the ball centered at 0 in $\mathbb{C}^n$ of radius r. Let U be a local chart in X biholomorphic to the unit ball in $\mathbb{C}^n$, let χ be a smooth function on X supported on $\mathbb{B}_{2/3} \subset U$ and $\chi = 1$ on $\mathbb{B}_{1/2}$, then $u := \chi(x) \log \|x\|$ is a well-defined quasi-psh function on X and $\{u = -\infty\}$ is non-empty. Put $d^c := i/(2\pi)(\overline{\partial} - \partial)$.

Lemma 2.5.2. *For every quasi-psh function u on a complex manifold X, there exist a smooth $(1,1)$-form η such that $dd^c u + \eta \geq 0$.*

Proof. Let $(\chi_j)_j$ be a partition of unity subordinated to some locally finite covering $(U_j)_j$ on X. Let $u = u_j + f_j$ on U_j where u_j is psh on U_j and f_j is smooth on U_j. Thus we get $dd^c u = dd^c u_j + dd^c f_j \geq dd^c f_j$ on U_j. It follows that

$$dd^c u = \sum_j \chi_j dd^c u \geq \sum_j \chi_j dd^c f_j =: \eta$$

which is a smooth $(1,1)$-form on X. This finishes the proof. $\square$

For a continuous real $(1,1)$-form η, a quasi-psh function u is said to be *η-psh* if $dd^c u + \eta \geq 0$ in the sense of currents.

Lemma 2.5.3. *Let T be a closed positive $(1,1)$-current on X. Then there exist a smooth closed $(1,1)$-form η on X and an η-psh function u such that $T = dd^c u + \eta$.*

Proof. Let $(\chi_j)_j$ be a partition of unity subordinated to some locally finite covering $(U_j)_j$ on X. We choose elements U_j of that covering to be relatively compact small enough local charts so that $T = dd^c u_j$ on U_j where u_j is psh on U_j (see Lemma 2.3.11). On $U_j \cap U_{j'}$, we have $dd^c(u_j - u_{j'}) = 0$. Hence by Corollary 2.3.10, $u_j - u_{j'}$ is smooth. Put

$$u := \sum_j \chi_j u_j.$$

We compute $dd^c u$ on U_{j_0}. Let $x_0 \in U_{j_0}$. Let J_0 be the set of j such that $x_0 \in U_j$. Observe that J_0 is finite because our covering is locally finite. Let W be a small neighborhood of x_0 such that $\overline{W} \cap \overline{U}_j = \varnothing$ for $j \notin J_0$. For $x \in W$ we have

$$\begin{aligned} u(x) &= \sum_{j\in J_0} \chi_j(x) u_j(x) \\ &= \big(\sum_{j\in J_0} \chi_j(x)\big) u_{j_0}(x) + \sum_{j\in J_0} \chi_j(x)(u_j(x) - u_{j_0}(x)) \\ &= u_{j_0}(x) + \sum_{j\in J_0} \chi_j(x)(u_j(x) - u_{j_0}(x)). \end{aligned}$$

Since the second term is smooth, we see that $dd^c u = dd^c u_{j_0} + \eta = T + \eta$ on W for some smooth form η. Hence the desired assertion follows. □

Proposition 2.5.4. *(the Lelong-Jensen formula) Let u be a psh function on an open subset Ω in $\mathbb{C}$. The for every disk $\mathbb{D}(z, r_1) \Subset \mathbb{D}(z, r_2) \Subset \Omega$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} u(z + r_2 e^{i\theta}) - \frac{1}{2\pi} \int_0^{2\pi} u(z + r_1 e^{i\theta}) = \int_{r_1}^{r_2} \frac{dr}{r} \int_{\mathbb{D}_r} dd^c u. \tag{2.5.1}$$

In particular,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r_2 e^{i\theta}) - \int_0^{r_2} \frac{dr}{r} \int_{\mathbb{D}_r} dd^c u.$$

Proof. Firstly notice that (2.5.1) holds for every smooth function in place of u. This can be seen by a direct computation using integration by parts. We leave it to readers. We explain how to get it for psh functions.

Let $(u_\epsilon)_\epsilon$ be a standard regularisation of u. Since u_ϵ is smooth, as observed above, we get

$$\frac{1}{2\pi} \int_0^{2\pi} u_\epsilon(z + r_2 e^{i\theta}) - \frac{1}{2\pi} \int_0^{2\pi} u_\epsilon(z + r_1 e^{i\theta}) = \int_{r_1}^{r_2} \frac{dr}{r} \int_{\mathbb{D}_r} dd^c u_\epsilon. \tag{2.5.2}$$

When $\epsilon \to 0$ the left-hand side tends to that of (2.5.1). We deal with the right-hand side. Since $\Delta u_\epsilon \to \Delta u$ weakly, for every r so that $dd^c u$ has no mass on $\partial \mathbb{D}_r$, we have

$$\int_{\mathbb{D}_r} dd^c u_\epsilon \to \int_{\mathbb{D}_r} dd^c u \tag{2.5.3}$$

as $\epsilon \to 0$. Since $dd^c u$ is a Radon measure, there are at most a countable number of $0 < r \leq r_2$ such that $dd^c u$ has mass on $\partial \mathbb{D}_r$. It follows that (2.5.3) holds for almost everywhere r. We infer that the right-hand side of (2.5.2) tends to that of (2.5.1) as $\epsilon \to 0$. Thus we get (2.5.1). The second desired equality follows by taking $r_1 \to 0$. The proof is finished. □

We have the following characterization of quasi-psh functions in terms of submean-type inequalities.

Proposition 2.5.5. *Let U be an open subset of $\mathbb{C}^n$ and η a continuous real $(1,1)$-form on U. A function $u : U \to [-\infty, \infty)$ is η-psh if and only if it is upper semi-continuous, not identically $-\infty$ and for every $x \in U$ and every complex line $L_v := \{x + tv : t \in \mathbb{C}\}$, for some $v \in \mathbb{C}^k \backslash \{0\}$, passing through x, we have*

$$u(x) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x + \epsilon e^{i\theta} v) d\theta + \int_0^{\epsilon} \frac{dt}{t} \int_{\{|s| \leq t\}} \eta_v, \tag{2.5.4}$$

for every constant $\epsilon > 0$ small enough, where $\eta_v(t)$ is the restriction of η to L_v which is identified with $\mathbb{C}$ via $t \longmapsto x + tv$.

Proof. Consider an η-psh function u. We need to verify (2.5.4). Let χ be a usual cut-off function used to define standard regularisation of u and let χ_r be the associated cut-off function for every constant $r > 0$. Recall

$$u^r(x) := \int_{\mathbb{C}^n} u(x - y) \chi_r(y) d \operatorname{Leb}(y)$$

which is smooth (we change a bit the notation for the standard regularisation here). We have $u^r \to u$ pointwise as $r \to 0$ because u can be written as the sum of a psh function and a smooth one. Denote by

$$\eta^r(x) := \int_{\mathbb{C}^n} \eta(x - y) \chi_r(y) \operatorname{vol}(y)$$

which converges uniformly to η as $r \to 0$ because η is continuous. We deduce that $dd^c u + \eta \geq 0$ if $dd^c u^r + \eta^r \geq 0$ for every r small. On the other hand, we have

$$dd^c u^r + \eta^r = \int_{\mathbb{C}^n} [dd^c u(\cdot - y) + \eta(\cdot - y)] \chi_r(y) \operatorname{vol}(y)$$

which is the convolution of the $(1,1)$-current $(dd^c u + \eta)$ with χ_r. Thus $dd^c u^r + \eta^r \geq 0$ if $dd^c u + \eta \geq 0$. Similarly, (2.5.4) holds if it holds for (u^r, η^r) in place of (u, η) for every small r. It follows that it suffices to prove (2.5.4) for smooth u and smooth η.

Hence we can assume u, η are smooth and follow standard arguments in [24]. Let $v \in \mathbb{C}^k$ and $x \in U$. Put $u_v(t) := u(x + tv)$. We get $dd^c u_v + \eta_v \geq 0$. The Lelong-Jensen formula for $u_v(t)$ gives

$$M_{\epsilon,v} - M_{\epsilon',v} = \int_{\epsilon'}^{\epsilon} \frac{dt}{t} \int_{\{|s| \leq t\}} dd^c u_v,$$

where $\epsilon > \epsilon'$ are positive constants and

$$M_{s,v} := \frac{1}{2\pi} \int_0^{2\pi} u_v(\epsilon e^{i\theta}) d\theta$$

for every constant $s > 0$. It follows that

$$M_{\epsilon',v} \leq M_{\epsilon,v} + \int_{\epsilon'}^{\epsilon} \frac{dt}{t} \int_{\{|s| \leq t\}} \eta_v.$$

Letting $\epsilon' \to 0$ in the last inequality gives (2.5.4) because u_v is continuous at 0.

Assume now (2.5.4). This combined with the hypothesis that $u \not\equiv -\infty$ implies $u \in L^1_{loc}$. Moreover, as in the case of psh functions, since u is upper semi-continuous, (2.5.4) also tells us that u is *strongly upper semi-continuous* in the sense that for every Borel subset A of U whose complement in U is of zero Lebesgue measure, we have

$$\limsup_{y \in A \to x} u(y) = u(x). \tag{2.5.5}$$

Indeed, by the upper semi-continuity of u, we have $\limsup_{y\in A\to x} u(y) \leq u(x)$. We only need to check the inverse inequality. Since the problem is local, we can assume η is bounded. Integrating (2.5.4) with respect to ϵ, we get

$$u(x) \leq \frac{1}{\pi\epsilon^2}\int_{\{|t|\leq\epsilon\}} u(x+tv)(\frac{i}{2}dt\wedge d\bar{t}) + \|\eta\|_{L^\infty} O(\epsilon^2),$$

for every ϵ small enough. Letting $\epsilon \to 0$ in the last inequality gives

$$u(x) \leq \limsup_{\epsilon\to 0} \frac{1}{\pi\epsilon^2}\int_{\{|t|\leq\epsilon\}} u(x+tv)(\frac{i}{2}dt\wedge d\bar{t}). \tag{2.5.6}$$

Let δ be a strictly positive constant. There exists a constant $\delta_1 > 0$ such that

$$u(y') \leq \limsup_{y\in A\to x} u(y) + \delta, \tag{2.5.7}$$

for every $y' \in A$ such that $\|y' - x\| \leq \delta_1$. Since the Lebesgue measure of $U\backslash A$ is zero, by Fubini's theorem, for almost everywhere $v \in \mathbb{C}^k\backslash\{0\}$, the set $(U\backslash A)\cap L_v$ is of Lebesgue measure zero in L_v. Using this, (2.5.6) and (2.5.7), we see that for almost everywhere $v \in \mathbb{C}^k\backslash\{0\}$ and $A_v := \{t : (x+tv) \in A\cap L_v\}$,

$$u(x) \leq \limsup_{\epsilon\to 0} \frac{1}{\pi\epsilon^2}\int_{\{|t|\leq\epsilon,\, t\in A_v\}} u(x+tv)(\frac{i}{2}dt\wedge d\bar{t}) \leq \limsup_{y\in A\to x} u(y) + \delta$$

for every constant $\delta > 0$. Letting $\delta \to 0$ in the last inequality implies

$$u(x) \leq \limsup_{y\in A\to x} u(y).$$

Thus (2.5.5) follows.

Consider first the case where $u \in \mathscr{C}^2$. Direct computations show

$$\epsilon^{-2}\big(M_{\epsilon,v} - u_v(0)\big) \to \pi dd^c u_v(0)/2$$

as $\epsilon \to 0$. Applying this to (2.5.4) gives $dd^c u_v(0) + \eta_v(0) \geq 0$. In other words, we get $dd^c u + \eta \geq 0$.

In general, let u^r, η^r be as above. Since $u \in L^1_{loc}$, $u^r \to u$ in L^1_{loc}. We see easily that (2.5.4) also holds for (u^r, η^r) in place of (u, η). By the above arguments, $dd^c u^r + \eta^r \geq 0$. Letting $r \to 0$ gives $dd^c u + \eta \geq 0$.

It remains to check that u is the sum of a psh function and a smooth one. To this end, we only need to work locally. Thus, we can assume there is a smooth function ψ on U

with $dd^c\psi \geq \eta$. We deduce $dd^c u_1 \geq 0$ for $u_1 := u + \psi$ which is also strongly semi-upper continuous in the above sense. Let u_1^r be the regularisation of u_1 defined in the same way as u^r. Notice that $u_1^r \to u_1$ in L^1_{loc} and u_1^r is psh and decreasing to some psh function u_1'. Hence, $u_1 = u_1'$ almost everywhere. Using this and (2.5.5) for u_1 in place of u yield that $u_1 = u_1'$ everywhere. In other words, u is quasi-psh. This ends the proof. □

The following extension result generalizes the similar property for psh functions.

Lemma 2.5.6. *Let U be an open subset in a complex manifold Y. Let η be a continuous real $(1,1)$-form on Y. Let ψ_1 be an η-psh function on U and ψ_2 an η-p.s.h function on Y such that $\limsup_{y\to x}\psi_1(y) \leq \psi_2(x)$ for every $x \in \partial U$. Define $\psi := \max\{\psi_1, \psi_2\}$ on U and $\psi := \psi_2$ on $Y\backslash U$. Then ψ is an η-psh function.*

Proof. This is a direct consequence of Proposition 2.5.5. □

Let η be a continuous real $(1,1)$-form. Denote by $\mathrm{PSH}(X,\eta)$ the set of η-psh functions on X. Let ω be a Hermitian metric on X, *i.e,* ω is a smooth real $(1,1)$-form on X such that ω can be written locally as

$$\omega = i \sum_{1\leq j,k\leq n} a_{jk} dz_j \wedge d\bar{z}_k,$$

where $[a_{jk}]_{j,k}$ is a positive definite Hermitian matrix. Since

$$\omega^n = |\det[a_{jk}]_{j,k}|^2\, vol_n$$

in the local coordinates $(z_1,\dots,z_n)$. The (n,n)-form ω^n defines a smooth volume form on X. In what follows we will use L^p-norms on X which are computed with respect to ω^n.

Proposition 2.5.7. *(Compactness for quasi-psh functions) Assume that X is compact. Let A_1, A_2, A_3 be the subset of $\mathrm{PSH}(X,\eta)$ consisting of u such that $\|u\|_{L^1(X)} \leq 1$, $\sup_X u = 0$, and $\int_X u\omega^n = 0$ respectively. Then A_j is compact in the L^1-topology (hence L^p-topology for every $p \geq 1$) for $1 \leq j \leq 3$.*

Proof. The fact that A_1 is compact follows directly from Theorem 2.4.2 and the compactness of X. We consider now A_2. Suppose that there exists a sequence $(u_j)_j \subset A_2$ such that $\|u_j\|_{L^1(X)} \to \infty$ as $j \to \infty$. Since $(u_j)_j$ is uniformly bounded from above, by Theorem 2.4.2 and extracting a subsequence if necessary, we get that either u_j converges uniformly to $-\infty$, or u_j converges in L^1 to some quasi-psh function. The second possibility cannot occur because $\|u_j\|_{L^1(X)} \to \infty$. So u_j converges uniformly to $-\infty$. Let $x_j \in X$ such that $u(x_j) = 0$. We can assume $x_j \to x_\infty$ as $j \to \infty$. Consider a local chart U around x_∞ and j big enough so that $x_j \in U$. By shrinking U, we can find a smooth psh function ψ on U such that $dd^c\psi \geq \eta$. Hence $u_j + \psi$ is psh on U. This combined with the submean inequality implies

$$u_j(x_j) + \psi(x_j) \leq 1/vol(D_j) \int_{D_j} (u_j(x) + \psi(x_j))d\,\mathrm{Leb},$$

where $D_j \Subset U$ is a small polydisk around x_j. Since f is smooth, we infer

$$0 \leq u_j(x_j) \leq C \int_U u_j d\,\mathrm{Leb} + C$$

for some constant C independent of j. Letting $j \to \infty$ gives a contradiction because the right-hand side tends to $-\infty$. Thus there exists a constant C such that $\|u\|_{L^1(X)} \leq C$ for every $u \in A_2$. Consequently A_2 is relatively compact. To see why A_2 is indeed compact, consider $(u_j)_j \subset A_2$ such that $u_j \to u$ in L^1. Since $u_j \leq 0$ for every j, and $u_j \to u$ almost everywhere (a subsequence), we obtain $u \leq 0$ on X. On the other hand, by (2.4.1), we have $\sup_X u \geq 0$. Hence $u \in A_2$, in other words, A_2 is compact.

We deal with A_3. It suffices to check that $\sup_X u$ is bounded uniformly for $u \in A_3$. In this case the compactness of A_3 follows from that of A_2. Let $(u_j)_j \subset A_3$, and $v_j := u_j - \sup_X u_j$ which belongs to A_2. Hence by extracting a subsequence, we can assume $v_j \to v \in A_2$ in L^1. It follows that

$$\int_X v\omega^n = \lim_{j\to\infty} \int_X v_j \omega^n = -\lim_{j\to\infty} \sup_X u_j \int_X \omega^n.$$

Hence $\sup_X u_j$ is uniformly bounded in j. This finishes the proof. □

The following is a nice application of the convexity of psh functions.

Lemma 2.5.8. *Let u be a psh function on an open subset Ω in $\mathbb{C}^n$. Let $M_u(z, r_1, \ldots, r_n)$ be the function defined in Theorem 2.1.12. Then $M_u(z, r_1, \ldots, r_n)$ is a continuous psh function in z for $r_1, \ldots, r_n$ fixed.*

Proof. It suffices to check that $M_u(z, r_1, \ldots, r_n)$ is continuous psh on every relatively compact subset U of Ω. Let U be such a set. Let $r_0 > 0$ be such that for every $z \in U$, the polydisk $(\mathbb{D}(z, r_0))^n \Subset \Omega$. Without loss of generality we can assume $r_0 = 1$.

By considering a sequence of smooth psh functions decreasing to u, we see that $M_u(z, r_1, \ldots, r_n)$ is the limit of some decreasing sequence of psh functions. Hence it is psh (hence upper semi-continuous). It remains to check that it is also lower semi-continuous. By Theorem 2.1.12, $M_u(z, r_1, \ldots, r_n)$ is convex in $(\log r_1, \ldots, \log r_n)$ for z fixed. Thus for $\lambda \in (0,1)$ we have

$$M_u(z, r_1, \ldots, r_n) \leq (1-\lambda) M_u(z, r_1^{(1-\lambda)^{-1}}, \ldots, r_n^{(1-\lambda)^{-1}}) + \lambda M_u(z, 1, \ldots, 1). \tag{2.5.8}$$

Fix λ small enough. Let $z = (z_1, \ldots, z_n)$. We see that if $z' = (z_1', \ldots, z_n')$ is closed enough to z, then

$$\mathbb{D}(z_1, r_1^{(1-\lambda)^{-1}}) \times \cdots \times \mathbb{D}(z_n, r_n^{(1-\lambda)^{-1}}) \Subset \mathbb{D}(z_1', r_1) \times \cdots \times \mathbb{D}(z_n', r_n).$$

Thus by the submean inequality we get

$$M_u(z, r_1^{(1-\lambda)^{-1}}, \ldots, r_n^{(1-\lambda)^{-1}}) \leq M_u(z', r_1, \ldots, r_n)$$

provided that z' is closed enough to z. Letting $z' \to z$ gives

$$M_u(z, r_1^{(1-\lambda)^{-1}}, \ldots, r_n^{(1-\lambda)^{-1}}) \leq \liminf_{z' \to z} M_u(z', r_1, \ldots, r_n).$$

This combined with (2.5.8) yields

$$M_u(z, r_1, \dots, r_n) \leq (1-\lambda) \liminf_{z' \to z} M_u(z', r_1, \dots, r_n) + \lambda M_u(z, 1, \dots, 1).$$

Letting $\lambda \to 0$ in the last inequality gives the desired lower semi-continuity. This finishes the proof. $\square$

Lemma 2.5.9. *Let $f : U \to V$ be a biholomorphism between to open subsets in $\mathbb{C}^n$. Let u be a psh function with zero Lelong number everywhere on V. Then $u_\epsilon - (u \circ f)_\epsilon \circ f^{-1}$ converges uniformly on compact subsets in U to 0 as $\epsilon \to 0$.*

Here u_ϵ and $(u \circ f)_\epsilon$ denote the standard regularisations of u and $u \circ f$ by using the same cut-off function.

Proof. It suffices to work on relatively compact subsets of V as in the proof of Lemma 2.5.8. Consider $z \in K \Subset V$. Note that

$$u_\epsilon(z) = \int_{[0,1]^n} M_u(z, \epsilon r_1, \dots, \epsilon r_n) \chi_1(r_1) \cdots \chi_n(r_n) dr_1 \cdots dr_n,$$

where $\int_{[0,1]} \chi_j(r) dr = 1$ for $1 \leq j \leq n$. Put

$$u_\epsilon(z, r_1, \dots, r_j) = \int_{[0,1]^{n-j}} M_u(z, \epsilon r_1, \dots, \epsilon r_n) \chi_{j+1}(r_{j+1}) \cdots \chi_n(r_n) dr_{j+1} \cdots dr_n.$$

We claim that

Claim. For every $1 \leq j \leq n$, $u_\epsilon(z, r_1, \dots, r_j) - u_\epsilon(z, r_1, \dots, r_{j+1})$ converges uniformly to 0 as $\epsilon \to 0$.

We prove Claim. We present the proof when $n = 1$. The general case is similar: we only have to write more cumbersome formulae. Write χ for χ_1 and r for r_1. Fix $r_0 > 0$ a constant such that $\mathbb{D}(z, r_0) \Subset \Omega$. As usual we can assume $r_0 = 1$. By convexity for $0 < r \leq 1$,

$$\begin{aligned} M_u(z, \epsilon) - M_u(z, \epsilon r) &\leq \frac{\log \epsilon - \log(\epsilon r)}{\log 1 - \log(\epsilon r)} (M_u(z, 1) - M_u(z, \epsilon r)) \\ &= \frac{-\log r}{\log 1 - \log(\epsilon r)} (M_u(z, 1) - M_u(z, \epsilon r)). \end{aligned} \tag{2.5.9}$$

Integrating the last inequality against $\chi(r) dr$ over $[0, 1]$ gives

$$M_u(z, \epsilon) - u_\epsilon(z) \leq \int_0^1 -\log r \frac{M_u(z, 1) - M_u(z, \epsilon r)}{-\log \epsilon - \log r} \chi(r) dr.$$

Since

$$\frac{M_u(z, 1) - M_u(z, \epsilon r)}{-\log \epsilon - \log r}$$

decreases to $\nu(u,z)=0$ everywhere. This convergence is uniform by Dini's theorem. This combined with Lebesgue's dominated convergence theorem shows that $M_u(z,\epsilon)-u_\epsilon(z)$ converges uniformly to 0 as $\epsilon\to 0$. The Claim is proved. Letting $\epsilon\to 0$ in (2.5.9), we also obtain that

$$M_u(z,\epsilon,\ldots,\epsilon)-M_u(z,\epsilon r,\ldots,\epsilon r)\to 0 \tag{2.5.10}$$

uniformly in $z\in K$ as $\epsilon\to 0$.

By Claim for $1\le j\le n$, we see that $u_\epsilon - M_u(\cdot,\epsilon,\ldots,\epsilon)$ converges uniformly to 0 as $\epsilon\to 0$. Now since f is diffeomorphism, there exists a constant $C>0$ such that

$$f\big(\mathbb{D}^n(z,\epsilon^n)\big)\Subset \mathbb{D}^n(f(z),C\epsilon),\quad \mathbb{D}^n(z,\epsilon)\Subset f\big(\mathbb{D}^n(f(z),C\epsilon)\big)$$

for every z. Using this and the maximum principle, we obtain

$$M_{u\circ f}(z,\epsilon,\ldots,\epsilon)\le M_u(\cdot,C\epsilon,\ldots,C\epsilon)\circ f(z).$$

Consequently,

$$\limsup_{\epsilon\to 0}\sup_K((u\circ f)_\epsilon\circ f^{-1}-u_\epsilon)=\limsup_{\epsilon\to 0}\sup_K(M_{u\circ f}(\cdot,\epsilon,\ldots,\epsilon)\circ f^{-1}-u_\epsilon)$$

which is

$$\le \limsup_{\epsilon\to 0}\sup_K\big(M_u(\cdot,C\epsilon,\ldots,C\epsilon)-M_u(\cdot,\epsilon,\ldots,\epsilon)\big)=0$$

by (2.5.10). Similarly by considering f^{-1} instead of f we can show that

$$\limsup_{\epsilon\to 0}\sup_K(u_\epsilon-(u\circ f)_\epsilon\circ f^{-1})=0.$$

Hence the desired assertion follows. The proof is finished. □

Theorem 2.5.10. *(Regularisation of quasi-psh functions) Let X be a complex manifold. Let X' be a relatively compact open subset on X. Let ω be a Hermitian metric on X, and η be a continuous real $(1,1)$-form on X. Let u be an η-psh function on X such that the Lelong numbers of u are all zero. Then there exist $(\epsilon_j)_j\subset\mathbb{R}_{\ge 0}$ converging to 0 and $u_j\in \mathrm{PSH}(X',\eta+\epsilon_j\omega)\cap\mathscr{C}^\infty(X')$ such that u_j decreases to u on X'.*

Proof. Fix a constant $\delta>0$. Cover $\overline{X}'$ by a finite number of local charts $U_1,\ldots,U_m$ biholomorphic to $\mathbb{D}^n$. Let $f_j:\mathbb{D}^n\to U_j$ be the biholomorphism defining the local chart U_j. Let $U'_j\Subset U''_j\Subset U_j$ be open subsets in X such that $(U'_j)_j$ covers X. By dividing U_j into smaller similar local charts and the continuity of η, we can assume that there is a smooth psh function ψ_j on U_j such that

$$0\le dd^c\psi_j-\eta\le\delta\omega$$

on U_j. Put

$$v_j:=u\circ f_j+\psi_j\circ f_j$$

which is psh on $\mathbb{D}^n$. Let $v_{j,\epsilon}$ be the standard regularisation of v_j (using the same cut off function χ for every j). We need to glue $v_{j,\epsilon}$ to obtain a global quasi-psh function. Let w_j

be a smooth nonpositive function on X such that $w_j = -1$ outside U''_j and $w_j = 0$ on U'_j, and $dd^c w_j \geq C f_j^* \omega$ on U_j for some constant $C > 0$. Let

$$u_\epsilon := \max_{1\leq j\leq m} \mathbf{1}_{U_j}\big(v_{j,\epsilon} \circ f_j^{-1} - \psi_j + \delta w_j/C\big)$$

We have that u_ϵ decreases to u as $\epsilon \to 0$ (using $w = 0$ on U'_j and $X = \cup_j U'_j$). Consider $1 \leq j_1, j_2 \leq m$. Let $\tau := f_{j_2}^{-1} \circ f_{j_1}$ which is a biholomorphic from $f_{j_1}^{-1}(U_{j_1} \cap U_{j_2})$ onto its image in $2\mathbb{D}^n$ and

$$\varphi_\epsilon := v_{j_1,\epsilon} - v_{j_2,\epsilon} \circ \tau = v_{j_1} * \chi_\epsilon - \Big((v_{j_1} \circ \tau^{-1}) * \chi_\epsilon\Big) \circ \tau + \Big((\psi_{j_1} \circ f_{j_2} - \psi_{j_1} \circ f_{j_2}) * \chi_\epsilon\Big) \circ \tau.$$

The third term tends uniformly to $(\psi_{j_1} \circ f_{j_1} - \psi_{j_1} \circ f_{j_1})$ as $\epsilon \to 0$. Whereas the difference of the first two terms converges uniformly to 0 by Lemma 2.5.9. Hence we deduce that for every $1 \leq j_1, j_2 \leq m$, the function $v_{j_1,\epsilon} \circ f_j^{-1} - v_{j_2,\epsilon} \circ f_{j_2}^{-1}$ converges uniformly to $\psi_{j_1} - \psi_{j_2}$ as $\epsilon \to 0$ on $U_{j_1} \cap U_{j_2}$.

Let $x_0 \in X$. Let J_0 the set of $1 \leq j \leq m$ such that $x_0 \in \partial U_j$. Let W be a small open neighborhood of x_0 such that $\overline{W} \cap \overline{U}''_j = \varnothing$ for every $j \in J_0$. By the above arguments and the fact that $w_j = -1$ outside $U''_j \Subset U_j$, if ϵ is small enough, then

$$u_\epsilon := \max_{j\notin J_0} \mathbf{1}_{U_j}\big(v_{j,\epsilon} \circ f_j^{-1} - \psi_j + \delta w_j/C\big)$$

on a small open neighborhood W of x_0. For $j \in J_0$, on W, we have

$$\mathbf{1}_{U_j}\big(v_{j,\epsilon} \circ f_j^{-1} - \psi_j + \delta w_j/C\big) = v_{j,\epsilon} \circ f_j^{-1} - \psi_j + \delta w_j/C$$

which is $(\eta + \epsilon\omega)$-psh. Hence u_ϵ is $(\eta + \epsilon\omega)$-psh on W, and hence on X because x_0 is arbitrary. The above arguments also show that u_ϵ is continuous. To get u_ϵ smooth, one just need to use a regularisation of the $\max$ function to replace the $\max$ function in the definition of u_ϵ. One can see [13, Page 43] for a specific construction: the function

$$G(t_1, \ldots, t_m) := \max\{t_1, \ldots, t_m\}$$

is convex and increasing in each variables, so the standard regularisation G_δ of G by a separate-variable cut-off function as we do before for psh function is also convex and increasing in each variable; put

$$u'_\epsilon := G_\epsilon\Big(\mathbf{1}_{U_j}\big(v_{1,\epsilon} \circ f_1^{-1} - \psi_1 + \delta w_1/C\big), \ldots, \mathbf{1}_{U_m}\big(v_{m,\epsilon} \circ f_m^{-1} - \psi_m + \delta w_m/C\big)\Big)$$

We leave it as an exercise to the readers to check that u'_ϵ is a smooth $(\eta+\epsilon\omega)$-psh function decreasing to u This finishes the proof. □

Corollary 2.5.11. *Let X be a compact complex manifold and ω be a Hermitian metric on X. Let u be an ω-psh function on X. Then there exist $u_j \in \mathrm{PSH}(X, \omega) \cap \mathscr{C}^\infty(X)$ such that u_j decreases to u on X.*

Proof. Assume for the moment u is bounded. Applying Theorem 2.5.10 to $X' = X$ and u, we obtain a sequence of smooth $(1+\epsilon_j)\omega$-psh $(u_j)_j$ decreasing to u, where ϵ_j decreases to 0. Using Proposition 2.5.7, and subtracting a big constant from u_j and u, we can assume $u_j \leq 0$ and $u \leq 0$. Hence $u_j/(1+\epsilon_j) \geq u_{j+1}/(1+\epsilon_{j+1})$ which decreases to u. So the desired assertion holds if u is bounded.

Consider now the general case. Let $u_k := \max\{u, -k\}$. Observe u_k bounded and decreases to u as $k \to \infty$. By the first part of the proof, we can find a sequence $(u_{kj})_j$ of smooth ω-psh functions decreasing to u_k as $j \to \infty$. Put $u_{(1)} := u_{11}$. We define $u_{(k)}$ inductively as follows. By Theorem 2.4.2 applied to u_k, u_{kj} (locally), we get

$$\limsup_{j\to\infty} \sup_X (u_{kj} - u_{(k-1)}) \leq \sup_X (u_k - u_{(k-1)}) \leq \sup_X (u_k - u_{k-1}) \leq 0.$$

Hence $u_{kj} \leq u_{(k-1)} + 1/k^2$ for $j \geq j_k$. Put $u_{(k)} := u_{kj_k}$. We see that the sequence

$$u'_{(k)} := u_{(k)} - \sum_{j=1}^{k} 1/j^2 + \sum_{j=1}^{\infty} 1/j^2$$

is decreasing and converges to u as $k \to \infty$. This finishes the proof. □

Notes. Lemma 2.5.3 is from [14]. Proposition 2.5.5 was proved in [37]. Theorem 2.5.10, Corollary 2.5.11 and their proof are taken from [10]. The other results are all standard; see [13, 24, 28].

Chapter 3

Monge-Ampère operators

3.1 Closed positive currents

Let $(z_1, \ldots, z_n)$ be the standard coordinates on $\mathbb{C}^n$. We orient $\mathbb{C}^n$ by using the standard volume form $vol_n := (i/2dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (i/2dz_n \wedge d\bar{z}_n)$. Let Ω be an open subset in $\mathbb{C}^n$. Recall that a *simple positive continuous* (p,p)*-form* on Ω is $(i\gamma_1 \wedge \overline{\gamma}_1) \wedge \cdots \wedge (i\gamma_p \wedge \overline{\gamma}_p)$, where $\gamma_1, \ldots, \gamma_p$ are $(1,0)$-form (with complex coefficients) on Ω. Every simple positive form is real. A *positive continuous* (p,p)*-form* is a form which is locally the limit of a sequence of linear combinations with nonnegative coefficients of simple positive continuous (p,p)-forms in $\mathscr{C}^0$ topology.

A continuous real (p,p)-form Ψ is said to be *weakly positive* if

$$\langle \Psi, \Phi \rangle \geq 0$$

for every positive continuous $(n-p, n-p)$-form Φ with compact support in Ω. *In standard literature on complex geometry, the notion of positivity corresponds to our weakly positivity, whereas strong positivity corresponds to our positivity. The choice of terminology in the lecture is consistent with the literature in complex dynamics.*

Note that weakly positive (n,n)-form is indeed positive by a bi-degree reason. And the positivity in this case means that for every real (n,n)-form Φ on Ω, Φ is positive if and only if for $x \in \Omega$, we have $\Phi(x) \geq 0$, *i.e,* $\Phi(x) = c vol_n$ for some constant $c \geq 0$.

Lemma 3.1.1. *Positive forms are weakly positive. The wedge products of positive forms are positive, the wedge product of a weakly positive form with a positive form is weakly positive.*

Note that the wedge product of weakly positive forms may fail to be weakly positive; see [13, Page 132].

Proof. It suffices to check that for every simple positive (n,n)-form α with compact support, we have $\int_{\mathbb{C}^n} \alpha \geq 0$. Write

$$\alpha = (i\gamma_1 \wedge \overline{\gamma}_1) \wedge \cdots \wedge (i\gamma_n \wedge \overline{\gamma}_n),$$

where $\gamma_1, \ldots, \gamma_n$ are $(1,0)$-form; $\gamma_j = \sum_{k=1}^n a_{jk} dz_k$. Let S_n be the set of permutations of $\{1, \ldots, n\}$. For $\sigma \in S_n$, put $a_\sigma := a_{\sigma(1)} \cdots a_{\sigma(n)}$, and $\bar{a}_\sigma$ is the complex conjugate of a_σ.

Direct computations show that

$$\begin{aligned}
\alpha &= \Big(\sum_{1\le k,l\le n} a_{1k}\bar{a}_{1l} dz_k \wedge d\bar{z}_l\Big) \wedge \cdots \wedge \Big(\sum_{1\le k,l\le n} a_{nk}\bar{a}_{nl} dz_k \wedge d\bar{z}_l\Big)\\
&= \sum_{\sigma,\tau\in S_n} a_\sigma \bar{a}_\tau (i dz_{\sigma(1)} \wedge d\bar{z}_{\tau(1)}) \wedge \cdots \wedge (i dz_{\sigma(n)} \wedge d\bar{z}_{\tau(n)})\\
&= \sum_{\sigma,\tau\in S_n} a_\sigma \bar{a}_\tau (-1)^{n(n-1)/2} i^n dz_{\sigma(1)} \wedge \cdots dz_{\sigma(n)} \wedge d\bar{z}_{\tau(1)} \wedge \cdots \wedge d\bar{z}_{\tau(n)}\\
&= \sum_{\sigma,\tau\in S_n} \mathrm{sign}\,(\sigma)\mathrm{sign}\,(\tau) a_\sigma \bar{a}_\tau (-1)^{n(n-1)/2} i^n dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n\\
&= \big|\det[a_{jk}]_{1\le j,k\le n}\big|^2\, vol_n \ge 0.
\end{aligned}$$

The desired assertion follows. □

The notions of weak positivity and positivity are dual as shown by the following lemma.

Lemma 3.1.2. *Let Ψ be a continuous (p,p)-form. Then the following two conditions are equivalent:*

(i) *We have*

$$\langle \Psi, \Phi \rangle \ge 0$$

for every weakly continuous $(n-p, n-p)$-form Φ with compact support on Ω,

(ii) *We have*

$$\Psi(z) \wedge \Phi(z) \ge 0$$

for every $z \in \Omega$ and every weakly continuous $(n-p, n-p)$-form Φ on Ω.

We also have similar statement by exchanging "weakly positivity" with "positivity" in the above statements. In particular, if Ψ satisfies the condition (i) or (ii), then Ψ is positive.

Proof. Clearly (ii) implies (i). We prove the converse assertion. Without loss of generality, we can assume $z = 0$. Let χ be the standard cut-off function and χ_ϵ as usual. Put $\Psi(z) \wedge \Phi(z) = f(z)\, vol_n$. Observe that

$$0 \le \langle \Psi, \chi_\epsilon(x)\Phi \rangle = \int_\Omega f(z)\chi_\epsilon(z)\, vol_n \to f(0)$$

as $z \to \infty$ because of the continuity of f. Hence $f(0) \ge 0$. This shows the equivalence between (i) and (ii).

We check the last desired assertion: if Ψ satisfies (ii), then it is positive. Assume Ψ satisfies (ii). Let $z_0 \in \Omega$. Let A_p be the real vector space of constant real (p,p)-forms in $\mathbb{C}^n$. For every differential (p,p)-form Φ on Ω, then $\Phi(z_0)$ belongs to A_p. Note that A_p is a real vector space of finite dimension.

Let C be the set of constant weakly positive continuous $(n-p, n-p)$-form. Let C' be the set of constant positive (p,p)-forms in $\mathbb{C}^n$. Observe that C and C' are closed convex sets, and C' is the closure of the convex hull of constant simple positive forms.

We note here a fact that we will not use that every element in C' can be written as a linear combination with nonnegative coefficients of constant simple positive forms (by Carathéodory's theorem on the convex hulls in finite dimensional Euclidean spaces).

Observe A_p and A_{n-p} are dual vector spaces by the scalar product

$$\langle \alpha, \beta \rangle := \alpha \wedge \beta / vol_n.$$

By the equivalence between (i) and (ii), for every $\alpha \in C$ and $\beta \in C'$ we have $\langle \alpha, \beta \rangle \geq 0$, and

$$C = \{\alpha \in A_{n-p} : \langle \alpha, \beta \rangle \geq 0, \quad \forall \beta \in A_p\}.$$

Thus by the Hahn-Banach theorem,

$$C' = \{\beta \in A_p : \langle \alpha, \beta \rangle \geq 0, \quad \forall \alpha \in A_{n-p}\}.$$

It follows that $\Psi(z) \in C'$ for every $z \in \Omega$. Combining this with an argument of partition of unity yields that Ψ is approximated by linearly combinations with nonnegative coefficients of simple positive forms in $\mathscr{C}^0$ topology. In other words, Ψ is positive. This finishes the proof. □

Corollary 3.1.3. *Let Ψ be a real continuous (p,p)-form. Then Ψ is positive if and only if for every constant weakly positive $(n-p, n-p)$-form Φ we have $\Psi(x) \wedge \Phi(x) \geq 0$ for every $x \in \Omega$. Similarly Ψ is weakly positive if and only if for every constant positive $(n-p, n-p)$-form Φ we have $\Psi(x) \wedge \Phi(x) \geq 0$ for every $x \in \Omega$.*

Proof. Follow directly from Lemma 3.1.2. □

Lemma 3.1.4. (i) *Let β be a continuous $(p, 0)$-form. Then the form $(i^{p^2} \beta \wedge \bar{\beta})$ is weakly positive.*

(ii) *Let $\alpha = i \sum_{j,k} a_{jk} dz_j \wedge d\bar{z}_k$ be a real $(1,1)$-form. Then α is weakly positive if and only if the Hermitian matrix $[a_{jk}]_{1 \leq j,k \leq n}$ is positive semidefinite. The last condition is also equivalent to the statement that α is positive. In particular, the notions of weak positivity and positivity coincide for forms of bidegree $(1,1)$ and $(n-1, n-1)$.*

Proof. We prove (i). Let α be a simple positive $(n-p, n-p)$-form. By Lemma 3.1.2, we need to check that $i^p \beta \wedge \bar{\beta} \wedge \alpha \geq 0$ at each point in Ω. Hence we can assume α is a constant form. By using a linear change of variables, we can assume that

$$\alpha = (i dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (i dz_{n-p} \wedge d\bar{z}_{n-p}).$$

Write $\beta = \sum_{I:|I|=p} a_I dz_I$. Thus for $I_0 := \{n-p+1, \ldots, n\}$, we have

$$i^{p^2} \beta \wedge \bar{\beta} \wedge \alpha = i^{p^2} |a_{I_0}|^2 dz_{I_0} \wedge d\bar{z}_{I_0} \wedge (i dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (i dz_{n-p} \wedge d\bar{z}_{n-p})$$

which is equal to

$$|a_{I_0}|^2 \, vol_n \geq 0.$$

Hence we get (i).

Let α be a real $(1,1)$-form. By similar arguments, we can check that α is weakly positive if and only if the Hermitian matrix $A := [a_{jk}]_{1\le j,k\le n}$ is positive semidefinite. To see why this means also that α is positive, we use the fact that every Hermitian matrix can be diagonalizable. Let $(\lambda_j)_{1\le j\le n}$ be eigenvalues of A. Since A is positive semidefinite, its eigenvalues λ_j are all nonnegative. Hence there exists a unitary matrix $U = [b_{jl}]_{1\le j,l\le n}$ ($UU^T = \mathrm{Id}$ and $U = \overline{U}^T$) such that UAU^T is the diagonal matrix whose diagonal is $[\lambda_1,\dots,\lambda_n]$. Let $U^T = [b^t_{jl}]_{j,l}$. We have $b_{jl} = b^t_{lj}$ and $b_{jl} = \bar b_{lj}$. Put $z_j = \sum_{l=1}^n b_{jl} z'_l$ for $1 \le j \le n$. We obtain new coordinates $(z'_1,\dots,z'_n)$ on $\mathbb{C}^n$. Direct computations give

$$\alpha = \sum_{j,k}\sum_{l,s} a_{jk} b_{jl}\bar b_{ks} dz'_l \wedge d\bar z'_s = \sum_{j,k}\sum_{l,s} a_{jk} b^t_{lj}\bar b_{sk} dz'_l \wedge d\bar z'_s$$

$$= \sum_{l,s}\sum_{k}\big(\sum_j a_{jk} b^t_{lj}\big)\bar b_{sk} dz'_l \wedge d\bar z'_s = \sum_{s=1}^n \lambda_s dz'_s \wedge d\bar z'_s$$

because UAU^T is the diagonal matrix whose diagonal is $[\lambda_1,\dots,\lambda_n]$. So the notion of positivity and weak positivity coincide for $(1,1)$-forms, this is also the case for $(n-1,n-1)$ forms because of duality (Corollary 3.1.3). This finishes the proof. □

Lemma 3.1.5. *Let $f : \Omega' \to \Omega$ be a holomorphic map. Let Φ be a (weakly) positive continuous (p,p)-form. Then $f^*\Phi$ is also (weakly) positive.*

Proof. When Φ is positive, the positivity of its pull-back by f is clear. We consider Φ weakly positive. By Corollary 3.1.3, it suffices to check that the constant form $f^*\Phi(z)$ is weakly positive for every $z \in \Omega'$. Fix $z_0 \in \Omega$. By the formula $f^*\Phi(z_0) = (df(z_0))^*\Phi$, the question is reduced to the case where f is linear.

As the next step, we use the following criteria: a constant real (p,p)-form is weakly positive if its restriction to every complex vector p-dimensional subspaces of $\mathbb{C}^n$ is so (proved by direct arguments from definition). This allows us to reduces the question to the case where Ω' is a vector space of dimension p and f is linear. In this case note that $f(\Omega')$ is a complex vector space of dimension at most p. If $L := f(\Omega')$ is of dimension $< p$, then since $f^*\Phi = f^*(\Phi|_L)$, we get $f^*\Phi = 0$ because $\Phi|_L = 0$. If $\dim L = p$, then $\Phi|_L$ is a weakly positive form of maximal bi-degree, hence Φ is positive. The desired assertion follows. □

Let T be a real current of bi-degree (p,p). We say that T is positive if $\langle T,\Phi\rangle \ge 0$ for every weakly positive smooth $(n-p,n-p)$-form Φ. This extends the notion of positivity to currents by Lemma 3.1.2. Le g be a smooth function with compact support in Ω. The *convolution* of T with g is defined by

$$T * g := \sum_{I,J} (T_{IJ} * g) dz_I \wedge d\bar z_J.$$

Let $(T_j)_j$ be a sequence of currents, and T is a current. We say that $T_j \to T$ weakly if $\langle T_j,\Phi\rangle \to \langle T,\Phi\rangle$ for every smooth Φ with compact support.

Lemma 3.1.6. *The following statements are true:*
(i) *the convolution $T * g$ is smooth, and*

$$d(T * g) = (dT) * g,$$

(ii) *for every smooth Φ with compact support in Ω,*

$$\langle T * g, \Phi \rangle = \langle T, \Phi * g_1 \rangle,$$

where $g_1(z) := g(-z)$,
(iii) *let χ be a smooth function with compact support in $\int_{\mathbb{C}^n} \chi \operatorname{vol}_n = 1$ and $\chi_\epsilon(z) := \epsilon^{-2n}\chi(z/\epsilon)$, then $T * \chi_\epsilon$ converges weakly to T as $\epsilon \to 0$ and if T is (weakly) positive, then $T * \chi_\epsilon$ is so,*
(iv) *if $(T_j)_j$ is a sequence of currents converging weakly to T as $j \to \infty$, then $T_j * g \to T * g$ in $\mathscr{C}^\infty$ topology.*

Proof. Everything follows from Lemma 1.5.5 except the positivity in (iii). To check it we argue as follows. By Corollary 3.1.3, if Φ is (weakly) positive, then $\Phi * g$ is so. Hence if T is positive, then $T * g$ is also so because (ii). □

Lemma 3.1.7. *T is positive if and only if $T \wedge \Phi$ is a positive distribution (hence a positive measure) for every constant weakly positive $(n-p, n-p)$-form Φ.*

Proof. The implication $\Rightarrow$ is clear. We check the converse one. Let T_ϵ be as in Lemma 3.1.6. We see from the proof of the last lemma that $T_\epsilon \wedge \Phi$ is positive form for every every constant weakly positive $(n-p, n-p)$-form Φ. By Corollary 3.1.3, T_ϵ is positive. Letting $\epsilon \to 0$ implies the desired assertion. □

We define weakly positive currents similarly, and a similar version of Lemma 3.1.7 also holds for weakly positive currents. Let $\omega := \sum_{j=1}^n i dz_j \wedge d\bar{z}_j$.

Lemma 3.1.8. *For every constant simple (p, p)-form α, there exists a constant $c > 0$ such that $c\omega^p - \alpha$ is positive.*

Proof. Let $\alpha := (i\gamma_1 \wedge \overline{\gamma}_1) \wedge \cdots \wedge (i\gamma_p \wedge \overline{\gamma}_p)$. Since γ_j has constant coefficients, and the form ω is strictly positive (in the sense that its coefficient matrix is positive definite), using Lemma 3.1.4, we infer that $c\omega - i\gamma_j \wedge \overline{\gamma}_j$ is positive for some big enough constant c. Hence $(c\omega - i\gamma_j \wedge \overline{\gamma}_j) \wedge (c\omega - i\gamma_j \wedge \overline{\gamma}_j)$ is again positive by Lemma 3.1.1. This finishes the proof. □

For every (p, p)-current T of order 0, and a compact $K \Subset \Omega$, we define

$$\|T\|_K := \sup_\Phi \langle T, \Phi \rangle$$

for every Borel $(n-p, n-p)$-form Φ whose coefficients are ≤ 1 and supported on K.

Proposition 3.1.9. *Every weakly positive (p, p)-current T is of order 0, hence is a form with measure coefficients. Moreover, for every compact K in Ω, there is a constant $C > 0$ such that*

$$\|T_{IJ}\|_K \leq C \|T \wedge \omega^{n-p}\|_K.$$

We call $T \wedge \omega^{n-p}$ *the trace measure of* T.

Proof. By Lemma 2.3.4, we can fix a basis of the space of constant $(n-p, n-p)$-forms which consists of constant simple positive $(n-1, n-1)$-forms $\alpha_1, \ldots, \alpha_M$. By rescaling and Lemma 3.1.8, we can assume that $\alpha_j \leq \omega^{n-p}$ for every j. By positivity, $T \wedge \alpha_j$ is a positive measure. By writing $dz_I \wedge d\bar{z}_J$ as a linear combination of these constant simple positive forms, we see that T_{IJ} can be written as a linear combination of $T \wedge \alpha_j$. It follows that for every function f, we get

$$\big|\langle T_{IJ}, f\rangle\big| \lesssim \sum_j \langle T \wedge \alpha_j, |f|\rangle \lesssim \langle T \wedge \omega^{n-p}, |f|\rangle.$$

Consequently, T_{IJ} are Radon measures and the desired inequality follows. This ends the proof. □

By the last result, for every positive current T, we can define $\langle T, \Phi\rangle$ for every continuous function Φ with compact support, or more generally for every bounded (Borel) measurable form Φ on Ω. The following result is simple but fundamental.

Lemma 3.1.10. *(Compactness of the space of positive currents) Let $(T_k)_k$ be a sequence of weakly positive currents of mass on compact subsets bounded uniformly. Then we can extract a subsequence $(T_{j_k})_k$ of $(T_k)_k$ such that T_{j_k} converges weakly to some current T as $k \to \infty$.*

Proof. The proof is a direct consequence of Proposition 3.1.9 and Lemma 1.6.2. □

Here are some basic operations on currents.

Lemma 3.1.11. *Let $f : X \to Y$ be a proper holomorphic map between complex manifolds. Let T be a current of bi-dimension (p, p) on X. Put*

$$\langle f_* T, \Phi\rangle := \langle T, f^* \Phi\rangle$$

for every smooth form Φ with compact support in Y. Then $f_* T$ is also a current of bi-dimension (p, p) which is (weakly) positive if T is so and f_* commutes with $d, \partial, \overline{\partial}$.

Proof. Direct. We leave it to readers. □

Lemma 3.1.12. *Let $f : X \to Y$ be a holomorphic submersion between complex manifolds. Let T be a (p, p)-current on Y. Put*

$$\langle f^* T, \Phi\rangle := \langle T, f_* \Phi\rangle$$

for every smooth form Φ with compact support in X. Then $f^* T$ is also a (p, p)-current which is (weakly) positive if T is so and f_* commutes with $d, \partial, \overline{\partial}$. Moreover f^* is the usual pull-back operator if acting on smooth forms.

Here $f_* \Phi$ is defined by integrating Φ along fibers of f.

Proof. It suffices to check that $f_* \Phi$ is well-defined and commute with $d, \partial, \overline{\partial}$. This can be directly seen by using partition of unity. □

We admit the following important results. Let X be a complex manifold. A subset A in X is said to be *pluripolar* if $A \subset \{u = -\infty\}$ for some quasi-psh function u on X. Such a set A is *complete pluripolar* if $A = \{u = -\infty\}$ for some quasi-psh function u on X.

Theorem 3.1.13. *Let A be a complete pluripolar subset in X. The following statements are true:*

(i) *For every closed (weakly) positive current T on X, the currents $\mathbf{1}_A T$ and $\mathbf{1}_{X\backslash A} T$ are closed positive,*

(ii) *Assume that A is closed. Let T be a closed (weakly) positive current on $X\backslash A$. Then T has locally finite mass around every point in A, and hence extends trivially through A to be a closed positive current on X. To be precise, for every smooth form Φ with compact support on X, put*

$$\langle T', \Phi\rangle := \langle T, \mathbf{1}_{X\backslash A}\Phi\rangle.$$

Then T' is a well-defined closed positive current on X.

We refer to [13, 34] for proofs and historical works. We knew that for every psh function u then $dd^c u$ is closed positive. Another important source of closed positive currents are currents of integration along analytic subsets defined as follows. We admit basic properties of analytic sets. Let Ω be an open subset in $\mathbb{C}^n$. Recall that a subset A in Ω is *an analytic subset in* Ω if for every $x \in \Omega$ there exists a small neighborhood U_x of x and a collection of holomorphic functions $(f_j)_{j\in J}$ defined on U_x such that $A \cap U_x = \cap_{j\in J}\{f_j = 0\}$. An analytic subset A in Ω is said to be *irreducible* if there exist no non-empty analytic subsets A_1, A_2 in Ω such that $A = A_1 \cup A_2$.

Proposition 3.1.14. (i) *Let A be an analytic subset in Ω. Then $A = \cup_{j\in J} A_j$, where A_j is an irreducible analytic subset in Ω and the family $(A_j)_{j\in J}$ is locally finite.*

(ii) *Let A be an irreducible analytic subset in Ω. Then there exists an analytic subset* $\mathrm{Sing}(A)$ *in Ω such that* $\mathrm{Sing}(A)$ *is a proper subset of A and for every $x \in$* $\mathrm{Reg}(A) :=$ $A\backslash\mathrm{Sing}(A)$*, there exists a small open neighborhood U_x of x in Ω satisfying that $A \cap U_x$ is a submanifold of dimension k independent of k in U_x.*

We call $\mathrm{Sing}(A)$ in (ii) *the singular part of A*, and $\mathrm{Reg}(A)$ *the regular part of A*. The number k is called *the dimension of A*. The notion of analytic sets and Proposition 3.1.14 are obviously extended to the setting where Ω is replaced by a complex manifold.

The *support* of a current T on X is the smallest closed subset B on X such that for $\langle T, \Phi\rangle = 0$ for every Φ compactly supported on $X\backslash B$. We will not use the following result in the next two chapters.

Theorem 3.1.15. *Let A be an irreducible analytic subset in X.*

(i) *If $\dim A < p$, then every closed positive current of bi-dimension (p,p) has no mass on A.*

$(i)'$ *If the support of a closed positive current of bi-dimension (p,p) is of zero $2p$-dimensional Hausdorff measure then this current is zero.*

(ii) *If $\dim A < p-1$, then every closed positive current of bi-dimension (p,p) on $X\backslash A$ can be extended trivially through A to be a closed positive current on X.*

(iii) *for every closed positive current* T *of bi-dimension* (p, p) *on* X*, if* T *is supported on* A*, then* T *is a current on* A*, that means there exists a closed positive current* T' *of bi-dimension* (p, p) *on* A*, and for* $i : A \to X$ *the natural inclusion, we have* $i_*T' = T$*.*

We need to do a bit more to define currents on analytic subsets. But we ignore this detail here. In the lecture we only use Theorem 3.1.15 (iii) when A is smooth. We refer to [2, 5] for a proof of Theorem 3.1.15 and information about historical works.

Theorem 3.1.16. *([8, 23, 39]) (Hironaka's desingularisation of analytic sets) Let* X *be a complex manifold and* A *an analytic subset in* X*. Then there exist a complex manifold* X' *of dimension* $\dim X$*, and a surjective proper holomorphic map* $p : X' \to X$ *and a simple normal crossing hypersurface* E *in* X' *such that* p *is biholomorphic on* $X' \backslash E$*, and* $\mathrm{Reg}(A) \cap p(E) = \varnothing$*, and the Euclidean topological closure of* $\overline{p^{-1}(A \backslash p(E))}$ *is a smooth complex submanifold of* X'*.*

Let A be an irreducible analytic subset of dimension k in X. For every smooth $2k$-form Φ with compact support in X, we put

$$\langle [A], \Phi \rangle := \int_{\mathrm{Reg} A} \Phi.$$

Corollary 3.1.17. *(Lelong)* $[A]$ *is a well-defined positive closed* (k, k)*-current.*

Proof. Assume for the moment A is smooth. By Stokes' theorem, $[A]$ is a closed current. Since the weak positivity is preserved by holomorphic maps, we get the positivity of $[A]$. Consider now the general case. Let $p : X' \to X$ be a map as in Theorem 3.1.16 desingularizing A. Put $A' := \overline{p^{-1}(A \backslash p(E))}$ which is a smooth submanifold of X'. Since p is isomorphic outside E and $\mathrm{Reg}(A) \cap p(E) = \varnothing$, we see that

$$\langle [A], \Phi \rangle := \int_{p^{-1}(\mathrm{Reg} A)} p^* \Phi = \int_{A'} p^* \Phi = \langle [A'], p^* \Phi \rangle.$$

Thus the desired assertion follows. $\square$

We call $[A]$ *the current of integration along* A.

3.2 Monge-Ampère of bounded psh functions

Let T be a closed positive current on an open subset Ω in $\mathbb{C}^n$. Let u be a psh function on Ω. Our goal is to study situations in which the product $dd^c u \wedge T$ can be defined.

Let σ_T be the trace measure of T. Let K be a compact subset on Ω. Since σ_T is of finite mass on K and u is bounded from above on K (and is a Borel function defined everywhere on Ω), the Lebesgue integral $\int_K u \sigma_T$ is well-defined but it can be equal to $-\infty$. When u is locally integrable with respect to σ_T (that means $\int_K u \sigma_T > -\infty$ for every $K \Subset \Omega$), the current uT is well-defined because every coefficients of T are (signed) measures whose variations are bounded by a constant times σ_T. Hence in this case we can put

$$dd^c u \wedge T := dd^c (uT).$$

Lemma 3.2.1. *Let $(\mu_k)_k$ be a sequence of Radon measures converging weakly to a Radon measure μ in Ω. Let u be a psh function and $(u_k)_k$ be a sequence of psh functions converging to u in L^1_{loc} in Ω. Let μ' be the limit of a convergent subsequence $(u_k\mu_k)_k$ as $k \to \infty$. Then we have $\mu' \leq u\mu$. If additionally $u_k \geq u$ and $\mu_k \leq \mu$ for every k, then $u_k\mu_k \to u\mu$ as $k \to \infty$.*

Here we view (positive) measures as functionals from the space of continuous functions with compact support in Ω to $[0, \infty]$ (in particular the functional identically equal to ∞ is also considered as a measure).

Proof. Without loss of generality we can assume $u_k\mu_k \to \mu'$ as $k \to \infty$. Let u_k^ϵ, u^ϵ are standard regularisations of u_k, u respectively. Observe that $u_k^\epsilon \to u^\epsilon$ uniformly on compact subsets in Ω as $k \to \infty$ and ϵ fixed. Moreover we have $u_k \leq u_k^\epsilon$ and $u \leq u^\epsilon$. Hence

$$\mu' = \lim_{k\to\infty} u_k\mu_k \leq \lim_{k\to\infty} u_k^\epsilon\mu_k = u^\epsilon\mu.$$

Letting $\epsilon \to 0$ gives the desired inequality. Now if we have $u_k \geq u$ and $\mu_k \leq \mu$, then (we can assume $u_k \leq 0$ by using Hartog's lemma)

$$u\mu \leq u_k\mu \leq u_k\mu_k \to \mu'.$$

Thus $\mu' = u\mu$ for every limit measure μ' of the sequence $(u_k\mu_k)_k$. Hence $u_k\mu_k \to u\mu$ as $k \to \infty$. This finishes the proof. □

Corollary 3.2.2. *Let $(T_k)_k$ be a sequence of closed positive (p, p)-currents converging weakly to a current T and $T_k \leq T$ for every k. Let u be a psh function locally integrable with respect to the trace measure of T. Let $(u_k)_k$ be a sequence of psh functions converging weakly to u in L^1_{loc} as $k \to \infty$ and $u_k \geq u$ for every k. Then $u_kT_k \to uT$ as $k \to \infty$.*

Proof. Let ω be the standard Kähler form in $\mathbb{C}^n$. By Hartogs' lemma and the local nature of the question, we can assume u_k, u are negative. Since $0 \geq u_k \geq u$ and $T_k \leq T$, we get $|u_k|T_k \leq |u|T$. Hence the sequence $(u_kT_k)_k$ is of mass bounded uniformly in compact subsets in Ω. Let T' is a limit current of the sequence $(u_kT_k)_k$ (the limit of a convergent subsequence of $(u_kT_k)_k$). Arguing as in the proof of Lemma 3.2.1, we obtain $T' \leq uT$. Let $\mu_k := T_k \wedge \omega^{n-p}$ and $\mu := T \wedge \omega^{n-p}$. Applying Lemma 3.2.1 to u_k, u, μ_k, μ gives $T' \wedge \omega^{n-p} = uT \wedge \omega^{n-p}$. Hence $T' = uT$. It follows that $u_kT_k \to uT$ as $k \to \infty$. □

Lemma 3.2.3. *Assume that u is locally integrable with respect to the trace measure of T. Then $dd^cu \wedge T$ is a closed positive current of bi-degree $(p+1, p+1)$ and it coincides with the usual wedge products of continuous forms with currents when u is $\mathscr{C}^2$. Moreover if $(u_k)_k$ is a sequence of psh function converging to u in L^1_{loc} and $u_k \geq u$, then we have*

$$dd^cu_k \wedge T \to dd^cu \wedge T$$

weakly as $k \to \infty$.

Proof. The first desired assertion is a direct consequence of the second one and the standard regularisation of psh functions. By Corollary 3.2.2, we get $u_kT \to uT$ as $k \to \infty$. We leave details for readers. □

Lemma 3.2.4. *Let u, v be psh functions which are locally integrable with respect to the trace measure of T. Assume that $u = v$ on an open subset U in Ω. Then $dd^c u \wedge T = dd^c v \wedge T$ on U.*

Proof. Using regularisation u_ϵ, v_ϵ of u, v by using the same convolution. Let $U_1 \Subset U$ be an open subset. Observe that $u_\epsilon = v_\epsilon$ on U_1 if ϵ is small enough. Hence $dd^c u_\epsilon \wedge T = dd^c v_\epsilon \wedge T$ on U_1. Letting $\epsilon \to 0$ gives the desired assertion. □

Now consider the following problem: let $u_1, \ldots, u_m$ be psh functions such that u_j is locally integrable with respect to $dd^c u_{j-1} \wedge \cdots \wedge dd^c u_1 \wedge T$, and u_{jk} a sequence of psh functions as in Lemma 3.2.3, is $dd^c u_m \wedge \cdots \wedge dd^c u_1 \wedge T$ (which is defined inductively) symmetric and continuous under $(u_{jk})_k$? What follows will give us some partial answer to this question.

Here is the first main result in this section.

Theorem 3.2.5. *Let S be a closed positive current on Ω. Let v be a psh function on Ω such that v is locally integrable with respect to the trace measure of S and $(v_k)_k$ a sequence of psh functions on Ω such that $v_k \to v$ in L^1_{loc} as $k \to \infty$ and $v_k \geq v$ for every k. Let $T := dd^c v \wedge S$ and $T_k := dd^c v_k \wedge S$. Let u_j be a bounded psh function on Ω for $1 \leq j \leq m$. Let $(u_{jk})_{k \in \mathbb{N}}$ be a sequence of uniformly bounded psh functions such that $u_{jk} \to u_j$ in L^1_{loc} as $k \to \infty$ and $u_{jk} \geq u_j$ for every j, k. Then we have*

$$u_{1k} dd^c u_{2k} \wedge \cdots \wedge dd^c u_{mk} \wedge T_k \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge T \tag{3.2.1}$$

as $k \to \infty$.

The above result was proved in [38]. It is a slightly more general version of a well-known convergence theorem in [6] when v_k, v are locally bounded.

Proof. By Hartog's lemma, v_k, u_{jk} are uniformly bounded from above in k on compact subsets of Ω for every j. Since the problem is local, we can assume that Ω is relatively compact open set with smooth boundary in $\mathbb{C}^n$, every psh function in questions is defined on an open neighborhood of $\overline{\Omega}$, $v_k, v \leq 0$ on U for every k and u_{jk}, u_j are all equal to a smooth psh function ψ outside some fixed compact subset of Ω such that $\psi = 0$ on $\partial\Omega$. To be more precise, we do it as follows.

Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$ and z the standard coordinate system in $\mathbb{C}^n$. We can assume $\Omega = \mathbb{B}_{1/2}$ the ball of radius $1/2$ centered at 0 in $\mathbb{C}^n$ and $-2 \leq u_{jk}, u_j \leq -1$ are defined on an open neighborhood of $\overline{\mathbb{B}}$, put

$$u'_{jk} := \max\{u_{jk}, M(\|z\|^2 - 1)\}, \quad u'_j := \max\{u_j, M(\|z\|^2 - 1)\},$$

where M is a big enough constant such that $M(\|z\|^2 - 1) \leq -3$ on $\mathbb{B}_{1/2}$. We see that

$$u'_{jk} = u_{jk}$$

on $\mathbb{B}_{1/2}$ and $u'_{jk} = M\|z\|^2$ on a small neighborhood of $\partial\mathbb{B}$ (because $\|z\|^2 - 1 = 0$ on $\partial\mathbb{B}$). Using Lemma 3.2.4 inductively, we obtain

$$u_{1k} dd^c u_{2k} \wedge \cdots \wedge dd^c u_{mk} \wedge T_k = u'_{1k} dd^c u'_{2k} \wedge \cdots \wedge dd^c u'_{mk} \wedge T_k$$

on $\mathbb{B}_{1/2}$ and similarly

$$u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge T = u_1' dd^c u_2' \wedge \cdots \wedge dd^c u_m' \wedge T$$

on $\mathbb{B}_{1/2}$. So we can reduce the setting to the case where we describe in the beginning of the proof. We claim that

$$Q_k := v_k dd^c u_{1k} \wedge \cdots \wedge dd^c u_{mk} \wedge S \to v dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S \tag{3.2.2}$$

as $k \to \infty$. In particular, this implies that v is locally integrable with respect to $dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S$. We will prove (3.2.1) and (3.2.2) simultaneously by induction on m. When $m = 0$, this is a direct consequence of Lemma 3.2.1. Assume that (3.2.1) and (3.2.2) hold for $(m-1)$ in place of m. Let

$$R_{j,k} := dd^c u_{jk} \wedge \cdots \wedge dd^c u_{mk} \wedge T_k$$

for $1 \le j \le m$. By induction hypothesis, we have

$$R_{j,k} \to R_j := dd^c u_j \wedge \cdots \wedge dd^c u_m \wedge T$$

for $j \ge 2$. Since u_{1k} is uniformly bounded on Ω, the family $u_{1k} R_{2,k}$ is of uniformly bounded mass. Let R_∞ be a limit current of the last family. Without loss of generality, we can assume $R_\infty = \lim_{k\to\infty} u_{1k} R_{2,k}$ and S is of bi-degree $(n-m, n-m)$. By Lemma 3.2.1, we have $R_\infty \le u_1 R_2$. Thus, in order to have $R_\infty = u_1 R_2$, we just need to check that

$$\int_\Omega R_\infty \ge \int_\Omega u_1 R_2 \tag{3.2.3}$$

(both sides are finite because of the assumption we made at the beginning of the proof). Since $\psi = 0$ on $\partial\Omega$ and $u_{1k} = \psi$ on outside a compact of Ω, we have

$$\int_\Omega u_{1k} R_{2,k} \to \int_\Omega R_\infty, \quad \int_\Omega \psi R_{2,k} \to \int_\Omega \psi R_2. \tag{3.2.4}$$

Let $u_{jk}^\epsilon, \psi^\epsilon$ be standard regularisations of u_{jk}, ψ respectively. Since $u_{jk} = \psi$ outside some compact of Ω, we have $u_{jk}^\epsilon = \psi^\epsilon$ outside some compact K of Ω, for ϵ small enough and K independent of j, k, ϵ. Consequently, $u_{jk}^\epsilon - \psi^\epsilon$ is supported in $K \Subset \Omega$. Note that since ψ is smooth, $\psi^\epsilon \to \psi$ in $\mathscr{C}^\infty$- topology. By integration by parts and the fact that $u_{jk} \ge u_j$ for $j = 1, 2$, we have

$$\begin{aligned}
\int_\Omega (u_1 - \psi) R_2 &\le \lim_{\epsilon\to 0} \int_\Omega (u_{1k}^\epsilon - \psi^\epsilon) R_2 = \lim_{\epsilon\to 0} \int_\Omega u_2 dd^c (u_{1k}^\epsilon - \psi^\epsilon) R_3 \\
&\le \lim_{\epsilon\to 0} \int_\Omega u_{2k}^\epsilon dd^c (u_{1k}^\epsilon - \psi^\epsilon) R_3 + \lim_{\epsilon\to 0} \int_\Omega (u_{2k}^\epsilon - u_2) dd^c \psi^\epsilon \wedge R_3 \\
&= \lim_{\epsilon\to 0} \int_\Omega (u_{1k}^\epsilon - \psi^\epsilon) dd^c u_{2k}^\epsilon \wedge R_3 + o_{k\to\infty}(1)
\end{aligned}$$

by induction hypothesis for $(m-1)$ of (3.2.1) and the fact that $\|dd^c\psi_\epsilon - dd^c\psi\|_{\mathscr{C}^0} = O(\epsilon)$. We now apply similar arguments to u_{3k} in place of u_{2k}. Precisely, as above we have

$$\begin{aligned}\int_\Omega (u^\epsilon_{1k} - \psi^\epsilon) dd^c u^\epsilon_{2k} \wedge R_3 &= \int_\Omega u_3 dd^c(u^\epsilon_{1k} - \psi^\epsilon) \wedge dd^c u^\epsilon_{2k} \wedge R_4 \\ &\le \int_\Omega u^\epsilon_{3k} dd^c(u^\epsilon_{1k} - \psi^\epsilon) \wedge dd^c u^\epsilon_{2k} \wedge R_4 \\ &+ \int_\Omega (u^\epsilon_{3k} - u_3) dd^c\psi^\epsilon \wedge dd^c u^\epsilon_{2k} \wedge R_4.\end{aligned}$$

Letting $\epsilon \to 0$ and applying the induction hypothesis to the second term in the right-hand side of the last inequality (noticing again that $\|dd^c\psi_\epsilon - dd^c\psi\|_{\mathscr{C}^0} = O(\epsilon)$), we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega (u^\epsilon_{1k} - \psi^\epsilon) dd^c u^\epsilon_{2k} \wedge R_3 &\le \lim_{\epsilon\to 0}\int_\Omega u^\epsilon_{3k} dd^c(u^\epsilon_{1k} - \psi^\epsilon) \wedge dd^c u^\epsilon_{2k} \wedge R_4 + o_{k\to\infty}(1) \\ &\le \lim_{\epsilon\to 0}\int_\Omega (u^\epsilon_{1k} - \psi^\epsilon) \wedge dd^c u^\epsilon_{2k} \wedge dd^c u^\epsilon_{3k} \wedge R_4 + o_{k\to\infty}(1).\end{aligned}$$

Put $R'^\epsilon_{2,k} := dd^c u^\epsilon_{2k} \wedge \cdots \wedge dd^c u^\epsilon_{mk}$. Repeating the above arguments for every u_{jk} ($j \ge 2$) and v, v_k gives

$$\begin{aligned}\int_\Omega (u_1 - \psi) R_2 &\le \lim_{\epsilon\to 0}\int_\Omega (u^\epsilon_{1k} - \psi^\epsilon) R'^\epsilon_{2,k} \wedge dd^c v \wedge S + o_{k\to\infty}(1) \\ &\le \lim_{\epsilon\to 0}\int_\Omega v dd^c(u^\epsilon_{1k} - \psi^\epsilon) \wedge R'^\epsilon_{2,k} \wedge S + o_{k\to\infty}(1) \\ &\le \lim_{\epsilon\to 0}\int_\Omega v_k dd^c(u^\epsilon_{1k} - \psi^\epsilon) \wedge R'^\epsilon_{2,k} \wedge S + \\ &+ \lim_{\epsilon\to 0}\int_\Omega (v_k - v) dd^c\psi^\epsilon \wedge R'^\epsilon_{2,k} \wedge S + o_{k\to\infty}(1) \\ &= \lim_{\epsilon\to 0}\int_\Omega (u^\epsilon_{1k} - \psi^\epsilon) \wedge R'^\epsilon_{2,k} \wedge dd^c v_k \wedge S + o_{k\to\infty}(1) \\ &= \int_\Omega (u_{1k} - \psi) \wedge R_{2,k} + o_{k\to\infty}(1)\end{aligned}$$

by (3.2.2) for $(m-1)$ and the usual convergence of Monge-Ampère operators. Letting $k \to \infty$ in the last inequality and using (3.2.4) give (3.2.3). Hence (3.2.1) for m follows.

It remains to prove (3.2.2) for m. Put $R'_{2,k} := dd^c u_{2k} \wedge \cdots \wedge dd^c u_{mk}$ and $R'_2 := dd^c u_2 \wedge \cdots \wedge dd^c u_m$. We check that Q_k is of uniformly bounded mass. Decompose

$$Q_k = v_k dd^c(u_{1k} - \psi) \wedge R'_{2,k} \wedge S + v_k dd^c\psi \wedge R'_{2,k} \wedge S.$$

The second term converges to $v dd^c\psi \wedge R'_2 \wedge S$ as $k \to \infty$ by induction hypothesis for $(m-1)$. Denote by $Q_{k,1}$ the first term. Let v^ϵ_k be standard regularizations of v_k. By integration by parts, we have

$$\begin{aligned}\int_\Omega Q^\epsilon_{k,1} &:= \int_U v^\epsilon_k dd^c(u_{1k} - \psi) \wedge R'_{2,k} \wedge S \\ &= \int_\Omega (u_{1k} - \psi) dd^c v^\epsilon_k \wedge R'_{2,k} \wedge S = (u_{1k} - \psi) R'_{2,k} \wedge dd^c v^\epsilon_k \wedge S\end{aligned}$$

which converges to $\int_\Omega (u_{1k} - \psi) R'_{2,k} \wedge dd^c v_k \wedge S$ as $\epsilon \to 0$ by (3.2.1) for m. Thus,

$$\int_\Omega Q_{k,1} = \int_U (u_{1k} - \psi) R'_{2,k} \wedge dd^c v_k \wedge S.$$

This combined with (3.2.1) for m again implies that $\int_\Omega Q_{k,1} \to \int_U (u_1 - \psi) R'_2 \wedge dd^c v \wedge S$ as $k \to \infty$. The last limit is equal to $\int_\Omega v dd^c(u_1 - \psi) \wedge R'_2$ by integration by parts which can be performed thanks to (3.2.1) for m. Thus, we have proved that Q_k is of uniformly bounded mass and

$$\int_\Omega Q_k \to \int_U v R_1$$

as $k \to \infty$. This combined with the fact that $vR_1 \geq Q_\infty$ for every limit current Q_∞ of the family $(Q_k)_k$ gives the desired assertion (3.2.2) for m. This finishes the proof. $\square$

The following two corollaries follow from the proof of Theorem 3.2.5.

Corollary 3.2.6. *Let S be a closed positive current on Ω. Let $u_1, \dots, u_m$ be psh function on Ω such that u_j is locally bounded for every $1 \leq j \leq m$ except possibly for one index. Then the current $dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S$, which is defined inductively as usual, is symmetric with respect to $u_1, \dots, u_m$ and satisfies the convergence under decreasing sequences.*

Corollary 3.2.7. *Let S be a closed positive current on Ω. Let $u_1, \dots, u_m$ be psh function on U such that u_j is locally bounded for every $1 \leq j \leq m$ except possibly for one index. Let u_0 be another psh function locally integrable with respect to S such that u_0 is locally bounded if there is an index $1 \leq j \leq m$ so that u_j is not locally bounded. Then $u_0 dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S$ is convergent under decreasing sequences and for every compact K in Ω, if we have $0 \leq u_1, \dots, u_m \leq 1$, then*

$$\|u_0 dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S\|_K \leq C \|u_0 S\|_\Omega \tag{3.2.5}$$

(Chern-Levine-Nirenberg inequality) for some constant C independent of $u_0, \dots, u_m, S$, in particular, in this case

$$\|dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S\|_{K \cap \{u_0 \leq -M\}} \leq C/M \|u_0 S\|_\Omega \tag{3.2.6}$$

for every constant $M > 0$.

Proof. Everything follows from the proof of Theorem 3.2.5 except (3.2.6). To see why (3.2.6) is true, one just notices that

$$\|dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S\|_{K \cap \{u_0 \leq -M\}} \leq M^{-1} \|u_0 dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S\|_K$$

which is $\leq C/M \|u_0 S\|_\Omega$ by (3.2.5). $\square$

Note that the usual Chern-Levine-Nirenberg inequality ([11]) was stated for $u_0 \equiv 1$. The inequality (3.2.5) was proved in [13] and [29].

Lemma 3.2.8. *(Cauchy-Schwarz inequality) Let η_1, η_2 be continuous $(1,0)$-form on Ω. Let T be positive current of bi-dimension $(1,1)$ with compact support on Ω. Then we have*

$$\int_\Omega \eta_1 \wedge \overline{\eta}_2 \wedge T \leq \left(\int_\Omega \eta_1 \wedge \overline{\eta}_1 \wedge T \right)^{1/2} \left(\int_\Omega \eta_2 \wedge \overline{\eta}_2 \wedge T \right)^{1/2}.$$

Proof. Consider the following positive semi-definite Hermitian form on the space of continuous $(1,0)$-forms on Ω:

$$\langle \eta_1, \eta_2 \rangle := \int_\Omega \eta_1 \wedge \overline{\eta}_2 \wedge T.$$

The desired inequality follows from the Cauchy-Schwarz inequality for the last Hermitian form. □

Let u be locally bounded psh and T be a closed positive current. We define (as in [6])

$$du \wedge d^c u \wedge T := dd^c u^2 \wedge T - u dd^c u \wedge T.$$

Note that since u is locally bounded, u^2 is the difference of two locally bounded psh functions (write $u^2 = (u+M)^2 - 2Mu - M^2$, where M is a constant such that $u + M \geq 0$). Hence $dd^c u^2 \wedge T$ is well-defined in the above sense. Let w be another locally bounded psh function. When T is of bi-dimension $(1,1)$ we define

$$2du \wedge d^c w \wedge T := dd^c(u+w)^2 \wedge T - (u+w)dd^c(u+w) \wedge T - du \wedge d^c u \wedge T - dw \wedge d^c w \wedge T.$$

One can see that the above definitions agree with the smooth case.

Lemma 3.2.9. (i) *The current $du \wedge d^c u \wedge T$ is positive, and if psh functions u_j decreases to u then $du_j \wedge d^c u_j \wedge T \to du \wedge d^c u \wedge T$ as $j \to \infty$. We also have a similar continuity property for $du \wedge d^c w \wedge T$ when T is of bi-dimension $(1,1)$.*

(ii) *(Cauchy-Schwarz inequality) if T is of bi-dimension $(1,1)$, then*

$$\int_\Omega du \wedge d^c w \wedge T \leq \Big(\int_\Omega du \wedge d^c u \wedge T \Big)^{1/2} \Big(\int_\Omega dw \wedge d^c w \wedge T \Big)^{1/2}.$$

(iii) *(Integration by parts formula) if T is of bi-dimension $(1,1)$ and χ is a smooth function with compact support in Ω, then*

$$\int_\Omega \chi du \wedge d^c w \wedge T = -\int_\Omega u d\chi \wedge d^c w \wedge T,$$

and if u' is a locally bounded psh function such that $u - u'$ is compactly supported on Ω then

$$\int_\Omega d(u-u') \wedge d^c w \wedge T = -\int_\Omega (u-u') dd^c w \wedge T.$$

Proof. Direct consequence of Theorem 3.2.5. □

3.3 Capacity and quasi-continuity

Let Ω be an open subset of $\mathbb{C}^n$. Let E be a Borel subset of Ω. The *capacity* cap(E, Ω) of E in Ω, which was introduced in [6], is given by

$$\text{cap}(E, \Omega) := \sup \Big\{ \int_E (dd^c u)^n : u \text{ is psh on } \Omega \text{ and } 0 \leq u \leq 1 \Big\}.$$

For a closed positive current T of bi-dimension (m,m) on Ω $(0 \leq m \leq n)$, we define

$$\text{cap}_T(E,\Omega) := \sup\left\{ \int_E (dd^c u)^m \wedge T : u \text{ is psh on } \Omega \text{ and } 0 \leq u \leq 1 \right\}.$$

We say that a sequence of functions $(u_k)_{k\in\mathbb{N}}$ *converges to* u *with respect to the capacity* cap_T *(relative in* Ω*)* if for any constant $\epsilon > 0$ and $K \Subset U$, we have $\text{cap}_T\big(\{|u_k - u| \geq \epsilon\} \cap K, U\big) \to 0$ as $k \to \infty$. We call cap_T *the* T*-capacity.* When $T = 1$, we simply refer to cap_T as capacity. The notion of relative cap_T was introduced in [29, 40]. By Corollary 3.2.7, $\text{cap}_T(E,\Omega) < \infty$ if E is relatively compact in Ω.

Lemma 3.3.1. *Let* E *be a Borel subset in* Ω*. The following are true:*

(i) *for every Borel set* $E' \subset E$*, then* $cap_T(E',\Omega) \leq cap_T(E,\Omega)$*.*

(ii) *If Borel sets* E_j *increases to* E*, then* $cap_T(E,\Omega) = \lim_{j\to\infty} cap_T(E_j,\Omega)$*.*

(iii) *for every Borel set* $E \subset \Omega$*, we have*

$$cap_T(E,\Omega) = \sup\{cap_T(K,\Omega) : K \textit{ compact subset in } E\}.$$

(iv) *if Borel sets* E_j *converges to a set* E *(in the sense that* $\mathbf{1}_{E_j}$ *converges pointwise to* $\mathbf{1}_E$ *as* $j \to \infty$*), then*

$$cap_T(E,\Omega) \leq \liminf_{j\to\infty} cap_T(E_j,\Omega).$$

Proof. The property (i) is clear. We check (ii). We can assume $\text{cap}_T(E,\Omega) < \infty$. The proof when $\text{cap}_T(E,\Omega) = \infty$ is similar. Let $\epsilon > 0$. There exists a psh function $0 \leq u \leq 1$ such that

$$\infty > \int_E (dd^c u)^m \wedge T \geq \text{cap}_T(E,\Omega) - \epsilon.$$

Since E_j increases to E, for j big enough we get

$$\text{cap}_T(E_j,\Omega) \geq \int_{E_j} (dd^c u)^m \wedge T \geq \int_E (dd^c u)^m \wedge T - \epsilon.$$

Hence the desired assertion (ii) follows. The (iii) is done analogously by using an extra property that $(dd^c u)^m \wedge T$ is a Radon measure: hence

$$\int_K (dd^c u)^m \wedge T \geq \int_E (dd^c u)^m \wedge T - \epsilon$$

for some compact K in E. Similarly we get (iv). □

Let U be an open subset in $\mathbb{C}^n$. We say that a subset A in U is *locally complete pluripolar* set if locally $A = \{\psi = -\infty\}$ for some psh function ψ.

Lemma 3.3.2. *Let* A *be a locally complete pluripolar set in* Ω*. Let* T *be a closed positive current of bi-dimension* (m,m) *on* Ω*. Assume that* T *has no mass on* A*. Then, we have* $cap_T(A,\Omega) = 0$*.*

Proof. The proof is standard. We present the details for readers' convenience. Since the problem is of local nature, we can assume that there is a negative psh function ψ on Ω such that $A = \{\psi = -\infty\}$. Let $u_1, \dots, u_m$ be bounded psh functions on Ω such that $0 \le u_j \le 1$ for $1 \le j \le m$. Let ω is the standard Kähler form on $\mathbb{C}^n$. Let $k \in \mathbb{N}$ and $\psi_k := k^{-1}\max\{\psi, -k\}$. We have $-1 \le \psi_k \le 0$. Let χ be a nonegative smooth function with compact support in Ω. Let $0 \le l \le m$ be an integer. Put

$$I_k := \int_\Omega \chi\psi_k dd^c u_1 \wedge \cdots \wedge dd^c u_l \wedge \omega^{m-l} \wedge T.$$

Since $\psi_k = -1$ on $\{\psi < -k\}$, in order to prove the desired assertion, it is enough to show that for every $0 \le l \le q$, we have

$$I_k \to 0 \tag{3.3.1}$$

as $k \to \infty$ uniformly in $u_1, \dots, u_l$. We will prove (3.3.1) by induction on l. Firstly, (3.3.1) is trivial if $l = 0$ because T has no mass on A. Assume that it holds for $(l-1)$. We prove it for l. Put

$$R := dd^c u_2 \wedge \cdots \wedge dd^c u_l \wedge \omega^{m-l} \wedge T.$$

By integration by parts, we have

$$I_k = \int_\Omega u_1\chi dd^c\psi_k \wedge R + \int_\Omega u_1\psi_k dd^c\chi \wedge R + 2\int_\Omega u_1 d\psi_k \wedge d^c\chi \wedge R.$$

Denote by $I_{k,1}, I_{k,2}, I_{k,3}$ the first, second and third term respectively in the right-hand side of the last equality. Since u_1 is bounded by 1, by integration by parts, we get

$$|I_{k,1}| \le C\int_{\mathrm{Supp}\chi} -\psi_k R \wedge \omega, \quad |I_{k,2}| \le C\int_{\mathrm{Supp}\chi} -\psi_k R \wedge \omega,$$

for some constant C depending only on χ. By induction hypothesis, we have

$$\lim_{k\to\infty}\int_{\mathrm{Supp}\chi} \psi_k R \wedge \omega = 0.$$

Thus $\lim_{k\to\infty} I_{k,j} = 0$ for $j = 1, 2$. To treat $I_{k,3}$, we use the Cauchy-Schwarz inequality to get

$$|I_{k,3}| \le \left(\int_{\mathrm{Supp}\chi} d\psi_k \wedge d^c\psi_k \wedge R\right)^{1/2}.$$

Let $0 \le \chi_1 \le 1$ be a smooth cut-off function compactly supported on U such that $\chi_1 = 1$ on $\mathrm{Supp}\chi$. Let $U_1 \Subset U$ be an open subset containing $\mathrm{Supp}\chi_1$. Since $d\psi_k \wedge d^c\psi_k \wedge R \ge 0$, we have

$$\begin{aligned}\int_{\mathrm{Supp}\chi} d\psi_k \wedge d^c\psi_k \wedge R &\leq \int_\Omega \chi_1 d\psi_k \wedge d^c\psi_k \wedge R \\ &= \int_\Omega \chi_1(dd^c\psi_k^2 - \psi_k dd^c\psi_k) \wedge R \\ &= \int_\Omega \chi_1 dd^c\psi_k^2 \wedge R - \int_\Omega \chi_1\psi_k dd^c\psi_k \wedge R \\ &= \int_\Omega \psi_k^2 dd^c\chi_1 \wedge R - \int_\Omega \chi_1\psi_k dd^c\psi_k \wedge R \\ &\lesssim \int_{\mathrm{Supp}\chi_1} -\psi_k R \wedge \omega + \int_\Omega \chi_1 dd^c\psi_k \wedge R \\ &\lesssim \int_{\mathrm{Supp}\chi_1} -\psi_k R \wedge \omega + \int_{U_1} -\psi_k R \wedge \omega.\end{aligned}$$

because $-1 \leq \psi_k \leq 0$ and $-\omega \lesssim dd^c\chi_1 \lesssim \omega$. We infer that

$$|I_{k,3}| \lesssim \bigg(\int_{U_1} -\psi_k R \wedge \omega\bigg)^{1/2}.$$

By induction hypothesis, $\lim_{k\to\infty} \int_{U_1} \psi_k R \wedge \omega = 0$. So $\lim_{k\to\infty} I_{k,3} = 0$. In conclusion, (3.3.1) follows. This finishes the proof. □

Lemma 3.3.3. *Let u and u' be locally bounded psh function such that $0 \leq u' \leq u \leq 1$. Let $K \Subset U \Subset \Omega$ be open subsets. Let T be a closed positive current of bi-dimension (m,m) such that $\|T\|_U \leq 1$. Then for every constant $\epsilon > 0$, we have*

$$\mathit{cap}_T(K \cap \{u - u' \geq \epsilon\}) \leq \epsilon^{-1} C\bigg(\int_U (u-u')(dd^c u')^m \wedge T\bigg)^{2^{-m}},$$

where $C > 0$ is a constant independent of u, u' and T.

Proof. We follow ideas presented in [29, Proposition 1.12]. Let $0 \leq v_1, \ldots, v_m \leq 1$ be psh function on Ω. Since the problem is local and $-1 \leq u, u', v_j \leq 0$ $(1 \leq j \leq m)$, we can assume that $\Omega \Subset \mathbb{C}^n$, the functions u, u', v_j are defined on an open neighborhood of $\overline{\Omega}$ and there exist a smooth psh function ψ defined on an open neighborhood of $\overline{\Omega}$ and an open neighborhood W of $\partial\Omega$ such that $K \subset \Omega \backslash W$ and $u' = u = v_j = \psi$ on W for every j. Let

$$T_l' := dd^c v_2 \wedge \cdots \wedge dd^c v_m \wedge T.$$

Observe $u' - u$ is of compact support in some open set $U_1 \Subset \Omega$ containing K. Hence, by integration by parts, we get

$$\begin{aligned}\int_{U_1} (u-u') dd^c v_1 \wedge T_l' &= -\int_{U_1} d(u-u') \wedge d^c v_1 \wedge T_l' \\ &\leq \bigg(\int_{U_1} d(u-u') \wedge d^c(u-u') \wedge T_l'\bigg)^{1/2} \bigg(\int_{U_1} dv_1 \wedge d^c v_1 \wedge T_l'\bigg)^{1/2}\end{aligned}$$

which is $\lesssim \left(\int_{U_1} d(u-u') \wedge d^c(u-u') \wedge T_l' \right)^{1/2}$ by the Chern-Levine-Nirenberg inequality. Denote by I the integral in the last quantity. We have

$$I = -\int_{U_1} (u-u') \wedge dd^c(u-u') \wedge T' \leq \int_{U_1} (u-u') \wedge dd^c u' \wedge T_l'.$$

Applying similar arguments to $v_2, \ldots, v_m$ consecutively and the right-hand side of the last inequality, we obtain that

$$\int_K (u-u') dd^c v_1 \wedge \cdots \wedge dd^c v_m \wedge T \leq C \left(\int_{U_1} (u-u')(dd^c u')^m \wedge T \right)^{2^{-m}}, \tag{3.3.2}$$

where C is independent of $u, u', v_1, \ldots, v_m$ and T. This finishes the proof. $\square$

We now give a definition which will be important later. Let $(T_k)_k$ be a sequence of closed positive currents of bi-dimension (m,m) on Ω. We say that $(T_k)_k$ satisfies *Condition* $(*)$ if $(T_k)_k$ is of uniformly bounded mass on compact subsets of Ω, and for every open set $U \subset \Omega$ and every bounded psh function u on U and every sequence $(u_k)_k$ of psh functions on U decreasing to u, we have

$$\lim_{k\to\infty} (u_k - u)(dd^c u)^m \wedge T_k = 0 \tag{3.3.3}$$

An obvious example for sequences satisfying Condition $(*)$ is constant sequences: $T_k = T$ for every k. By Theorem 3.2.5, for every sequence of psh functions $(v_k)_k$ on Ω such that v_k converges to some psh v in L^1_{loc} as $k \to \infty$ and $v_k \geq v$ for every k, then the sequence $T_k := dd^c v_k \wedge S$ satisfies Condition $(*)$.

Theorem 3.3.4. *(Strong quasi-continuity of bounded psh functions) Let $(T_l)_l$ be a sequence of closed positive currents satisfying Condition $(*)$. Let u be a* bounded *psh function on U and $(u_k)_k$ a sequence of psh functions on Ω decreasing to u. Then for every constant $\epsilon > 0$ and every compact K in U, we have cap$_{T_l}(\{|u_k - u| \geq \epsilon\} \cap K) \to 0$ as $k \to \infty$ uniformly in l. In particular, for every constant $\epsilon > 0$, there exists an open subset U of Ω such that cap$_{T_l}(U, \Omega) < \epsilon$ for every l and the restriction of u to $\Omega \backslash U$ is continuous.*

Consider the case where $T_l = T$ for every l. Then, the above theorem give a quasi-continuity with respect to cap$_T$ for bounded psh function which is *stronger* than the usual one for general psh functions with respect to cap (see [6]). We refer to Theorem 3.3.4 as a (uniform) strong quasi-continuity of bounded psh functions.

Proof. Let $K \Subset U_1 \Subset \Omega$. Let T_l be of bi-dimension (m,m). By Hartog's lemma and the boundedness of u, we obtain that u_k is uniformly bounded in k in compact subsets of Ω. Hence this allows us to apply Lemma 3.3.3 to u_k, u to obtain

$$\text{cap}_{T_l}(K \cap \{u_k - u \geq \epsilon\}) \leq C \left(\int_{U_1} (u_k - u)(dd^c u)^m \wedge T_l \right)^{2^{-m}}, \tag{3.3.4}$$

where C is independent of k (note that the mass of T_l on compact subsets of Ω is bounded uniformly in l). Let

$$H_{k,l} := \int_{U_1} (u_k - u)(dd^c u)^m \wedge T_l.$$

We need to prove that $H_{k,l}$ converges to 0 as $k \to \infty$ uniformly in l. Suppose that this is not the case. This means that there exists a constant $\epsilon > 0$, $(k_s)_s \to \infty$ and $(l_s)_s \to \infty$ such that $H_{k_s,l_s} \geq \epsilon$ for every s. However, by Condition $(*)$, we get $(u_{k_s} - u)(dd^c u)^m \wedge T_{l_s} \to 0$ as $s \to \infty$. This is contradiction. Hence the first desired assertion follows.

We prove the second desired assertion, let $K \Subset \Omega$ and $(u_k)_k$ a sequence of smooth psh functions defined on an open neighborhood of K decreasing to u. Let $\epsilon > 0$ be a constant. Since $u_k \to u$ in cap_{T_l} as $k \to \infty$ uniformly in l, there is a sequence $(j_k)_k$ converging to ∞ for which

$$\text{cap}_{T_l}\big(K \cap \{u_{j_k} > u + 1/k\}, \Omega\big) \leq \epsilon 2^{-k}$$

for every $k, l \in \mathbb{N}^*$. Consequently, for $K_\epsilon := K \backslash \cup_{k=1}^\infty \{u_{j_k} > u + 1/k\}$, we have that $\text{cap}_{T_l}(K \backslash K_\epsilon, \Omega) \leq \epsilon$ and u_{j_k} is convergent uniformly on K_ϵ. Hence u is continuous on K_ϵ.

Let $(U_s)_s$ be an increasing exhaustive sequence of relatively compact open subsets of Ω and $K_s := \overline{U}_s \backslash U_{s-1}$ for $s \geq 1$, where $U_0 := \varnothing$. Observe that K_l is compact, $\Omega = \cup_{s=1}^\infty K_s$ and

$$K_s \cap \overline{\cup_{s' \geq s+2} K_{s'}} = \varnothing \tag{3.3.5}$$

for every $s \geq 1$. By the previous paragraph, there exists a compact subset K'_s of K_s such that $\text{cap}_{T_l}(K_s \backslash K'_s, \Omega) \leq \epsilon 2^{-s}$ and u is continuous on K'_s. Observe that $K' := \cup_{s=1}^\infty K'_s$ is closed in Ω and u is continuous on K' because of (3.3.5). We also have $\Omega \backslash K' \subset \cup_{s=1}^\infty (K_s \backslash K'_s)$. Hence $\text{cap}_{T_l}(U \backslash K', \Omega) \leq \epsilon$ for every l. The proof is finished. □

As one can expect, the above quasi-continuity of bounded psh functions allows us to treat, to certain extent, these functions as continuous functions with respect to closed positive currents.

Corollary 3.3.5. *Let $R_k := dd^c v_{1k} \wedge \cdots \wedge dd^c v_{mk} \wedge T_k$ and $R := dd^c v_1 \wedge \cdots \wedge dd^c v_m \wedge T$, where v_{jk}, v_j are uniformly bounded psh functions on Ω and T_k, T closed positive currents of bi-degree (p, p). Let u be a bounded psh function on Ω and χ a continuous function on $\mathbb{R}$. Assume that $R_k \to R$ as $k \to \infty$ on Ω and $(T_k)_k$ satisfies Condition $(*)$. Then we have*

$$\chi(u) R_k \to \chi(u) R$$

as $k \to \infty$. In particular, the last convergence holds when $T_k = T$ for every k or $T_k = dd^c w_k \wedge S$, $T = dd^c w \wedge S$, where S is a closed positive current, w is a psh function locally integrable with respect to S and w_k is a psh function converging to w in L^1_{loc} as $k \to \infty$ so that $w_k \geq w$ for every k.

Proof. The problem is local. Hence we can assume Ω is relatively compact in $\mathbb{C}^n$. Since u is bounded, using Theorem 3.3.4, we have that u is uniformly quasi-continuous with respect to the family cap_{T_k} with $k \in \mathbb{N}$. This means that given $\epsilon > 0$, we can find an open subset U' of Ω such that $\text{cap}_{T_k}(U', \Omega) < \epsilon$ and $u|_{\Omega \backslash U'}$ is continuous. Let $\tilde{u}$ be a bounded continuous function on U extending $u|_{\Omega\backslash U'}$ (see [32, Theorem 20.4]). We have $\chi(\tilde{u})R_k \to \chi(\tilde{u})R$ because $\chi, \tilde{u}$ are continuous. Moreover,

$$\big\|\big(\chi(\tilde{u}) - \chi(u)\big)R_k\big\| \lesssim \|R_k\|_{\Omega\backslash U'} \leq \mathrm{cap}_{T_k}(\Omega\backslash U', \Omega) < \epsilon$$

(we used here the boundedness of Ω) and a similar estimate also holds for $\big(\chi(\tilde{u})-\chi(u)\big)R$. The desired assertion then follows. This finishes the proof. □

The following result is a well-known convergence property of Monge-Ampère operators in [6].

Theorem 3.3.6. *Let u_j be a locally bounded psh function on Ω for $1 \leq j \leq m$. Let $(u_{jk})_{k\in\mathbb{N}}$ be a sequence of locally bounded psh functions increasing to u_j in L^1_{loc} as $k \to \infty$. Then we have*

$$u_{1k} dd^c u_{2k} \wedge \cdots \wedge dd^c u_{mk} \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m$$

as $k \to \infty$.

Proof. The proof follows that of [29, Theorem 1.15]. First of all, observe that if $u_{jk} \nearrow u_j$ almost everywhere then, we have $u_{jk} \leq u_{j(k+1)} \leq u_j$ pointwise on U. Since the problem is local, as in the proof of Theorem 3.2.5, we can assume that u_{jk}, u_j are all equal to some smooth psh function ψ outside some set $K \Subset \Omega$ on Ω. Let

$$S_{jk} := dd^c u_{jk} \wedge \cdots \wedge dd^c u_{mk}, \quad S_j := dd^c u_j \wedge \cdots \wedge dd^c u_m.$$

We prove by induction in j that

$$u_{(j-1)k} S_{jk} \to u_{(j-1)} S_j \tag{3.3.6}$$

k and for every $2 \leq j \leq m+1$ (by convention we put $S_{(m+1)k} = S_{m+1} := 1$). The claim is clear for $j = m+1$. Suppose that it holds for $(j+1)$. We need to prove it for j. Let $R_{j\infty}$ be a limit current of $u_{(j-1)k}S_{jk}$ as $k \to \infty$. By induction hypothesis (3.3.6) for $(j+1)$ instead of j, $S_{jk} \to S_j$ as $k \to \infty$. This combined with the fact that the sequence $(u_{jk})_k$ converges in L^1_{loc} to u_j gives

$$R_{j\infty} \leq u_{j-1} S_j$$

(one can see [19, Proposition 3.2]). Fix $s \in \mathbb{N}$. Let ω be the standard Kähler form in $\mathbb{C}^n$. For $k \geq s$, by integration by parts,

$$\begin{aligned}
\liminf_{k\to\infty} \int_\Omega u_{(j-1)k} S_{jk} \wedge \omega^{n-m+j-1} &\geq \liminf_{k\to\infty} \int_\Omega u_{(j-1)s} S_{jk} \wedge \omega^{n-m+j-1} \\
&= \int_\Omega u_{(j-1)s} S_j \wedge \omega^{n-m+j-1} \\
&= \int_\Omega u_{(j-1)s} dd^c u_j \wedge S_{(j+1)} \wedge \omega^{n-m+j-1} \\
&= \int_\Omega u_j dd^c u_{(j-1)s}) \wedge S_{(j+1)} \wedge \omega^{n-m+j-1}
\end{aligned}$$

which converges to

$$\int_\Omega u_j dd^c u_{(j-1)} \wedge S_{(j+1)} \wedge \omega^{n-m+j-1} = \int_\Omega u_{(j-1)s} S_j \wedge \omega^{n-m+j-1}$$

by induction hypothesis and Corollary 3.3.5. This finishes the proof. □

Lemma 3.3.7. *(Negligible sets are of zero capacity) Negligible sets are Borel sets of zero capacity.*

Proof. Let $(u_j)_{j\in J}$ be a family of psh functions bounded uniformly from above. Let E be the set of $x \in \Omega$ such that $(\sup_{j\in J} u_j)^*(x) > \sup_{j\in J} u_j(x)$. We nede to prove that $\text{cap}(E,\Omega) = 0$. By Choquet's lemma, we can assume J is countable, and $u := (\sup_{j\in J} u_j)^*$ is L^1 limit of an increasing sequence $(u_j)_j$ of psh functions. Observe that

$$E = \cup_{s,t\in\mathbb{Q}}\big\{x : (\sup_{j\in J} u_j)^* \geq s > t \geq \sup_{j\in J} u_j\big\}.$$

Since each of these sets in the last union is Borel, so is E. We first assume that $(u_j)_j$ is uniformly locally bounded on Ω. By Theorem 3.3.6, we get

$$u_j(dd^c v)^n \geq u(dd^c v)^n$$

as $j \to \infty$ for every bounded psh function v. On the the hand, by Lebesgue's monotone convergence theorem, $u_j(dd^c v)^n \to (\lim_{j\to\infty} u_j)(dd^c v)^n$. Hence the set $\{x : u(x) > \lim_{j\to\infty} u_j(x)\}$ is of zero measure with respect to $(dd^c v)^n$.

Consider the general case where $(u_j)_j$ is not necessarily uniformly locally bounded. Let u be as above, and J is countable, the family $(u_j)_j$ is uniformly bounded from above. Let $A := \{u_1 = -\infty\}$. We already know that A is of zero capacity by Lemma 3.3.2. Let M be a big integer. Consider $u_{jM} := \max\{u_j, -M\}$, and $u_M := (\sup_j u_{jM})^*$. Observe that

$$\{x : u(x) > \sup_j u_j(x)\}\backslash A \subset \bigcup_{M\in\mathbb{N}} \big\{x \in \Omega : u_M(x) > \sup_j u_{jM}(x)\big\}.$$

This combined with the first part of the proof implies that $\{x : u > \sup_j u_j\}$ is of zero capacity. □

Just by replacing the usual quasi-continuity of psh functions by the stronger one given in Theorem 3.3.4 for bounded psh functions, we immediately obtain results similar to those in [7]. We state here results we will use later.

Lemma 3.3.8. *(similar to [7, Lemma 4.1]) Let Ω be an open subset in $\mathbb{C}^n$. Let T be a closed positive current on Ω and $u_j, u_{jk}, u'_j, u'_{jk}$ bounded psh functions on Ω for $k \in \mathbb{N}$ and $1 \leq j \leq m$, where $m \in \mathbb{N}$. Let $q \in \mathbb{N}^*$ and v_j, v'_j bounded psh functions on Ω for $1 \leq j \leq q$. Put $W := \cap_{j=1}^q \{v_j > v'_j\}$. Assume that*

$$R_k := dd^c u_{1k} \wedge \cdots \wedge dd^c u_{mk} \wedge T \to R := dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T$$

and

$$R'_k := dd^c u'_{1k} \wedge \cdots \wedge dd^c u'_{mk} \wedge T \to R' := dd^c u'_1 \wedge \cdots \wedge dd^c u'_m \wedge T$$

as $k \to \infty$ and

$$\mathbf{1}_W R_k = \mathbf{1}_W R'_k \tag{3.3.7}$$

for every k. Then we have $\mathbf{1}_W R = \mathbf{1}_W R'$.

Proof. The problem is clear if W is open, for example, when v_j is continuous for $1 \le j \le q$. In the general case, we will use the strong quasi-continuity to modify v_j. Since the problem is local, we can assume that Ω is bounded. Let $\epsilon > 0$ be a constant. By Theorem 3.3.4, we can find bounded continuous functions $\tilde{v}_j$ on Ω such that $\mathrm{cap}_T(\{\tilde{v}_j \neq v_j\}, U) < \epsilon$. Put $\tilde{W} := \cap_{j=1}^q \{\tilde{v}_j > v'_j\}$ which is an open set. The choice of $\tilde{v}_j$ combined with the definition of cap_T yields that

$$\|\mathbf{1}_W R - \mathbf{1}_{\tilde{W}} R\|_\Omega \le \epsilon, \quad \|\mathbf{1}_W R_k - \mathbf{1}_{\tilde{W}} R_k\|_\Omega \le \epsilon.$$

We also have similar estimates for R', R'_k. By this and (3.3.7), we get $\|\mathbf{1}_{\tilde{W}} R_k - \mathbf{1}_{\tilde{W}} R'_k\|_\Omega \le 2\epsilon$. This combined with the fact that $\tilde{W}$ is open yields that $\|\mathbf{1}_{\tilde{W}} R - \mathbf{1}_{\tilde{W}} R'\|_\Omega \le 2\epsilon$. Thus $\|\mathbf{1}_W R - \mathbf{1}_W R'\|_\Omega \le 4\epsilon$ for every ϵ. The desired equality follows. This finishes the proof. □

Theorem 3.3.9. *Let Ω be an open subset in $\mathbb{C}^n$. Let T be a closed positive current on Ω and u_j, u'_j bounded psh functions on Ω for $1 \le j \le m$, where $m \in \mathbb{N}$. Let v_j, v'_j be bounded psh functions on Ω for $1 \le j \le q$. Assume that $u_j = u'_j$ on $W := \cap_{j=1}^q \{v_j > v'_j\}$ for $1 \le j \le m$. Then we have*

$$\mathbf{1}_W dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = \mathbf{1}_W dd^c u'_1 \wedge \cdots \wedge dd^c u'_m \wedge T. \tag{3.3.8}$$

Proof. We give here a complete proof for the readers' convenience. Let $\epsilon > 0$ be a constant. Put $u''_j := \max\{u_j, u'_j - \epsilon\}$ and $\tilde{W} := \cap_{j=1}^m \{u_j > u'_j - \epsilon\}$. By hypothesis, $W \subset \tilde{W}$. We will prove that

$$\mathbf{1}_{\tilde{W}} dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = \mathbf{1}_{\tilde{W}} dd^c u''_1 \wedge \cdots \wedge dd^c u''_m \wedge T. \tag{3.3.9}$$

Since the problem is local, we can assume there is a sequence of uniformly bounded smooth psh functions $(u_{jk})_k$ decreasing to u_j for $1 \le j \le m$. Since $\tilde{W}_k := \{u_{jk} > u'_j - \epsilon\}$ is open, we have

$$\mathbf{1}_{\tilde{W}_k} dd^c u_{1k} \wedge \cdots \wedge dd^c u_{mk} \wedge T = \mathbf{1}_{\tilde{W}_k} dd^c \max\{u_{1k}, u'_j - \epsilon\} \wedge \cdots \wedge dd^c\{u_{mk}, u'_j - \epsilon\} \wedge T.$$

This together with the inclusion $\tilde{W} \subset \tilde{W}_k$ gives

$$\mathbf{1}_{\tilde{W}} dd^c u_{1k} \wedge \cdots \wedge dd^c u_{mk} \wedge T = \mathbf{1}_{\tilde{W}} dd^c \max\{u_{1k}, u'_j - \epsilon\} \wedge \cdots \wedge dd^c\{u_{mk}, u'_j - \epsilon\} \wedge T.$$

Using this and Lemma 3.3.8, we obtain (3.3.9) by considering $k \to \infty$. In particular, we get

$$\mathbf{1}_W dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = \mathbf{1}_W dd^c u''_1 \wedge \cdots \wedge dd^c u''_m \wedge T.$$

Letting $\epsilon \to 0$ and using Lemma 3.3.8 again gives

$$\mathbf{1}_W dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = \mathbf{1}_W dd^c \max\{u_1, u'_1\} \wedge \cdots \wedge dd^c \max\{u_m, u'_m\} \wedge T.$$

The last equality still holds if we replace u_j in the left-hand side by u'_j by using similar arguments. So the desired equality follows. The proof is finished. □

Remark 3.3.10. *Recall that a quasi-psh function u on Ω is, by definition, locally the sum of a psh function and a smooth one. We can check that results presented above have their analogues for quasi-psh functions.*

Here is an integration by parts formula which will be useful later.

Lemma 3.3.11. *(Integration by parts formula II) Let $\chi \in \mathscr{C}^3(\mathbb{R})$ and w_1, w_2 bounded psh functions on an open subset Ω of $\mathbb{C}^n$. Let Q be a closed positive current of bi-dimension $(1,1)$ on Ω. Then we have*

$$dd^c\chi(w_2) \wedge Q = \chi''(w_2) dw_2 \wedge d^c w_2 \wedge Q + \chi'(w_2) dd^c w_2 \wedge Q \tag{3.3.10}$$

and the operator $w_1 dd^c \chi(w_2) \wedge Q$ is continuous (in the usual weak topology of currents) under decreasing sequences of smooth psh functions converging to w_1, w_2. Consequently, if f is a smooth function with compact support in U, then the equality

$$\int_\Omega f w_1 dd^c \chi(w_2) \wedge Q = \int_\Omega \chi(w_2) dd^c(f w_1) \wedge Q \tag{3.3.11}$$

holds. Moreover, for f as above, we also have

$$\int_\Omega f\chi(w_2) dd^c w_1 \wedge Q = -\int_\Omega \chi(w_2) df \wedge d^c w_1 \wedge Q - \int_\Omega f \chi'(w_2) dw_2 \wedge d^c w_1 \wedge Q. \tag{3.3.12}$$

Proof. Clearly, all of three desired equalities follows from the integration by parts if w_1, w_2 are smooth. The arguments below essentially say that both sides of these equalities are continuous under sequences of smooth psh functions decreasing to w_1, w_2. This is slightly non-standard due to the presence of Q even when χ is convex.

First observe that (3.3.11) is a consequence of the second desired assertion because both sides of (3.3.11) are continuous under a sequence of smooth psh functions decreasing to w_2. We prove (3.3.10). The desired equality (3.3.10) clearly holds if w_2 is smooth. In general, let $(w_2^\epsilon)_\epsilon$ be a sequence of standard regularisations of w_2. Recall that $dd^c\chi(w_2) \wedge Q$ is defined to be $dd^c\big(\chi(w_2)Q\big)$ which is equal to the limit of $dd^c\big(\chi(w_2^\epsilon)Q\big)$ as $\epsilon \to 0$. By (3.3.10) for w_2^ϵ in place of w_2, we see that $dd^c\big(\chi(w_2^\epsilon)Q\big)$ is of uniformly bounded mass. As a result, $dd^c\chi(w_2) \wedge Q$ is of order 0. Thus $w_1 dd^c \chi(w_2) \wedge Q$ is well-defined. Put

$$I(w_1, w, w_2) := w_1 \chi''(w) dw_2 \wedge d^c w_2 \wedge Q + w_1 \chi'(w) dd^c w_2 \wedge Q.$$

Recall that $I(1, w_2^\epsilon, w_2^\epsilon) \to dd^c \chi(w_2) \wedge Q$. By Corollary 3.3.5, we have

$$I(w_1, w_2, w_2^\epsilon) \to I(w_1, w_2, w_2) \tag{3.3.13}$$

as $\epsilon \to 0$. On the other hand, since χ'' is in $\mathscr{C}^1$, we get

$$|\chi''(w_2^\epsilon) - \chi''(w_2)| \lesssim (w_2^\epsilon - w_2), \quad |\chi'(w_2^\epsilon) - \chi'(w_2)| \lesssim (w_2^\epsilon - w_2).$$

This combined with the convergence of Monge-Ampère operators under decreasing sequences tells us that

$$\big(I(w_1, w_2^\epsilon, w_2^\epsilon) - I(w_1, w_2, w_2^\epsilon)\big) \to 0 \tag{3.3.14}$$

as $\epsilon \to 0$. Combining (3.3.14) and (3.3.13) gives that $I(w_1, w_2^\epsilon, w_2^\epsilon) \to I(w_1, w_2, w_2)$ as $\epsilon \to 0$. Letting $w_1 \equiv 1$ in the last limit, we get (3.3.10). The second desired assertion also follows. We prove (3.3.12) similarly. The proof is finished. $\square$

3.4 Comparison principle

Let Ω be a bounded domain in $\mathbb{C}^n$. Let ω_0 be the standard Kähler form on $\mathbb{C}^n$. Let u, v be bounded psh functions on Ω.

Theorem 3.4.1. *(Comparison principle I) Let $1 \le k \le n$ be an integer. Let T be a closed positive $(n-k, n-k)$-current on Ω. Assume that $\liminf_{x\to\partial\Omega}(u(x) - v(x)) \ge 0$. Then we have*

$$\int_{\{u<v\}} (dd^c v)^k \wedge T \le \int_{\{u<v\}} (dd^c u)^k \wedge T.$$

Proof. By considering $u + \epsilon$ in place of u, and letting $\epsilon \to 0$, we can assume that $\liminf_{x\to\Omega}(u(x) - v(x)) \ge \epsilon > 0$ for some constant ϵ. Hence the set $A := \{u \le v + \delta/2\}$ is relatively compact in Ω. Hence there exists a relatively compact subset U in Ω such that $A \subset U$. Take a cut-off function χ with compact support in Ω such that $0 \le \chi_j \le 1$ and $\chi = 1$ on $\overline{A}$. By integration by parts and the fact that $dd^c\chi = 0$ on U and Theorem 3.3.9, we get

$$\begin{aligned}\int_\Omega \chi(dd^c \max\{u, v\})^k \wedge T &= \int_\Omega dd^c\chi \wedge (dd^c \max\{u, v\})^{k-1} \wedge T\\ &= \int_{\{u>v\}} dd^c\chi \wedge (dd^c \max\{u, v\})^{k-1} \wedge T\\ &= \int_{\{u>v\}} dd^c\chi \wedge (dd^c u)^{k-1} \wedge T.\end{aligned}$$

Letting $\{\chi = 1\}$ converge to Ω gives

$$\int_\Omega (dd^c \max\{u, v\})^k \wedge T = \int_\Omega (dd^c u)^k \wedge T.$$

By this and Theorem 3.3.9 again,

$$\begin{aligned}\int_{\{u<v\}} (dd^c v)^k \wedge T &= \int_{\{u<v\}} (dd^c \max\{u, v\})^k \wedge T\\ &= \int_\Omega (dd^c \max\{u, v\})^k \wedge T - \int_{\{u\ge v\}} (dd^c \max\{u, v\})^k \wedge T\\ &\le \int_\Omega (dd^c \max\{u, v\})^k \wedge T - \int_{\{u>v\}} (dd^c \max\{u, v\})^k \wedge T\\ &\le \int_\Omega (dd^c \max\{u, v\})^k \wedge T - \int_{\{u>v\}} (dd^c \max\{u, v\})^k \wedge T\\ &= \int_\Omega (dd^c u)^k \wedge T - \int_{\{u>v\}} (dd^c \max\{u, v\})^k \wedge T = \int_{\{u\le v\}} (dd^c v)^k \wedge T.\end{aligned}$$

We replace u by $u + \epsilon$ in the last inequality, and by letting $\epsilon \to 0$ we obtain the desired inequality. $\square$

Corollary 3.4.2. *(Domination principle) Let the notations and the hypothesis be as in Theorem 3.4.1. Then if $(dd^c u)^k \wedge T \le (dd^c v)^k \wedge T$ and $T \ge c\omega_0^{n-k}$ for some constant $c > 0$, then $u \ge v$.*

Proof. Let $\rho \geq 0$ be a smooth psh function on $\mathbb{C}^n$ such that $dd^c\rho \geq \omega_0$ on Ω. Suppose that $E := \{u < v\}$ is non-empty. Observe $E \subset \{u < v + \rho\}$. We claim that E is of strictly positive Lebesgue measure. Consider $x_0 \in E$. We have $u(x_0) < v(x_0)$. Let $\delta := v(x_0) - u(x_0) > 0$. By the upper semi-continuity of u,

$$u(x) \leq u(x_0) + \delta/3$$

for $x \in \mathbb{B}(x_0, \epsilon_\delta)$ for some constant $\epsilon_\delta > 0$. On the other hand, applying the submean inequality to v at x_0 yields that $v(x) \geq v(x_0) - \delta/3$ for x in a subset A of positive Lebesgue measure in $\mathbb{B}(x_0, \epsilon_\delta)$. Thus $A \subset E$. Since A is of positive Lebesgue measure, so is E.

By Theorem 3.4.1 applied to $v + \rho, u$, we get

$$\int_{\{u<v+\rho\}} (dd^c v + dd^c \rho)^k \wedge T \leq \int_{\{u<v+\rho\}} (dd^c u)^k \wedge T \leq \int_{\{u<v+\rho\}} (dd^c v)^k \wedge T.$$

On the other hand by the choice of ρ and the hypothesis the left-hand side of the last inequality is

$$\geq \int_{\{u<v+\rho\}} (dd^c v)^k \wedge T + \int_{\{u<v+\rho\}} \omega_0^n > \int_{\{u<v+\rho\}} (dd^c v)^k \wedge T.$$

This gives a contradiction. Hence E is empty. The desired assertion follows. □

Let ω be a Hermitian metric on Ω. Let $u_1, \ldots, u_m$ be bounded ω-psh function on Ω. By using a local smooth psh function ψ such that $dd^c\psi \geq \omega$ and writing $dd^c u_j + \omega = dd^c(u + \psi) + \omega - dd^c\psi$, we can define $R := (dd^c u_1 + \omega) \wedge \cdots \wedge (dd^c u_m + \omega)$ as in the case where u_j's are psh. In particular, R is continuous under decreasing sequences to u_j's (see [15]). By Theorem 2.5.10, we can find smooth ω-psh functions $(u_{jk})_k$ decreasing to u. Hence $(dd^c u_{1k} + \omega) \wedge \cdots \wedge (dd^c u_{mk} + \omega)$ converges weakly to R as $k \to \infty$. As a consequence, R is independent of the choice ψ and $R \geq 0$. We thus obtain a well-defined positive current R.

Corollary 3.4.3. *([15]) (Domination principle II) Let u, v be bounded ω-psh functions on Ω such that $\liminf_{x \to \partial\Omega}(u(x) - v(x)) \geq 0$, and $(dd^c u + \omega)^n \leq (dd^c v + \omega)^n$ on Ω. Then $u \geq v$ on Ω.*

Consider from now on a compact complex manifold X, and a Hermitian metric ω on X. For every Borel set $A \subset X$, define

$$\text{cap}_{BTK}(A) := \sup\Big\{ \int_A (dd^c\varphi + \omega)^n : \varphi \ \omega\text{-psh}, 0 \leq \varphi \leq 1 \text{ on X}\Big\}.$$

By Lemma 3.4.6 below, $\text{cap}_{BTK}(A)$ is always finite. It is also clear that if we use another Hermitian metric to define cap_{BTK}, then the resulted capacity is equivalent to that associated to ω.

Let $(U_j)_{1 \leq j \leq N}$ and $(U'_j)_{1 \leq j \leq N}$ be finite open coverings of X such that $\overline{U}_j$ is smooth and contained in some local chart of X biholomorphic to a polydisc for every $1 \leq j \leq N$, $U_j = \{\psi_j < 0\}$ for some psh function ψ_j defined on an open neighborhood of $\overline{U}_j$ with $\partial U_j = \{\psi_j = 0\}$ and $U'_j \Subset U_j$ for $1 \leq j \leq N$. In practice, it suffices to take U_j, U'_j to be balls and ψ_j are the differences of radius functions and constants.

Lemma 3.4.4. *([29, 15]) There exists strictly positive constants c_1, c_2 such that for every $A \subset X$ we have*

$$c_1 \sum_{j=1}^{N} \textit{cap}\big(A \cap U'_j, U_j\big) \le \textit{cap}_{BTK}(A) \le c_2 \sum_{j=1}^{N} \textit{cap}\big(A \cap U'_j, U_j\big).$$

Proof. Put $A'_j := A \cap U'_j$ which is a relatively compact subset of U_j. We have $\cup_j A'_j = A$. The second desired inequality is obvious from the definitions of capacities. We prove now the first desired inequality.

Fix an index $1 \le j \le N$. By our choice of U_j, for every psh function $0 \le u \le 1$ on U_j, we can find another psh function $-1 \le \tilde{u} \le 0$ on U_j satisfying $\tilde{u} = u - 1$ on some open neighborhood of $\overline{U}'_j$ and $\tilde{u} = 0$ on ∂U_j. Such a $\tilde{u}$ can be chosen to be $\max\{u-1, A\psi_j\}$ for some constant A big enough. Clearly,

$$\int_{A'_j} (dd^c u)^k = \int_{A'_j} (dd^c \tilde{u})^k.$$

Since $-1 \le \tilde{u} \le 0$ and $\tilde{u} = 0$ on ∂U_j, there is a quasi-psh function $\tilde{u}_1$ on X such that $dd^c \tilde{u}_1 + C\omega \ge 0$ for some constant C independent of $\tilde{u}$ and $\tilde{u}_1 = \tilde{u}$ on some open neighborhood of $\overline{U}'_j$ and $|\tilde{u}_1|$ is bounded by a constant independent of $\tilde{u}$. We deduce that

$$\int_{A'_j} (dd^c u)^k = \int_{A'_j} (dd^c \tilde{u}_1)^k \le \int_{A'_j} (dd^c \tilde{u}_1 + C\omega)^k \le C' \mathrm{cap}_{BTK}(A'_j),$$

for some constant C' independent of u. Consequently, $\mathrm{cap}_{BT}(A'_j, U_j) \le C^k \mathrm{cap}_{BTK}(A'_j)$. Summing over $1 \le j \le N$ in the last inequality gives the first desired inequality. This finishes the proof. $\square$

Proposition 3.4.5. *([30, Theorem 0.2]) Let φ, ψ be bounded ω-p.s.h functions on X. Let $0 < \epsilon < 1$ and $m_\epsilon := \inf_X(\varphi - (1-\epsilon)\psi)$. Then there exists a big constant $B > 0$ depending only on ω, n such that for every constant $0 < s < \epsilon^3/(16B)$ we have*

$$\int_{\{\varphi < (1-\epsilon)\psi + m_\epsilon + s\}} \big((1-\epsilon) dd^c\psi + \omega\big)^n \le (1 + C\epsilon^{-k} s) \int_{\{\varphi < (1-\epsilon)\psi + m_\epsilon + s\}} (dd^c\varphi + \omega)^n,$$

where C is a constant depending only on n, B.

A consequence of the last result is the following.

Lemma 3.4.6. *([15, 30]) Let M be a positive number. Then there exists a constant $c_M > 0$ such that for every ω-psh function φ with $|\varphi| \le M$, we have*

$$0 < \int_X (dd^c\varphi + \omega)^n \le c_M. \tag{3.4.1}$$

However, we don't know whether

$$\inf_{\{\varphi : |\varphi| \le M\}} \int_X (dd^c\varphi + \omega)^n > 0?$$

Proof. The second desired inequality is proved in [15] by using integration by parts. The first one is observed in [30]. To see it, it is enough to notice that by choosing $\epsilon := 1/2$ and $s > 0$ small enough in Proposition 3.4.5, for every ω-psh ψ with $0 \le \psi \le s$ and φ as in the hypothesis, we have

$$\int_{\{\varphi<\inf_X \varphi+s\}} \big(dd^c\psi+\omega\big)^n \le c\int_{\{\varphi<(1-\epsilon)\psi+m_\epsilon+2s\}} \big((1-\epsilon)dd^c\psi+\omega\big)^n \le c_s\int_X (dd^c\varphi+\omega)^n$$

because

$$\{\varphi<\inf_X\varphi+s\}\subset\{\varphi<(1-\epsilon)\psi+m_\epsilon+2s\},$$

where c, c_s are constants independent of ψ and c_s might depend on s. It follows that there is a strictly positive constant c'_s satisfying

$$\int_X (dd^c\varphi+\omega)^n \ge c'_s \mathrm{cap}_{BTK}\big(\{\varphi<\inf_X\varphi+s\}\big) \tag{3.4.2}$$

which is strictly positive because it is the capacity of a non-empty open set. The proof is finished. □

3.5 Locally pluripolar sets

We assume the following important result. Let ω be a Hermitian form on $\mathbb{C}^n$. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$.

Theorem 3.5.1. *(Dirichlet's problem) Let φ be a continuous function on $\partial\mathbb{B}$ and let $f \in Ł^p(\omega^n)$. Then there exists a unique $u \in \mathrm{PSH}(\mathbb{B})\cap\mathscr{C}^0(\overline{\mathbb{B}})$ such that $(dd^cu+\omega)^n = f\omega^n$ and $u=\varphi$ on $\partial\mathbb{B}$.*

This is a special case of [30, Theorem 4.2] (see references in this paper for historical works).

Corollary 3.5.2. *Let φ, f be as in Theorem 3.5.1. Then there exists a unique $u \in \mathrm{PSH}(\mathbb{B})\cap\mathscr{C}^0(\overline{\mathbb{B}})$ such that $(dd^cu)^n = f\omega^n$ and $u=\varphi$ on $\partial\mathbb{B}$.*

Proof. Let $\omega := dd^c\|z\|^2$ which is Kähler. By writing $dd^cu = dd^c(u-\|z\|+1)+\omega$ and noticing that $u-\|z\|+1$ is equal to u on $\partial\mathbb{B}$, we see that the desired assertion is a direct consequence of Theorem 3.5.1. □

Let X be a compact complex manifold of dimension n. Let ω be a Hermitian metric on X.

Proposition 3.5.3. *Let φ be an ω-psh function on X. Let $\mathbb{B}$ be a local chart in X biholomorphic to a unit ball in $\mathbb{C}^n$. Then there exists a bounded ω-psh function u on X such that*

$$(dd^cu+\omega)^n=0$$

on $\mathbb{B}$ and $u=\varphi$ on $X\backslash\mathbb{B}$, and $u\ge\varphi$ on X.

Using Corollary 3.5.2 as the proof below, we see that this result also holds if we consider psh functions in place of ω-psh functions on X and $dd^c u$ in place of $dd^c u + \omega$.

Proof. By Corollary 2.5.11, there exist a sequence of smooth ω-psh functions φ_j on X decreasing to φ. Applying Theorem 3.5.1 to φ_j and $\mathbb{B}$ gives an ω-psh function u_j on $\mathbb{B}$ such that u_j is continuous on $\overline{\mathbb{B}}$ and $(dd^c u_j + \omega)^n = 0$ and $u_j = \varphi_j$ on $\partial\mathbb{B}$. Let $u'_j := \max\{u_j, \varphi_j\}$ on $\mathbb{B}$ and $u'_j := \varphi_j$ on $X \backslash \mathbb{B}$. By Lemma 2.5.6, u'_j is an ω-psh function on X.

Since $u_j \geq u_{j+1}$ on $\partial\mathbb{B}$ and $(dd^c u_j + \omega)^n \leq (dd^c u_{j+1} + \omega)^n$ on $\mathbb{B}$, using Domination principle (Corollary 3.4.3) implies $u_j \geq u_{j+1}$ on $\mathbb{B}$. It follows that $(u'_j)_j$ is a decreasing sequence, and $u'_j \geq \varphi_j \geq \varphi$ for every j. Hence $u := \lim_{j\to\infty} u'_j$ exists and is an ω-psh function on X. By Theorem 3.2.5, $(dd^c u + \omega)^n = \lim_{j\to\infty}(dd^c u_j + \omega)^n = 0$. It is also clear by construction that $u = \varphi$ outside $\mathbb{B}$. This finishes the proof. □

A subset A of X is *locally pluripolar* if for every point x in A there is an open neighborhood U_x of x in X and a psh function φ on U_x for which $A \cap U_x \subset \{\varphi = -\infty\}$. A subset A of X is *pluripolar* if $A \subset \{\varphi = -\infty\}$ for some quasi-psh function φ in X.

Since we already know that if A is locally pluripolar in U_j, then cap$(A, U_j) = 0$, we get cap$_{BTK}(A) = 0$ if A is locally pluripolar in X. Let (u_j) be a family of psh functions on an open subset U of $\mathbb{C}^k$ locally bounded from above. Define $u := \sup_j u_j$ and $u^* := \sup_j^* u_j$ the upper semi-continuous regularisation of u. The set $\{u < u^*\}$ is of zero capacity cap by Lemma 3.3.7. For $A \subset X$,

$$\text{cap}_{ADS}(A) := \inf\{\exp(\sup_A \varphi) : \varphi \ \omega\text{-psh on } X, \ \sup_X \varphi = 0\}.$$

Lemma 3.5.4. *cap$_{ADS}(A) = 0$ if and only if A is pluripolar on X.*

Proof. If $A \subset \{\varphi = -\infty\}$ for some quasi-psh φ, it is clear that cap$_{ADS}(A) = 0$. Consider now

$$\text{cap}_{ADS}(A) = 0. \tag{3.5.1}$$

Recall that there exists a constant c such that for every ω-psh function φ with the normalization condition $\sup_X \varphi = 0$, we have

$$\|\varphi\|_{L^1(X)} \leq c. \tag{3.5.2}$$

We refer to [24, 16, 15] for a proof. Using (3.5.1), there exists a sequence of ω-psh functions (φ_n) with $\sup_X \varphi_n = 0$ such that $\sup_A \varphi_n \leq -n^3$. Put

$$\varphi := \sum_{n=1}^{\infty} \frac{\varphi_n}{n^2}$$

which is a well-defined quasi-psh function because of (3.5.2). On the other hand,

$$\sup_A \varphi \leq \sum_{n=1}^{\infty} \frac{-n^3}{n^2} = -\infty.$$

It means that $A \subset \{\varphi = -\infty\}$. This finishes the proof. □

Let $(\varphi_j)_{j\in J}$ be a family of ω-psh functions on X uniformly bounded from above. Define

$$\varphi_J := \sup_{j\in J} \varphi_j.$$

Lemma 3.5.5. *φ_J^* is an ω-psh function.*

Proof. We will use Proposition 2.5.5 to check the desired claim. It is enough to work locally as in the situation of Proposition 2.5.5. Note that we already have that φ_J^* is upper semi-continuous by its definition. Let $v \in \mathbb{C}^k\backslash\{0\}$ and ϵ a small positive constant. Applying Proposition 2.5.5 to φ_j gives

$$\varphi_j(x) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi_j(x+\epsilon e^{i\theta}v)d\theta + \int_0^{\epsilon} \frac{dt}{t}\int_{\{|s|\le t\}} \omega_{x,v},$$

where $\omega_{x,v}$ is the restriction of ω to the line $L_{x,v} := \{x+tv : t \in \mathbb{C}\}$. Taking the supremum over $j \in J$ in the last inequality implies

$$\varphi_J(x) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi_J(x+\epsilon e^{i\theta}v)d\theta + \int_0^{\epsilon} \frac{dt}{t}\int_{\{|s|\le t\}} \omega_{x,v}. \tag{3.5.3}$$

Let $x_\infty \in X$. Consider a sequence $(x_n) \subset X$ converging to x_∞ such that

$$\varphi_J^*(x_\infty) = \lim_{n\to\infty} \varphi_J(x_n).$$

Applying (3.5.3) to $x = x_n$ and letting $n \to \infty$, we obtain

$$\varphi_J^*(x_\infty) \le \limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \varphi_J(x_n+\epsilon e^{i\theta}v)d\theta + \limsup_{n\to\infty} \int_0^{\epsilon} \frac{dt}{t}\int_{\{|s|\le t\}} \omega_{x_n,v}.$$

The second term in the right-hand side of the last inequality is equal to

$$\int_0^{\epsilon} \frac{dt}{t}\int_{\{|s|\le t\}} \omega_{x_\infty,v}$$

because ω is smooth. This combined with the fact that

$$\limsup_{n\to\infty} \varphi_J(x_n+\epsilon e^{i\theta}v) \le \varphi_J^*(x_\infty+\epsilon e^{i\theta}v)$$

and Fatou's lemma yields

$$\varphi_J^*(x_\infty) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi_J^*(x_\infty+\epsilon e^{i\theta}v)d\theta + \int_0^{\epsilon} \frac{dt}{t}\int_{\{|s|\le t\}} \omega_{x_\infty,v}.$$

The desired assertion now follows by Proposition 2.5.5. This finishes the proof. □

As in the local setting, $\{\varphi_J^* > \varphi_J\}$ is of zero capacity (see Lemma 3.3.7). We will present below an important case of $(\varphi_j)_{j\in J}$ and its associated extremal function φ_J^*.

Let A be a *non-pluripolar* subset of X. As in the local setting or in the Kähler case, we introduce the following extremal ω-psh function:

$$T_A := \sup\big\{\varphi \ \omega\text{-psh} : \varphi \leq 0 \text{ on } A\big\}.$$

It is clear that $T_A \geq 0$. Let T_A^* be the upper semi-continuous regularisation of T_A. We can check that

$$\text{cap}_{ADS}(A) = \exp(-\sup_X T_A). \tag{3.5.4}$$

Thus T_A is bounded from above because A is non-pluripolar. We deduce that T_A^* is a bounded ω-psh function and $Q_A := \{T_A^* > T_A\}$ is of zero capacity. Since $T_A = 0$ on A, we get $T_A^* = 0$ on $A \backslash Q_A$.

Proposition 3.5.6. *Let A be a nonpluripolar compact subset of X. We have*

$$(dd^c T_A^* + \omega)^n = 0 \tag{3.5.5}$$

on $X \backslash A$.

Proof. By Choquet's lemma (Lemma 1.2.5), there exists an increasing sequence of ω-psh function φ_j for which $T_A^* = (\lim_{j\to\infty} \varphi_j)^*$. Let $\mathbb{B}$ be a ball in X such that $\mathbb{B} \cap A = \varnothing$. By Proposition 3.5.3, we obtain an ω-psh function φ_j' such that $(dd^c\varphi_j' + \omega)^n = 0$ on $\mathbb{B}$ and $\varphi_j' \geq \varphi_j$ on X and $\varphi_j' = \varphi_j$ on $X \backslash \mathbb{B}$. Observe that φ_j' increases to T_A by the construction and the definition of T_A. It follows that $(dd^c T_A^* + \omega)^n = 0$ on $\mathbb{B}$ by Theorem 3.7.4. The desired assertion hence follows. □

Proposition 3.5.7. *Let A be a nonpluripolar compact subset of X. Then there exist strictly positive constants c_1, λ_1 independent of A such that*

$$\exp\big(-\lambda_1 \mathit{cap}_{BTK}^{-1}(A)\big) \leq \mathit{cap}_{ADS}(A) \leq c_1 \exp\big(-M_A^{1/n} \mathit{cap}_{BTK}^{-1/n}(A)\big). \tag{3.5.6}$$

where $M_A := \int_X (dd^c T_A^* + \omega)^n > 0$.

Note that $M_A > 0$ because of Lemma 3.4.6.

Proof. Since A non-pluripolar, T_A^* is a bounded ω-psh function. By (3.5.4), the desired inequalities are equivalent to the following:

$$\lambda_1 \text{cap}_{BTK}^{-1}(A) \geq \sup_X T_A \geq c_1' + M_A^{1/n} \text{cap}_{BTK}^{-1/n}(A), \tag{3.5.7}$$

where $c_1' := -\log c_1$.

We prove now the first inequality of (3.5.7). We can assume $\sup_X T_A > 0$ because otherwise the desired inequality is trivial for any $\lambda_1 \geq 0$. Put $\varphi_A := T_A^* - \sup_X T_A^*$ which is an ω-psh function with $\sup_X \varphi_A = 0$. It follows that

$$\|\varphi_A\|_{L^p} \lesssim 1 \tag{3.5.8}$$

for every $p \geq 1$ by Proposition 2.5.7.

Let φ be an ω-psh function such that $0 \leq \varphi \leq 1$. Since $(\sup_X T_A)^{-1}\varphi_A = -1$ on $A \backslash Q_A$, and cap$_{BTK}(Q_A) = 0$, we obtain

$$\int_A (dd^c\varphi + \omega)^n \leq (\sup_X T_A)^{-1} \int_X [-\varphi_A](dd^c\varphi + \omega)^n \lesssim (\sup_X T_A)^{-1} \|\varphi_A\|_{L^1} \tag{3.5.9}$$

for every φ with $0 \leq \varphi \leq 1$ by the Chern-Levine-Nirenberg inequality (Corollary 3.2.7). Combining (3.5.9) with (3.5.8) gives the first inequality of (3.5.7). It remains to prove the second one.

Recall that $-1 \leq (\sup_X T_A)^{-1}\varphi_A \leq 0$ and $(\sup_X T_A)^{-1}\varphi_A$ is an $(\sup_X T_A)^{-1}\omega$-psh function. Hence $(\sup_X T_A)^{-1}\varphi_A$ is ω-psh if $(\sup_X T_A)^{-1} \leq 1$. Consider the case where $(\sup_X T_A)^{-1} \leq 1$. By definition of cap$_{BTK}$, we get

$$\text{cap}_{BTK}(A) \geq (\sup_X T_A)^{-n} \int_A (dd^c\varphi_A + \omega)^n = (\sup_X T_A)^{-n} \int_A (dd^c T_A^* + \omega)^n \tag{3.5.10}$$

By Proposition 3.5.6, we have

$$\int_A (dd^c T_A^* + \omega)^n = \int_X (dd^c T_A^* + \omega)^n.$$

Hence the second inequality of (3.5.7) follows if $(\sup_X T_A)^{-1} \leq 1$. When $(\sup_X T_A)^{-1} \geq 1$, then $T_A^* - 1 \leq 0$ on X and ≤ -1 on $A \backslash Q_A$. We infer that

$$\text{cap}_{BTK}(A) = \text{cap}_{BTK}(A \backslash Q_A) \geq \int_A (dd^c T_A^* + \omega)^n > 0$$

which combined with the fact that $\sup_X T_A \geq 0$ yields the second inequality of (3.5.7) in this case. The proof is finished. □

Theorem 3.5.8. *A subset on a compact complex manifold is pluripolar if and only if it is locally pluripolar.*

One can apply this result to subsets of $\mathbb{C}^n$ because $\mathbb{C}^n$ is an open subset of $\mathbb{P}^n$ a compact complex manifold. By the above theorem, there exist abundantly non-continuous quasi-psh functions on a compact complex manifold. This is a fact which probably cannot be seen directly because unlike projective manifolds, a general compact complex manifold might have very few hypersurfaces.

Proof. First observe that a countable union of pluripolar sets is again a pluripolar set. Indeed, let $(V_k)_{k \in \mathbb{N}}$ be a countable family of pluripolar sets on X. Hence we have $V_k \subset \{\varphi_k = -\infty\}$ for some ω-p.s.h function φ_k with $\sup_X \varphi_k = 0$. Define

$$\varphi := \sum_{n=1}^{\infty} \varphi_k / k^2$$

which is of bounded L^1-norm because $\|\varphi_k\|_{L^1}$ is uniformly bounded in k. Hence φ is a quasi-psh function and $V_k \subset \{\varphi = -\infty\}$ for every k.

Let V be a locally pluripolar set. We need to prove V is pluripolar. If V is compact, the desired claim is a direct application of (3.5.6). For the general case, we need some more arguments.

By Lindelöf's property, we can cover V by at most countably many sets of form $\{\varphi_j = -\infty\}$ for some p.s.h functions φ_j on some open subset U_j of X. Hence in order to prove the desired assertion, we only need to consider $V = \{\varphi = -\infty\}$ for some psh function φ in an open subset U of X which is biholomorphic to a ball in $\mathbb{C}^k$.

Let U_1 be a relatively compact open subset of U. Suppose that $V \cap U_1$ is not pluripolar. Hence $T^*_{V\cap U_1}$ is a bounded ω-p.s.h function. Consider a decreasing sequence of smooth psh functions $(\varphi_k)_{k\in\mathbb{N}}$ defined on an open neighborhood of $\overline{U}_1$ converging pointwise to φ. For every positive integer N, put

$$V_{k,N} := \{\varphi_k \le -N\} \cap \overline{U}_1$$

which is a compact subset increasing in k. Hence $(T^*_{V_{k,N}})_{k\in\mathbb{N}}$ is a decreasing sequence of ω-psh functions which converges pointwise to an ω-psh function T_N.

Since $\{\varphi_k < -N\}$ is open, $T^*_{V_{k,N}} = T_{V_{k,N}} = 0$ on $\{\varphi_k < -N\} \cap U_1$. Thus $T_N = 0$ on $\{\varphi < -N\} \cap U_1$ which contains $V \cap U_1$. We infer that

$$0 \le T_N \le T^*_{V\cap U_1}$$

for every N. This combined with the fact that $(T_N)_{N\in\mathbb{N}}$ is increasing gives

$$0 \le T_\infty := (\lim_{N\to\infty} T_N)^* \le T^*_{V\cap U_1} \tag{3.5.11}$$

and T_∞ is an ω-psh function. Applying (3.5.6) to $A := V_{k,N}$ we get

$$\sup_X T^*_{V_{k,N}} \ge c_1' + M_{k,N}^{1/n} \text{cap}_{BTK}(V_{k,N})^{-1/n} \tag{3.5.12}$$

where $M_{k,N} := \int_X (dd^c T^*_{V_{k,N}} + \omega)^n$. By the convergence of Monge-Ampère operators, we have

$$\lim_{k\to\infty} M_{k,N} = \int_X (dd^c T_N + \omega)^n =: M_N, \quad \lim_{N\to\infty} M_N = \int_X (dd^c T_\infty + \omega)^n =: M_\infty \tag{3.5.13}$$

Note that $M_\infty > 0$ by Lemma 3.4.6. On the other hand, since φ_k decreases pointwise to φ as $k \to \infty$, there exists a constant c independent of k, N such that

$$\text{cap}_{BTK}(V_{k,N}) \le cN^{-1}$$

by the Chern-Levine-Nirenberg inequality (Corollary 3.2.7). This together with (3.5.13) and (3.5.12) implies

$$\sup_X T_N \ge c_1' + cM_N^{1/n} N^{1/n}, \tag{3.5.14}$$

for some constant $c > 0$ independent of N. Letting $N \to \infty$ in the last inequality and using (3.5.13), (3.5.11), we get

$$\sup_X T^*_{V\cap U_1} \ge \sup_X T_\infty = \infty.$$

This is a contradiction. Hence $V \cap U_1$ is pluripolar for every relatively compact open subset U_1 of U. It follows that V is pluripolar. This finishes the proof. □

3.6 Regularity of capacity

We recall the following important theorem due to Choquet. Let X be a locally compact separable metric space. Let c be a nonnegative real function defined the set of all subsets of X. The function c is said to be *a generalized capacity* if the following properties are satisfied:

(i) for $E_1 \subset E_2$, $c(E_1) \le c(E_2)$,
(ii) if E_j increases to E, then $c(E_j)$ increases to $c(E)$,
(iii) if compact K_j decreases to K, then $c(K_j)$ decreases to $c(K)$.
For every subset E in X we put

$$c^*(E) = \inf\{c(U) : U \text{ open subset in } X \text{ containing } E\}$$

and

$$c_*(E) = \sup\{c(K) : K \text{ compact subset in } E\}.$$

Here is a well-known result of Choquet.

Theorem 3.6.1. *([9, 12]) Let c be a generalized capacity on X. Then for every Borel subset E, we have*

$$c(E) = c_*(E). \tag{3.6.1}$$

This result is true for any locally compact topological space X. But in the scope of the lecture, it is enough for us to consider locally compact separable metric spaces.

Proof. We give a sketch of the proof for readers' convenience. Recall that a F_σ subset in X is a countable union of closed subsets in X, and $F_{\sigma\delta}$ is a countable intersection of F_σ subsets in Ω. Using Property (ii) of c, we get $c(E) = c_*(E)$ if E is a closed subset in X because F is a countable union of compact subsets in X. Similarly (3.6.1) holds true if E is F_σ and $F_{\sigma\delta}$ set. The desired assertion follows from it and the following claim asserting that every Borel set is the image of a $F_{\sigma\delta}$ set under a continuous map.

Claim. For every Borel subset E in X, there exist a compact metric space Y and a continuous map $f : Y \to X$ and a $F_{\sigma\delta}$ subset A in Y such that $f(A) = E$.

We prove the Claim. We avoid using Tychonoff's theorem in this proof. Let $\mathscr{A}$ be the set of subsets E in X satisfying the claim. Observe that open subsets of X belong to $\mathscr{A}$. In order to obtain the desired assertion, it suffices to check that $\mathscr{A}$ is an σ-algebra. Let $(E_j)_{j\in\mathbb{N}}$ be in $\mathscr{A}$. Put $E = \cap_j E_j$. Let $f_j : Y_j \to X$ be a continuous map such that $f(A_j) = E_j$ for some $F_{\sigma\delta}$ subset A_j in Y_j. Consider the space $Y := \prod_j Y_j$ with the usual topology which is the coarsest topology making every projection $p_j : Y \to Y_j$ to its component Y_j to be continuous. Recall that open subsets in Y is given by $\prod_{j\in J} U_j \times \prod_{j\notin J} Y_j$, where U_j are an open subset in Y_j. This topology is metrizable by the following metric:

$$d(x,y) := \sum_{j=1}^{\infty} \frac{d_j(x_j,y_j)}{2^j(1+d_j(x_j,y_j))},$$

where d_j denotes the metric on Y_j, and $x = (x_j)_{j\in\mathbb{N}}$ and $y = (y_j)_{j\in\mathbb{N}}$ are elements in Y. With this metric, one can check that Y is compact because Y_j is so. Let

$$A_j = \cap_{s=1}^{\infty} \cup_{k=1}^{\infty} K_{skj},$$

where K_{skj} is compact in Y_j (notice Y_j is compact) and K_{skj} is increasing in k as s, j fixed. Hence

$$\prod_j A_j = \cap_s \cup_{k_1,\ldots,k_s} K_{sk_11} \times \cdots \times K_{sk_ss} \times Y_{r+1} \times \cdots$$

which is a $F_{\sigma\delta}$ in Y. Put

$$A := \{y \in \prod_j A_j : f_j \circ p_j(y) = f_{j'} \circ p_{j'}(y), \forall j, j'\}$$

which is closed subset of $\prod_j A_j$. Let $f : A \to X$ be given by $f(y) := f_1 \circ p_1(y)$. Using the metric structure, we extend f naturally to a continuous map from $\overline{A}$ to X. Note that $\overline{A}$ is a compact metric space, and $f(A) = \cap_j f_j(A_j) = \cap_j E_j$. Hence $E \in \mathscr{A}$.

Consider now $E := \cup_j E_j$. Let $\{y^0\}$ be a point-set. Let $Y_j' := Y_j \cup \{y^0\}$ which is again a compact metric space, and $Y' := \prod_j Y_j'$. Let $p_j : Y' \to Y_j'$ be the natural projection from Y_j' to Y_j'. Set

$$A := \cup_j \{y^0\} \times \cdots \times \{y^0\} \times A_j \times \{y^0\} \times \cdots$$

which is formally the disjoint union of A_j. Observe A is $F_{\sigma\delta}$ in Y'. Consider $f : A \to X$ given by

$$f(y^0, \ldots, y^0, y_j, y^0, \ldots) := f_j(y_j).$$

We see that $f(A) = \cup_j E_j$. Extends f to $\overline{A}$ as before. We thus proved Claim. This finishes the proof. □

Recall that a domain Ω is said to be *hyperconvex* if there exists a continuous psh function $h < 0$ on Ω such that for every constant $c < 0$, the set $\{h < c\}$ is relatively compact in Ω. *From now on we assume that Ω is bounded and hyperconvex.* We follow partly the presentation in [9, 29]. Let E be a subset in Ω. The *relative extreme function* of E in Ω is defined by

$$u_{E,\Omega} := \sup\{u \text{ psh in } \Omega : u \le 0, u \le -1 \text{ on } E\}.$$

Note that $u_{E,\Omega} = -1$ on E and $-1 \le u_{E,\Omega} \le 0$. By Choquet's lemma, $u^*_{E,\Omega}$ is the limit of an increasing sequence of negative psh functions on Ω. Hence $u^*_{E,\Omega}$ is bounded psh on Ω.

Lemma 3.6.2. (i) *If $E_1 \subset E_2$, then $u_{E_2,\Omega} \le u_{E_1,\Omega}$,*

(ii) *If $E \subset \Omega_1 \subset \Omega_2$, then $u_{E,\Omega_2} \le u_{E,\Omega_1}$*

(iii) *If compact K_j decreases to K, then $(\lim_j u^*_{K_j,\Omega})^* = u^*_{K,\Omega}$.*

Proof. The first two desired claims are trivial. It remains to prove the third desired claim. Put

$$u := (\lim_j u_{K_j,\Omega})^*.$$

Since $u_{K_j,\Omega} \leq u_{K,\Omega}$, we get $u \leq u_{K,\Omega}$. It remains to prove the converse inequality. By Choquet's lemma, there is a sequence of psh functions $(u_k)_k$ increasing almost everywhere to $u^*_{E,\Omega}$ such that $u_k \leq 0$ on Ω and $u_k \leq -1$ on K. Let $\epsilon > 0$ be a constant. Since K_j compact decreases to K and $K \subset \{u_k < 1-\epsilon\}$, we see that for k fixed, and j big enough, $K_j \subset \{u_k < 1-\epsilon\}$. It follows that $(1-\epsilon)^{-1}u_k \leq u^*_{K_j,\Omega}$. Letting $j \to \infty$ gives $u \geq (1-\epsilon)^{-1}u_k$ for every k. Letting $k \to \infty$ and then $\epsilon \to 0$, we obtain that $u \geq u^*_{E,\Omega}$ almost everywhere. Hence $u \geq u^*_{E,\Omega}$. This finishes the proof. □

Proposition 3.6.3. *Let E be a relatively compact subset in a bounded hyperconvex domain Ω. We have*

$$\mathit{cap}^*(E,\Omega) = \int_\Omega (dd^c u^*_{E,\Omega})^n.$$

If compact K_j decreases to K, then

$$\lim_{j\to\infty} \mathit{cap}(K_j,\Omega) = \mathit{cap}(K,\Omega) = \mathit{cap}^*(K,\Omega).$$

Proof. Since $-1 \leq u^*_{E,\Omega} \leq 0$, we get

$$\mathrm{cap}(\overline{E},\Omega) \geq \int_{\overline{E}} (dd^c u^*_{E,\Omega})^n = \int_\Omega (dd^c u^*_{E,\Omega})^n.$$

We prove the converse inequality. Let h be a psh function on Ω such that $h < 0$ on Ω and $\{h < c\} \Subset \Omega$ for every constant $c < 0$. Since E is relatively compact, by rescaling, we can assume $h < -1$ on E. Thus $h \leq u^*_{E,\Omega}$. We infer that $u^*_{E,\Omega}$ is an exhaustion psh function for Ω, *i.e,* the open set $\{u^*_{E,\Omega} < c\}$ is relatively compact in Ω for every constant $c < 0$. In particular, $\liminf_{x\to\partial\Omega} u^*_{E,\Omega} = 0$ (hence $\limsup_{x\to\partial\Omega} u^*_{E,\Omega} = 0$ because $u^*_{E,\Omega} \leq 0$). Let $-1 \leq u \leq 0$ be a psh function on Ω. Let $A := \{u^*_{E,\Omega} > u_{E,\Omega}\}$. We already know that

$$\mathrm{cap}(A,\Omega) = 0 \tag{3.6.2}$$

by Lemma 3.3.7. Let $0 < \epsilon < 1$ be a constant. Observe that $E \subset \{u_{E,\Omega} < (1-\epsilon)u\}$. Using this and the comparison principle (Theorem 3.4.1) gives

$$(1-\epsilon)^n \int_{\{u^*_{E,\Omega} < (1-\epsilon)u\}} (dd^c u)^n \leq \int_{\{u^*_{E,\Omega} < u\}} (dd^c u^*_{E,\Omega})^n \leq \int_\Omega (dd^c u^*_{E,\Omega})^n.$$

Combining this with (3.6.2) yields

$$(1-\epsilon)^n \int_E (dd^c u)^n \leq \int_{\overline{E}} (dd^c u^*_{E,\Omega})^n.$$

Taking the supremum over every u and letting $\epsilon \to 0$ give

$$\mathrm{cap}(E,\Omega) \leq \int_\Omega (dd^c u^*_{E,\Omega})^n.$$

Hence we obtain

$$\mathrm{cap}(E,\Omega) = \int_E (dd^c u^*_{E,\Omega})^n \tag{3.6.3}$$

if E is compact. Consider now a relatively compact open subset U in Ω. Let K_j be a sequence of compact subsets increasing to U. Observe $u_{K_j,\Omega}$ decreases to $u_{U,\Omega}$. By a similar equality as (3.6.2), $u^*_{K_j,\Omega}$ decreases to a psh function $u_\infty \geq u^*_{U,\Omega}$. Moreover for every open $B \Subset U$, we have $u^*_{K_j,\Omega} = -1$ on B if j big enough. Hence $u_\infty = -1$ on U. It follows that $u_\infty = u^*_{U,\Omega}$, and $u^*_{K_j,\Omega}$ decreases to $u^*_{U,\Omega}$. Using this and applying (3.6.3) to K_j and taking $j \to \infty$ gives

$$\text{cap}(U,\Omega) = \limsup_{j\to\infty} \int_\Omega (dd^c u^*_{K_j,\Omega})^n = \int_\Omega (dd^c u^*_{U,\Omega})^n$$

by Theorem 3.2.5 (notice here that $(dd^c u^*_{K_j,\Omega})^n$ is supported on a fixed compact subset in Ω because $U \Subset \Omega$).

Let $E \Subset \Omega$ be an arbitrary set, and $U \Subset \Omega$ open set containing E. Observe that $0 > u^*_{E,\Omega} \geq u^*_{U,\Omega}$ on Ω. Hence using comparison principle, we get

$$\int_\Omega (dd^c u^*_{E,\Omega})^n \leq \int_\Omega (dd^c u^*_{U,\Omega})^n = \text{cap}(U,\Omega)$$

Taking the infimum over every open U containing E gives

$$\int_\Omega (dd^c u^*_{E,\Omega})^n \leq \text{cap}^*(E,\Omega).$$

Now let $(u_j)_j$ be a sequence of psh functions increasing to $u_{E,\Omega}$. Hence E is contained in the open subset $G_j := \{u_j < -\lambda_j\}$, where the constant λ_j increases to 1. We can assume that u_j is an exhaustion psh of Ω (by replacing u_j by $\max\{u_j, h\}$). We infer that $G_j \Subset \Omega$ and $u^*_{G_j,\Omega}$ increases to $u^*_{E,\Omega}$ almost everywhere. Theorem 3.3.6 implies that

$$\int_\Omega (dd^c u^*_{E,\Omega})^n = \lim_{j\to\infty} \int_\Omega (dd^c u^*_{G_j,\Omega})^n = \lim_{j\to\infty} \text{cap}(G_j,\Omega) \geq \text{cap}^*(E,\Omega).$$

The last desired assertion follows from the first part of the proof and Lemma 3.6.2. This finishes the proof. □

Lemma 3.6.4. *Let $E \Subset \Omega$. The following statements are equivalent:*
(i) *E is pluripolar in Ω,*
(ii) *$u_{E,\Omega} = 0$,*
(iii) *cap$^*(E,\Omega) = 0$.*

Proof. The equivalence between (i) and (ii) can be obtained by using directly the definition of $u_{E,\Omega}$. By Proposition 3.6.3, we get

$$\text{cap}^*(E,\Omega) = \int_\Omega (dd^c u^*_{E,\Omega})^n,$$

and we see that (ii) implies (iii). Assume now (iii). Hence $(dd^c u^*_{E,\Omega})^n = 0$ Recall that $\lim_{x\to\partial\Omega} u^*_{E,\Omega} = 0$, see the proof of Proposition 3.6.3. Using this and Corollary 3.4.2 implies (ii). □

Theorem 3.6.5. *Negligible sets are pluripolar.*

Proof. Let $(u_j)_{j\in J}$ be a family of psh functions bounded uniformly from above on Ω. Since the problem is local, we can assume Ω is hyperconvex. Let E be the set of $x \in \Omega$ such that $u := (\sup_{j\in J} u_j)^*(x) > \sup_{j\in J} u_j(x)$. By Choquet's lemma, we can assume J is countable, and $u := (\sup_{j\in J} u_j)^*$ is L^1 limit of an increasing sequence $(u_j)_j$ of psh functions. Let $\epsilon > 0$ be a constant. By quasi-continuity, there is an open subset $U = U(\epsilon)$ in X such that $\text{cap}_{BTK}(U) \le \epsilon$ and u, u_j is continuous on $X\backslash U$. Observe that

$$E\backslash U = \cup_{s,t\in\mathbb{Q}}\big\{x \in X\backslash U : u(x) \ge s > t \ge \sup_{j\in J} u_j\big\} = \cup_{s,t\in\mathbb{Q}}K_{s,t}.$$

Observe that $K_{s,t}$ are compact. Hence

$$E \subset U \cup \cup_{s,t\in\mathbb{Q}}K_{s,t}, \quad K_{s,t} \subset E.$$

Since $\text{cap}(E,\Omega) = 0$ (Lemma 3.3.7), we get $\text{cap}(K_{st},\Omega) = 0$. Thus $\text{cap}^*(K_{st},\Omega) = 0$ by Proposition 3.6.3. This combined with the fact that $\text{cap}(U_\epsilon,\Omega) < \epsilon$ yields that there is an open subset U' in Ω such that $\text{cap}(U',\Omega) \le 2\epsilon$ and $E \subset U'$. We infer that $\text{cap}^*(E,\Omega) = 0$. In particular $\text{cap}^*(E\cap K,\Omega) = 0$ for every compact K in Ω. Combining this with Lemma 3.6.4 shows that $E \cap K$ is pluripolar in Ω. Hence $E = \cup_j(E\cap K_j)$ is pluripolar, where $\Omega = \cup_j K_j$ and $K_j \Subset \Omega$. This finishes the proof. □

Theorem 3.6.6. *The function cap*$^*(\cdot,\Omega)$ *is a generalized capacity on* Ω*, and we have*

$$\text{cap}(E,\Omega) = \text{cap}^*(E,\Omega) = \text{cap}_*(E,\Omega) \tag{3.6.4}$$

for every Borel subset E *in* Ω*. If* E *is relatively compact Borel subset in* Ω*, then*

$$\text{cap}(E,\Omega) = \int_\Omega (dd^c u^*_{E,\Omega})^n. \tag{3.6.5}$$

Proof. The first and third desired properties for a generalized capacity holds for $\text{cap}^*(\cdot,\Omega)$ thanks to Proposition 3.6.3. Let E_j be subsets in Ω increasing to E. We need to prove that $\text{cap}^*(E_j,\Omega)$ increases to $\text{cap}^*(E,\Omega)$. Without loss of generality, we can assume $E_j \Subset \Omega$. Let $\epsilon > 0$ be a constant. By the proof of Theorem 3.6.5, we know that there is an open subset U in Ω such that $\text{cap}^*(U,\Omega) \le \epsilon$ such that

$$\cup_{j=1}^\infty\{u^*_{E_j,\Omega} > u_{E_j,\Omega}\} \subset U.$$

Let $0 < r < 1$ be a constant. Put $U_j := \{u^*_{E_j,\Omega} < -r\}$. By the proof of Proposition 3.6.3, we know that $U_j \Subset \Omega$ and

$$\lim_{x\to\partial\Omega} u^*_{E_j,\Omega} = \lim_{x\to\partial\Omega} u^*_{U_j,\Omega} = 0.$$

This combined with the fact that $r^{-1}u^*_{E_j,\Omega} \le u^*_{U_j,\Omega}$ and the comparison principle implies that

$$\text{cap}^*(U_j,\Omega) = \int_\Omega (dd^c u^*_{U_j,\Omega})^n \le r^{-n}\int_\Omega (dd^c u^*_{E_j,\Omega})^n = r^{-n}\text{cap}^*(E_j,\Omega).$$

Note that U_j is increasing in j, and $V := U \cup \cup_j U_j$ contains E. Hence $U \cup U_j$ increases to V. We obtain thus

$$\text{cap}^*(E, \Omega) \le \text{cap}(V, \Omega) \le \text{cap}(U, \Omega) + r^{-n} \liminf_{j \to \infty} \text{cap}(E_j, \Omega).$$

Letting $r \to 1$ in the last inequality yields that $\text{cap}^*(E_j, \Omega)$ increases to $\text{cap}^*(E, \Omega)$. Hence $\text{cap}^*(\cdot, \Omega)$ is a generalized capacity. The equality (3.6.4) is deduced immediately from this and the fact that

$$\text{cap}^*(K, \Omega) = \text{cap}(K, \Omega)$$

for every compact K in Ω.

The equality (3.6.5) follows directly from (3.6.4) and Proposition 3.6.3. This finishes the proof. □

3.7 Continuity of Monge-Ampère operators: continued

Let Ω be an open subset in $\mathbb{C}^n$.

Theorem 3.7.1. *Let u_j, u_{jk} be as in Theorem 3.3.6. Let $(u'_{jk})_k$ be a sequence of locally bounded psh functions on Ω for $1 \le j \le m$ such that $u'_{jk} \ge u_{jk}$ and $u'_{jk} \to u_j$ in L^1_{loc} as $k \to \infty$. Then we have*

$$u'_{1k} dd^c u'_{2k} \wedge \cdots \wedge dd^c u'_{mk} \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m$$

as $k \to \infty$.

Proof. Since the problem is local, as usual, we can assume that u'_{jk}, u_{jk}, u_j are all equal to some smooth psh function ψ outside some set $K \Subset \Omega$ on Ω. Let

$$S'_{jk} := dd^c u'_{jk} \wedge \cdots \wedge dd^c u'_{mk}, \quad S_j := dd^c u_j \wedge \cdots \wedge dd^c u_m.$$

We let S_{jk} as in the proof of Theorem 3.3.6. We prove by induction in j that

$$u'_{(j-1)k} S'_{jk} \to u_{(j-1)} S'_j$$

as $k \to \infty$ and for every $2 \le j \le m+1$ (by convention we put $S'_{(m+1)k} = S'_{m+1} := 1$). The claim is clear for $j = m+1$. Suppose that it holds for $(j+1)$. We need to prove it for j. Let $R_{j\infty}$ be a limit current of $u'_{(j-1)k} S'_{jk}$ as $k \to \infty$. By induction hypothesis (3.3.6) for $(j+1)$ instead of j, $S'_{jk} \to S_j$ as $k \to \infty$. This combined with the fact that the sequence $(u'_{jk})_k$ converges in L^1_{loc} to u_j gives

$$R_{j\infty} \le u_{j-1} S_j.$$

On the other hand, since $u'_{jk} \geq u_{jk}$, we get

$$\begin{aligned}
\int_\Omega u'_{(j-1)k} S'_{jk} \wedge \omega^{n-m+j-1} &\geq \int_\Omega u_{(j-1)k} S'_{jk} \wedge \omega^{n-m+j-1} \\
&= \int_\Omega u'_{jk} dd^c u_{(j-1)k} \wedge S'_{(j+1)k} \wedge \omega^{n-m+j-1} \\
&= \int_\Omega u_{jk} dd^c u_{(j-1)k} \wedge S'_{(j+1)k} \wedge \omega^{n-m+j-1} \\
&\cdots \\
&\geq \int_\Omega u_{(j-1)k} S_{jk} \wedge \omega^{n-m+j-1} \to \int_\Omega u_{j-1} S_k \wedge \omega^{n-m+j-1}
\end{aligned}$$

as $j \to \infty$ by Theorem 3.3.6. Hence $R_{j\infty} = u_{j-1} S_j$. This finishes the proof. □

Corollary 3.7.2. *Let $(u_j)_j$ be a sequence of psh function uniformly bounded increasing to a psh function u. Let $(u'_j)_j$ be a sequence of psh function uniformly bounded such that $u'_j \geq u_j$ and u'_j converges to u in L^1_{loc}. Then u'_j converges to u in capacity.*

Proof. Let $\epsilon > 0$ be a constant. Observe that

$$\{|u - u'_j| \geq \delta\} \subset \{u'_j - u_j \geq \delta\} \cup \{u - u_j \geq \delta\}.$$

Let $K \Subset U \Subset \Omega$ be open sets. Since $(u_j)_j, (u'_j)_j$ are uniformly bounded, using Lemma 3.3.3 gives

$$\text{cap}(K \cap \{u - u_j \geq \delta\}, \Omega) \leq \delta^{-1} C \bigg(\int_U (u - u_j)(dd^c u_j)^n \bigg)^{2^{-n}}$$

and

$$\text{cap}(K \cap \{u'_j - u_j \geq \delta\}, \Omega) \leq \delta^{-1} C \bigg(\int_U (u'_j - u_j)(dd^c u_j)^n \bigg)^{2^{-n}}$$

where $C > 0$ is a constant independent of j. The right-hand sides of both inequality tend to 0 as $j \to \infty$ thanks to Theorem 3.7.1. The desired assertion hence follows. □

Proposition 3.7.3. *Let $(u_j)_{j \in J}$ be a family of uniformly bounded continuous psh functions, and $u := (\sup_j u_j)^*$. Let $\epsilon > 0$ be a constant. Then there exists a closed subset A in Ω such that cap$(A, \Omega) < \epsilon$ and u is continuous on $\Omega \backslash A$.*

Note here that the usual quasi-continuity of psh functions implies that psh function are continuous outside an open subset A with small capacity. In the above statement, A is closed. This fact might be useful in practice.

Proof. One just needs to use Corollary 3.7.2 and argue as in the proof of Theorem 3.3.4. We will obtain a closed subset A in U such that cap$(A, U) < \epsilon$ and u is continuous on $U \backslash A$. Notice that the exceptional set A is closed because in the current setting the sequence u'_j increases to u (almost everywhere), while in the setting of Theorem 3.3.4 we have a sequence decreasing to u. □

The following continuity property of Monge-Ampère operators covers both those for increasing and decreasing sequences (Theorems 3.3.6 and 3.2.5).

Theorem 3.7.4. *Let $\Omega \subset \mathbb{C}^n$ be an open set. Let $(T_k)_k$ be a sequence of closed positive currents satisfying Condition $(*)$ so that T_k converges to a closed positive current T on Ω as $k \to \infty$. Let u_j be a locally bounded psh function on Ω for $1 \le j \le m$. Let $(u_{jk})_{k\in\mathbb{N}}, (u'_{jk})_{k\in\mathbb{N}}$ be sequences of locally bounded psh functions converging to u_j in L^1_{loc} as $k \to \infty$ such that $u'_{jk} \ge u_{jk}$. Then, the convergence*

$$u'_{1k} dd^c u'_{2k} \wedge \cdots \wedge dd^c u'_{mk} \wedge T_k \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge T \tag{3.7.1}$$

as $k \to \infty$ holds provided that one of the following two conditions is fulfilled for each j:

(i) $u_{jk}(x) \nearrow u_j(x)$ for every $x \in \Omega$ as $k \to \infty$,

(ii) $u_{jk}(x) \nearrow u_j(x)$ for almost everywhere $x \in \Omega$ (with respect to the Lebesgue measure) and T has no mass on pluripolar sets.

Proof. Assume for the moment that (3.7.1) holds for u_{jk} in place of u'_{jk}. Then arguing as in the proof of Theorem 3.7.1 gives (3.7.1). Hence it remains to check (3.7.1) for u_{jk} in place of u'_{jk}. Let

$$S_{jk} := dd^c u_{jk} \wedge \cdots \wedge dd^c u_{mk} \wedge T_k, \quad S_j := dd^c u_j \wedge \cdots \wedge dd^c u_m \wedge T.$$

We prove by induction in j that

$$u_{(j-1)k} S_{jk} \to u_{(j-1)} S_j \tag{3.7.2}$$

k and for every $2 \le j \le m+1$ (by convention we put $S_{(m+1)k} := T_k$ and $S_{m+1} := T$). The claim is true for $j = m+1$. Suppose that it holds for $(j+1)$. We need to prove it for j. Let $R_{j\infty}$ be a limit current of $u_{(j-1)k} S_{jk}$ as $k \to \infty$. By induction hypothesis (3.7.2) for $(j+1)$ instead of j, $S_{jk} \to S_j$ as $k \to \infty$. This combined with the fact that the sequence $(u_{jk})_k$ converges in L^1_{loc} to u_j gives

$$R_{j\infty} \le u_{j-1} S_j \tag{3.7.3}$$

(Lemma 3.2.1). On the other hand, since $(u_{jk})_k$ is increasing, using Corollary 3.3.5, we obtain

$$\liminf_{k\to\infty} u_{(j-1)k} S_{jk} \ge \liminf_{k\to\infty} u_{(j-1)s} S_{jk} = u_{(j-1)s} S_j$$

for every $s \in \mathbb{N}$. Letting $s \to \infty$ in the last inequality gives

$$R_{j\infty} \ge (\lim_{s\to\infty} u_{(j-1)s}) S_j = u_{j-1} S_j + (\lim_{s\to\infty} u_{(j-1)s} - u_{j-1}) S_j. \tag{3.7.4}$$

Recall that the set of $x \in U$ with $u_{j-1}(x) > \lim_{s\to\infty} u_{(j-1)s}(x)$ is empty in the setting of (i) and is a pluripolar set in the setting of (ii) by Theorem 3.6.5. Hence (3.7.2) follows from Lemma 3.3.2, (3.7.4) and (3.7.3). The proof is finished. □

The following remark will be important in next chapters.

Remark 3.7.5. *By the above proof and Lemma 3.3.2, Property (ii) of Theorem 3.7.4 still holds if instead of requiring T has no mass on pluripolar sets, we assume the following two conditions:*

(i) *T has no mass on $A_j := \{x \in U : u_j(x) \neq \lim_{k\to\infty} u_{jk}(x)\}$ for every $1 \leq j \leq m$ and,*

(ii) *the set A_j is locally complete pluripolar for every j.*

We conclude this section with the following result.

Corollary 3.7.6. *Let $(u_j)_j$ be a sequence of psh function uniformly bounded increasing to a psh function u. Let $(u'_j)_j$ be a sequence of psh function uniformly bounded such that $u'_j \geq u_j$ and u'_j converges to u in L^1_{loc}. Then u_j converges to u in capacity cap$_T$ for every closed positive current T having no mass on pluripolar sets.*

Proof. Similar to the proof of Corollary 3.7.2. □

We now present an important setting where we can define Monge-Ampère operators for unbounded psh functions. Let ω be the standard Kähler form on $\mathbb{C}^n$. Let Ω be an open subset in $\mathbb{C}^n$. Let T be a closed positive current of bi-degree (p, p) with $p < n$ on Ω. Let u be a psh function on Ω. The *unbounded locus* $L(u)$ of u is the set of $x \in \Omega$ such that u is unbounded in every open neighborhood of x. Observe that $L(u)$ is closed, and it is empty if u is bounded. When $u = \log\|x - a\|$ for $a \in \Omega$, then $L(u) = \{a\}$.

Lemma 3.7.7. *Assume that $L(u) \cap \mathrm{Supp}T$ is compact in Ω. Then u is locally integrable with respect to the trace measure of T. Hence the current $dd^c u \wedge T := dd^c(uT)$ is well-defined.*

Note that the hypothesis that $p < n$ is necessary. When $p = n$, take T be the the Dirac mass at a, then $\log\|x - a\|$ is not locally integrable with respect to δ_a. A weaker version of Lemma 3.7.7 was presented in [13, Page 151] requiring an extra assumption that $L(u) \cap \mathrm{Supp}T$ is contained in a Stein open subset in Ω. This condition is actually superfluous as the proof below shows.

Proof. We give a sketch of proof, and refer to [13] for detailed arguments. By wedging T with ω^{n-p-1}, we can assume that T is of bi-dimension $(1, 1)$. Let $K \Subset U \Subset \Omega$ be open subsets such that U contains $L(u) \cap \mathrm{Supp}T$. Since u is bounded from above on compact subsets, the desired assertion is equivalent to checking that

$$\int_K uT \wedge \omega > -\infty. \tag{3.7.5}$$

Let $0 \leq \chi \leq 1$ be a smooth cut-off function with compact support in U and $\chi = 1$ on an open neighborhood of $L(u) \cap \mathrm{Supp}T$. Let ψ be a smooth psh function on Ω such that $dd^c\psi(x) > 0$ for every $x \in \Omega$. We can take for example $\psi(x) = \|x\|^2$. Since ψ is strongly psh, in order to get (3.7.5), it suffices to check that

$$I := \int_\Omega \chi u T \wedge dd^c\psi > -\infty.$$

By integration by parts we get

$$\begin{aligned} I_\epsilon &:= \int_\Omega \chi u_\epsilon T \wedge dd^c\psi \qquad (3.7.6)\\ &= \int_\Omega \chi\psi dd^c u_\epsilon \wedge T + \int_\Omega dd^c\chi \wedge u_\epsilon \psi T + 2\int_\Omega \psi du_\epsilon \wedge d^c\chi \wedge T\\ &= \int_\Omega \chi\psi dd^c u_\epsilon \wedge T + \int_\Omega dd^c\chi \wedge u_\epsilon \psi T - 2\int_\Omega u_\epsilon d\psi \wedge d^c\chi \wedge T. \end{aligned}$$

Let I_1, I_2, I_3 be the first, second and third term in the right-hand side. Let $0 \le \chi_1 \le 1$ be a smooth cut-off function with compact support in Ω such that $\chi_1 = 1$ on $U \cap \mathrm{Supp}\chi$. We have

$$\begin{aligned} I_1 &= \int_{\mathrm{Supp}\chi} \chi\psi dd^c u_\epsilon \wedge T \lesssim \int_\Omega \chi_1 dd^c u_\epsilon \wedge T\\ &= \int_\Omega u_\epsilon dd^c \chi_1 \wedge T\\ &= \int_{\mathrm{Supp}\chi_1 \backslash L(u)} u_\epsilon dd^c\chi_1 \wedge T \to \int_{\mathrm{Supp}\chi_1 \backslash L(u)} u dd^c \chi_1 \wedge T > -\infty \end{aligned}$$

because u is uniformly bounded on $\mathrm{Supp}\chi_1 \backslash L(u)$. We treat I_2, I_3 similarly. So we get $I := \lim_{\epsilon\to 0} I_\epsilon > -\infty$. This finishes the proof. □

Note that similar arguments also allow us to prove the following. We leave it as an exercises for readers.

Lemma 3.7.8. *Let X be a complex manifold with a Hermitian metric ω such that there exists a bounded psh function ψ on X such that for every compact K in X, there exists a constant c_K satisfying $dd^c\psi \ge c_K\omega$ on K. Let η be a continuous $(1,1)$-form. Then every η-psh function is locally integrable with respect to every closed positive current of bi-degree (p,p) with $p < \dim X$.*

As in the case of bounded psh functions, we have the following continuity property.

Theorem 3.7.9. *Let T be a closed positive current of bi-degree (p,p) on Ω with $p < n$. Let $u_1, \dots, u_m$ ($m \le n-p$) be psh functions on Ω such that $L(u_j) \cap \mathrm{Supp}T \Subset \Omega$ for every j. Let $(u_{jk})_k$ be a sequence of psh functions on Ω converging to u_j in L^1_{loc} and $u_{jk} \ge u_j$ for every j, k. Then we have*

$$u_{1k} dd^c u_{2k} \wedge \cdots \wedge dd^c u_{mk} \wedge T \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge T$$

weakly as $k \to \infty$.

Theorem 3.7.9 was proved in [13, Page 152] under an extra assumption that Ω is Stein. We note that the usual arguments as in Theorem 3.2.5 or in [13, Page 152] don't work directly because Ω is not Stein.

Proof. Put

$$Q_{jk} := u_{jk} dd^c u_{(j+1)k} \wedge \cdots \wedge dd^c u_{mk} \wedge T, \quad Q_j := u_j dd^c u_{j+1} \wedge \cdots \wedge dd^c u_m \wedge T.$$

We prove by induction on j that $Q_{jk} \to Q_j$ as $k \to \infty$. When $j = m$, it is clear thanks to Lemma 3.7.7. Assume it is true for $j+1$. We prove the desired assertion for j. By the proof of Lemma 3.7.7, the mass on compact subsets of the family $(Q_{jk})_k$ is uniformly bounded. Let Q'_j be a limit current of $(Q_{jk})_k$ as $k \to \infty$. As usual we note that $Q'_j \le Q_j$. Let $0 \le \chi \le 1$ be a cut-off function with compact support in Ω such that $\chi = 1$ on an open neighborhood of $L := \cup_{j=1}^m L(u_j) \cap \mathrm{Supp}T$. We will prove that $\chi Q'_j = \chi Q_j$. We already have $\chi Q'_j \le \chi Q_j$. We prove the converse inequality. By wedging T with $\omega^{n-p-m+j}$, we can assume that Q is of bi-degree (n,n). Let u_j^ϵ and u_{jk}^ϵ be standard regularisation of u_j, u_{jk} respectively. Observe $u_j \le u_j^\epsilon \le u_{jk}^\epsilon$. By this and integration by parts as in (3.7.6) one obtains

$$\begin{aligned}\int_\Omega \chi Q_j &\le \int_\Omega \chi u_{jk}^\epsilon dd^c u_{(j+1)} \wedge dd^c Q_{j+2}\\ &= \int_\Omega \chi u_{j+1} dd^c u_{jk}^\epsilon \wedge dd^c Q_{j+2} + \int_\Omega u_{jk}^\epsilon u_{j+1} dd^c\chi \wedge dd^c Q_{j+2}\\ &- 2\int_\Omega u_{jk}^\epsilon du_{j+1} \wedge d^c\chi \wedge dd^c Q_{j+2}.\end{aligned}$$

Letting $\epsilon \to 0$ and noticing that u_j are locally bounded on an open neighborhood of the supports of $dd^c\chi$ and $d\chi$ ($\{\chi \equiv 1\}$ contains an open neighborhood of $\cup_{j=1}^m L(u_j) \cap \mathrm{Supp}T$ gives

$$\begin{aligned}\int_\Omega \chi Q_j \le \liminf_{\epsilon\to 0} \int_\Omega \chi u_{j+1} dd^c u_{jk}^\epsilon \wedge dd^c Q_{j+2} &+ \int_{\Omega\setminus L} u_{jk} u_{j+1} dd^c\chi \wedge dd^c Q_{j+2}\\ &- 2\int_{\Omega\setminus L} u_{jk} du_{j+1} \wedge d^c\chi \wedge dd^c Q_{j+2}.\end{aligned}$$

Let A_{1k}, A_{2k}, A_{3k} be the first, second and third term in the sum in the right-hand side of the last inequality. We define A'_{2k}, A'_{3k} by substituting u_{j+1} by $u_{(j+1)k}$ in the integral defining A_{2k}, A_{3k} respectively. By Corollary 3.3.5 and noticing again that u_{jk}, u_j are locally bounded on $\Omega\setminus L$, we infer

$$\lim_{k\to\infty}(A_{2k} - A'_{2k}) = \lim_{k\to\infty}(A_{3k} - A'_{3k}) = 0. \tag{3.7.7}$$

On the other hand, by induction hypothesis, the current $dd^c u_{jk}^\epsilon \wedge dd^c Q_{j+2}$ converges weakly to $dd^c u_{jk} \wedge dd^c Q_{j+2}$ as $\epsilon \to 0$. This combined with the inequality $u_{j+1} \le u_{(j+1)k}$ gives

$$A_{1k} \le A'_{1k} := \int_\Omega \chi u_{(j+1)k} dd^c u_{jk} \wedge dd^c Q_{j+2}.$$

Thus we conclude that

$$\int_\Omega \chi Q_j \le \liminf_{k\to\infty}(A'_{1k} + A'_{2k} + A'_{3k}) = \int_\Omega \chi u_{jk} dd^c u_{(j+1)k} \wedge dd^c Q_{j+2}.$$

Repeating this procedure for $u_{j+2}, \dots, u_m$, we obtain that

$$\int_\Omega \chi Q_j \le \int_\Omega \chi Q_{jk}.$$

Hence $\chi Q_j = \chi Q_{jk}$. This finishes the proof. □

It is now a right time to introduce the following notion.

Definition 3.7.10. *We say that the intersection of $dd^c u_1, \dots, dd^c u_m, T$ is* classically well-defined *if for every $u'_j \geq u_j$ for $1 \leq j \leq m$, and every non-empty subset $J = \{j_1, \dots, j_s\}$ of $\{1, \dots, m\}$, we have that u'_{j_s} is locally integrable with respect to the trace measure of T and inductively, u'_{j_r} is locally integrable with respect to the trace measure of $\bigwedge_{t=r+1}^{s} dd^c u'_{j_t} \wedge T$ for $r = s-1, \dots, 1$, and*

$$u_{j_1 k} dd^c u_{j_2 k} \wedge \dots \wedge dd^c u_{j_s k} \wedge T \to u_{j_1} dd^c u_{j_2} \wedge \dots \wedge dd^c u_{j_s} \wedge T$$

as $k \to \infty$, where $(u_{jk})_k$ is a any sequence of psh functions converging to u_j in L^1_{loc} such that $u_{jk} \geq u_j$.

The above definition is independent of the local potential of $dd^c u_j$. See [25, 27] for variants of this definition.

Proposition 3.7.11. *Let $u_1, \dots, u_m$ be psh functions such that $dd^c u_1, \dots, dd^c u_m, T$ have a classical intersection. Let $(u_{jk})_k$ be a sequence of psh functions converging to u_j in L^1_{loc} such that $u_{jk} \geq u_j$. Then the sequence $Q_k := \bigwedge_{j=1}^{m} dd^c u_{jk} \wedge T$ satisfies the Condition $(*)$. In particular, $(Q_k)_k$ satisfies the property mentioned in Theorem 3.7.4.*

Proof. We will use arguments similar to those in the proof of Theorem 3.2.5. Notice that by hypothesis, the intersection $\bigwedge_{j=1}^{m} dd^c u_j \wedge T$ is symmetric in $u_1, \dots, u_m$. Let v be a bounded function on Ω and $(v_k)_k$ a sequence decreasing to v as $k \to \infty$. We need to check that

$$(v_k - u)(dd^c v)^{m'} \wedge T_k \to 0$$

as $k \to \infty$ for every integer $m' \geq 0$. As usually (by the uniform boundedness of v_k, v) we can assume that $v_k \leq 0, v \leq 0$, Ω is a ball in $\mathbb{C}^n$ and there is a smooth psh function ψ on Ω such that

$$v_k = v = \psi$$

outside some compact subset in an open neighborhood of $\overline{\Omega}$, and all functions and currents in consideration are defined on an open neighborhood of $\overline{\Omega}$ in $\mathbb{C}^n$, . Since the arguments are more or less standard modulo what we have gone so far, in what follows we only present the complete proof for the case where $m' = 0$ and $m = 1$, *i.e,* we will prove that

$$v_k Q_k \to vQ. \tag{3.7.8}$$

The general case follows from analogous argument with a bit more messy writing. Let R be a limit current of the family $(v_k Q_k)_k$ as $k \to \infty$. Put $Q := dd^c u_1 \wedge T$. We have $R \leq uQ$ by Lemma 3.2.1. We will prove that the masses of R and uQ on Ω are equal. To this end, without loss of generality, we can assume that Q is of bi-dimension (n, n). Let $v_{k,\epsilon}, u_\epsilon, \psi_\epsilon$ be standard regularisations of v_k, u, ψ (on an open neighborhood of $\overline{\Omega}$). Since $v - \psi = 0$ outside some compact subset in Ω, we obtain

$$v_{k,\epsilon} = v_\epsilon = \psi_\epsilon$$

outside some compact subset in Ω if ϵ is small enough. Since $v \leq v_\epsilon \leq v_{k,\epsilon}$, we get

$$\int_\Omega vQ \leq \int_\Omega v_{k,\epsilon} Q = \int_\Omega (v_{k,\epsilon} - \psi_\epsilon) Q + \int_\Omega \psi_\epsilon Q.$$

By integration by parts, we get

$$\begin{aligned}
\int_\Omega (v_{k,\epsilon} - \psi_\epsilon) Q &= \int_\Omega u_1 dd^c (v_{k,\epsilon} - \psi_\epsilon) \wedge T \\
&= \int_\Omega u_1 dd^c v_{k,\epsilon} \wedge T - \int_\Omega u_1 dd^c \psi_\epsilon \wedge T \\
&\leq \int_\Omega u_{1k} dd^c v_{k,\epsilon} \wedge T - \int_\Omega u_1 dd^c \psi_\epsilon \wedge T \\
&= \int_\Omega u_{1k} dd^c (v_{k,\epsilon} - \psi_\epsilon) \wedge T + \int_\Omega u_{1k} dd^c \psi_\epsilon \wedge T - \int_\Omega u_1 dd^c \psi_\epsilon \wedge T \\
&= \int_\Omega (v_{k,\epsilon} - \psi_\epsilon) dd^c u_{1k} \wedge T + \int_\Omega u_{1k} dd^c \psi_\epsilon \wedge T - \int_\Omega u_1 dd^c \psi_\epsilon \wedge T
\end{aligned}$$

because $u_1 \leq u_{1k}$. Letting $\epsilon \to 0$ yields

$$\int_\Omega vQ \leq \int_\Omega v_k Q_k.$$

The desired assertion hence follows. This finishes the proof. □

We have known that the intersection of $dd^c u_1, \dots, dd^c u_m, T$ is classically well-defined if $L(u_j) \cap \mathrm{Supp} T \Subset \Omega \subset \mathbb{C}^n$. We present now another very important setting where the classical intersection of closed positive $(1,1)$-current is well-defined. We start with a basic lemma. For every constant $0 < \rho < 1$, let $\mathbb{D}_\rho$ denote the disk of radius ρ centered at 0 in $\mathbb{C}$.

Lemma 3.7.12. *Let $0 < \rho_1 < \rho_2 < \rho_3 < 1$ be constants. Then there exists a smooth subharmonic function $v \geq 0$ on $\mathbb{D} \backslash \mathbb{D}_{\rho_1}$ such that* $\mathrm{Supp} v \Subset \mathbb{D}$ *and*

$$v > 0, \quad dd^c v > 0$$

in $\mathbb{D}_{\rho_3} \backslash \mathbb{D}_{\rho_2}$.

Proof. Let $v_1(z) := 1/|z|^2$. We have $dd^c v_1 > 0$ on $\mathbb{D}_{\rho_3} \backslash \mathbb{D}_{\rho_2}$. Let $\rho_3 < \rho_3' < 1$ be a constant and $c := 1/\rho_3'^2$. Let χ be a smooth convex increasing function such that $\chi = 0$ on $(-\infty, c]$ and $\chi'(t) > 0$ for $t \geq 1/\rho_3^2$. Put $v := \chi(v_1)$. Then v is subharmonic, $v(z) = 0$ for $|z| \geq \rho_3'$ and

$$dd^c v \geq \chi'(v_1) dd^c v_1 > 0$$

on $\mathbb{D}_{\rho_3} \backslash \mathbb{D}_{\rho_2}$. This finishes the proof. □

One can see that the above proof doesn't work in higher dimension due to the Hartogs' phenomena. Let $n \geq 2$. Let $0 < r, r_1 < 1$ be constants. Let

$$H := \{(z, w) \in \mathbb{D}^{n-p} \times \mathbb{D}^p : \|w\|' < r\} \cup \{(z, w) \in \mathbb{D}^{n-p} \times \mathbb{D}^p : r_1 < \|z\|' < 1\},$$

where $z = (z_1, \ldots, z_{n-p})$, $\|z\|' := \max\{|z_1|, \ldots, |z_{n-p}|\}$, and $w = (w_1, \ldots, w_p)$, $\|w\|' := \max\{|w_1|, \ldots, |w_p|\}$. The set H is called *the standard* $(n-p, p)$ *Hartogs figure*. Observe

$$\mathbb{D}^n \backslash H = \{(z, w) \in \mathbb{D}^n : \|z\|' \leq r_1, \|w\|' \geq r\}.$$

We put $\widehat{H} := \mathbb{D}^n$. Generally, the image H of every standard $(n-p, p)$ Hartogs' figure under a biholomorphism Ψ from $\mathbb{D}^n$ to a bounded domain in $\mathbb{C}^n$ is called a Hartog's figure, and we also put $\widehat{H} := \Psi(\mathbb{D}^n)$. Here is a deep estimate concerning the mass of currents on Hartogs figures.

Theorem 3.7.13. *(Oka's inequality for currents) ([19]) Let H be the standard $(n-p, p)$ Hartogs's figure. Let $0 < \rho < 1$ be a constant. Then there exists a constant C_ρ such that for every negative current Q of bi-dimension (p, p) with $dd^c Q \geq 0$, we have*

$$\|Q\|_{\mathbb{D}^n_\rho} + \|dd^c Q\|_{\mathbb{D}^n_\rho} \leq C_\rho \|Q\|_H.$$

We will apply the above result to $Q := uT$, where T is a closed positive current and u is a negative psh function.

Proof. Before going into details, we observe that the size of the Hartogs figure plays no role here, that means that the specific values r_1, r are not important for our below estimates. At the end of the proof we will need to deform slightly our Hartogs figure but this will cause no problem at all. Let

$$W_j := \{(z, w) \in \mathbb{D}^{n-p}_{r_1} \times \mathbb{D}^p_\rho : \rho \geq |w_j| \geq r\}$$

for $1 \leq j \leq p$ and

$$H' := \{(z, w) \in \mathbb{D}^{n-p} \times \mathbb{D}^p : r_1 < \|z\|' < 1\}.$$

Observe that

$$\mathbb{D}^n_\rho \backslash H = \cup_{j=1}^p W_j. \tag{3.7.9}$$

Hence it suffices to estimate the mass of currents on W_j. Put $W := W_1$. We claim that

$$\|Q\|_W + \|dd^c Q\|_W \leq C_\rho \|Q\|_{H'}. \tag{3.7.10}$$

Assume for the moment that (3.7.10) is true. Then one just needs to apply consecutively (3.7.10) for $w_2, \ldots, w_k$ in place of w_1 to get the desired inequality. It remains to check (3.7.10).

Let $r_1 < r_1' < 1$ and $0 < r' < r < \rho' < 1$ be constants. Let $0 \leq \chi_1(z), \chi_2(w) \leq 1$ be smooth cut-off functions such that

$$\mathrm{Supp}\chi_1 \Subset \{\|z\|' < r_1'\}, \quad \mathrm{Supp}\chi_2 \Subset \{r' < \|w\|' < \rho'\},$$

and $\chi_1 = 1$ on an open neighborhood of $\{\|z\|' \leq r_1\}$. We will choose χ_2 explicitly later. Let

$$\Omega := \mathbb{D}^{n-p}_{r_1'} \times \{r' < \|w\|' < \rho'\}.$$

Observe $\mathrm{Supp}(\chi_1\chi_2) \Subset \Omega$. Let S be a smooth closed positive form on Ω. Observe that

$$\int_\Omega \chi_1\chi_2 dd^c Q \wedge S = \int_\Omega dd^c(\chi_1\chi_2)Q \wedge S.$$

It follows that

$$\int_\Omega \chi_1\chi_2 dd^c Q \wedge S + \int_\Omega dd^c\chi_2 \wedge (-Q) \wedge S = \int_\Omega dd^c\big((1-\chi_1)\chi_2\big)(-Q) \wedge S. \tag{3.7.11}$$

By Lemma 3.7.12, we can choose a smooth subharmonic function $v(w_1)$ on $\{|w_1| > r/2\}$ such that $dd^c v(w_1) > 0$ and $v > 0$ on $\{r' \le |w_1| \le \rho'\}$. Let $w' := (w_2, \dots, w_k)$ and

$$\chi_2(w) := v(w_1) + \chi_3(w'),$$

where $0 \le \chi_3(w') \le 1$ is a smooth function with compact support in $\mathbb{D}^{p-1}$ and $\chi_3 = 1$ on $\mathbb{D}^{n-p-1}_\rho$. Let $S := (dd^c\|w'\|^2)^{p-1}$. Note that

$$dd^c\chi_2 \wedge S \gtrsim (dd^c\|w\|^2)^p$$

on $\{r' \le |w_1| \le \rho'\} \times \mathbb{D}^{p-1}$ because every bad terms from $dd^c\chi_2$ are canceled when wedging with S. Using this and (3.7.11), we obtain that

$$\|dd^c Q \wedge (dd^c\|w'\|^2)^p\|_W + \|Q \wedge (dd^c\|w\|^2)^p\|_W \lesssim \int_\Omega dd^c\big((1-\chi_1)\chi_2\big)(-Q) \wedge S \lesssim \|Q\|_{H'} \tag{3.7.12}$$

because $dd^c\big((1-\chi_1)\chi_2\big) = 0$ outside H'. The last inequality is almost what we want. The remaining issue is that $\|Q \wedge (dd^c\|w\|^2)^p\|_W$ is less than $\|Q\|_W$ because we need to take into account

$$Q \wedge (dd^c(\|w\|^2 + \|z\|^2))^p$$

when computing $\|Q\|_W$. We bypass this problem by applying (3.7.12) to small generic perturbed coordinates of w. To be precise, we can consider

$$\tilde{w}_j := w_j + \epsilon t_j$$

for $1 \le j \le p$, where t_j is one of $z_1, \dots, z_{n-p}, w_1, \dots, w_p$. Denote by $\mathcal{A}$ the set of such coordinate systems and the original coordinate system w itself. Since ϵ is small, we still have

$$|\tilde{w}_1| \ge \tilde{r}$$

for some fixed constant $\tilde{r} > 0$ for $\tilde{w} \in W$. This allows us to apply (3.7.12) to these new coordinates $\tilde{w}'$. Hence we obtain

$$\|dd^c Q \wedge (dd^c\|w'\|^2)^p\|_W + \|Q \wedge (dd^c\|\tilde{w}\|^2)^p\|_W \lesssim \int_\Omega dd^c\big((1-\chi_1)\chi_2\big)(-Q) \wedge S \lesssim \|Q\|_{H'}. \tag{3.7.13}$$

Summing up (3.7.13) over $\mathcal{A}$ and noticing that $dd^c\|\tilde{w}\|^2 + dd^c\|w\|^2 \gtrsim dd^c\|t_j\|^2$, we get (3.7.10). The proof is finished. $\square$

Lemma 3.7.14. *Let u, v be psh functions on $\mathbb{D}^n$. Let $1 \le p \le n-1$ be an integer. Let T be a closed positive current of bi-dimension $(p+1,p+1)$ on $\mathbb{D}^n$. Let $0<\rho<1$ be a constant. Let $(u_k)_k$ and $(v_k)_k$ be sequences of psh functions converging to u, v in $L^1_{loc}(\mathbb{D}^n)$ respectively such that $v_k \ge v$ and $u_k \ge u$ for every k. Assume that v is locally bounded on H and u is locally integrable with respect to the trace measure of T. Then, the following two properties hold:*

(i) v, v_k are locally integrable with respect to the trace measure of T, and u, u_k are locally integrable with respect to the trace measure of $dd^c v \wedge T, dd^c v_k \wedge T$, respectively, and $u_k dd^c v_k \wedge T$ converges weakly to $u dd^c v \wedge T$ as $k \to \infty$ in $\mathbb{D}^n$,

(ii) if $(T_k)_k$ is a sequence of closed positive currents of bi-dimension $(p+1,p+1)$ such that u_k is locally integrable with respect to the trace measure of T_k and $u_k T_k \to uT$ as $k \to \infty$ and for every bounded psh function w, we have $w dd^c u_k \wedge T_k \to w dd^c u \wedge T$ as $k \to \infty$, then $u_k dd^c v_k \wedge T_k \to u dd^c v \wedge T$ as $k \to \infty$.

Proof. We first check (i). By Theorem 3.7.13 applied to $v_\epsilon T$ (we can, as usual, assume that $v \le 0$ and here v_ϵ is the standard regularisation of v), the functions v, v_k are locally integrable with respect to T. Combining this with Lemma 3.2.1 implies that $v_k T \to vT$. In particular $dd^c v_k \wedge T$ converges to $dd^c v \wedge T$ as $k \to \infty$. By Theorem 3.2.5 or Corollary 3.2.7, u is locally integrable with respect to the trace measure of $dd^c v \wedge T$ and $dd^c v_k \wedge T$ on H. Since $u_k \ge u$, we also obtain that u_k is locally integrable with respect to the trace measure of $dd^c v_k \wedge T$ on H. Using this and Theorem 3.7.13 again, we see that the functions u_k, u are locally integrable with respect to the trace measure of $dd^c v_k \wedge T, dd^c v \wedge T$ respectively.

Put $Q := u dd^c v \wedge T$. Now let the notations be as in the proof of Theorem 3.7.13. By Theorem 3.7.13 and the Chern-Levine-Nirenberg inequality and the fact that v_k is uniformly bounded on fixed compact subset in H, we see that the current $u_k dd^c v_k \wedge T$ is of mass bounded uniformly on compact subsets in $\mathbb{D}^n$. Let R be a limit current of the family $(u_k dd^c v_k \wedge T)_k$ as $k \to \infty$. As usual we get

$$R \le u dd^c v \wedge T.$$

We check the inverse inequality. Since $R = u dd^c v \wedge T$ on H by Corollary 3.2.7, using (3.7.9), we see that it suffices to prove that

$$\int_\Omega \chi_1 \chi_2 R = \int_\Omega \chi_1 \chi_2 u dd^c v \wedge T.$$

Let u^ϵ, u^ϵ_k be standard regularisations of u, u_k respectively. We define similarly v^ϵ, v^ϵ_k. Let $S := (dd^c \log \|w'\|^2)^{p-1}$. Applying (3.7.11) to $Q_{k,\epsilon} := u^\epsilon_k dd^c v_k \wedge T$ gives

$$\int_\Omega \chi_1 \chi_2 dd^c Q_{k,\epsilon} \wedge S + \int_\Omega dd^c \chi_2 \wedge (-Q_{k,\epsilon}) \wedge S = \int_\Omega dd^c\big((1-\chi_1)\chi_2\big)(-Q_{k,\epsilon}) \wedge S.$$

Let $A_{1,\epsilon}, A_{2,\epsilon}, A_{3,\epsilon}$ be the first, second, and third term from left to right. We obtain similar $A'_{1,\epsilon}, A'_{2,\epsilon}, A'_{3,\epsilon}$ for $Q'_k := u^\epsilon_k dd^c v \wedge T$. Since $dd^c\big((1-\chi_1)\chi_2\big) \Subset H$, we infer

$$\lim_{k\to\infty} \lim_{\epsilon \to 0} (A_{3,\epsilon} - A'_{3,\epsilon}) = 0.$$

We are going to prove a similar assertion for $A_{1,\epsilon}$. Using $u \le u_k \le u_k^\epsilon$ and $dd^c\chi_2 \wedge S \ge 0$ (recall $S = (dd^c\|w'\|^2)^{p-1}$ and χ_2 is chosen explicitly as in the proof of Theorem 3.7.13), we get

$$A'_{1,\epsilon} = \int_\Omega \chi_1 v dd^c\chi_2 \wedge dd^c u_k^\epsilon \wedge T + \text{ terms with } d\chi_1$$

which is

$$\le \int_\Omega \chi_1 v_k dd^c\chi_2 \wedge dd^c u_k^\epsilon \wedge T \wedge S + \text{ terms with } d\chi_1.$$

We don't have to worry about terms with $d\chi_1$ because its support lies in H, so everything is ok. We deduce that

$$\lim_{k\to\infty}\lim_{\epsilon\to 0}(A_{1,\epsilon} - A'_{1,\epsilon}) \ge 0.$$

It follows that

$$\lim_{k\to\infty}\lim_{\epsilon\to 0}(A_{2,\epsilon} - A'_{2,\epsilon}) \le 0.$$

In other words,

$$\lim_{k\to\infty}\lim_{\epsilon\to 0}\Big(\int_\Omega dd^c\chi_2 \wedge u_k^\epsilon dd^c v_k \wedge T \wedge S - \int_\Omega dd^c\chi_2 \wedge u_k^\epsilon dd^c v \wedge T \wedge S\Big) \ge 0.$$

Now by perturbing Ω and the Hartogs's figure by using generic Euclidean change of coordinates as in the end of the proof of Theorem 3.7.13, there exists a smooth strictly positive $(n-p, n-p)$-form Φ on an open neighborhood of $\overline{\Omega}$ (*i.e,* $\Phi \gtrsim \omega^{n-p}$) such that

$$\lim_{k\to\infty}\lim_{\epsilon\to 0}\Big(\int_\Omega u_k^\epsilon dd^c v_k \wedge T \wedge \Phi - \int_\Omega u_k^\epsilon dd^c v \wedge T \wedge \Phi\Big) \ge 0.$$

Using this and the inequality $u_k \ge u$ gives

$$\lim_{k\to\infty}\lim_{\epsilon\to 0}\Big(\int_\Omega u_k^\epsilon dd^c v_k \wedge T \wedge \Phi - \int_\Omega u dd^c v \wedge T \wedge \Phi\Big) \ge 0. \tag{3.7.14}$$

Now we perturb a bit Ω such that the trace measures of $dd^c v_k \wedge T, dd^c v \wedge T, R$ have no mass on $\partial\Omega$. Using this and letting $\epsilon \to 0$ in (3.7.14) yield

$$\int_\Omega R \wedge \Phi - \int_\Omega u dd^c v \wedge T \wedge \Phi = \lim_{k\to\infty}\lim_{\epsilon\to 0}\Big(\int_\Omega u_k dd^c v_k \wedge T \wedge \Phi - \int_\Omega u dd^c v \wedge T \wedge \Phi\Big) \ge 0. \tag{3.7.15}$$

Hence $R = udd^c v \wedge T$, and (i) follows. The second desired assertion (ii) is obtained by arguing similarly as above with $\tilde{Q}_{k,\epsilon} := u_k^\epsilon dd^c v_k \wedge T_k$ in place of $Q_{k,\epsilon}$. Notice that we need to use the hypothesis on T_k when comparing $\tilde{A}'_{1,\epsilon}$ and $\tilde{A}_{1,\epsilon}$ (analogous versions of $A'_{1,\epsilon}$ and $A_{1,\epsilon}$). This finishes the proof. □

Let $\mathcal{H}_m$ denote the m-dimensional Hausdorff measure on $\mathbb{C}^n$. Recall that $\mathcal{H}_{2n}$ is proportional to the Lebesgue measure on $\mathbb{C}^n$. For basic material on Hausdorff measures, one can consult [18].

Lemma 3.7.15. *Let Ω be an open subset in $\mathbb{C}^n$. Let E be a closed subset in Ω such that $\mathcal{H}_{2p}(E)=0$. Then for every $x\in E$, there exists a $(n-p,p)$ Hartogs' figure H in Ω such that $\overline{H}\cap E=\varnothing$ and $x\in\widehat{H}$.*

Proof. Let L be an affine $(n-p)$-dimensional complex subspace in $\mathbb{C}^n$. Let L_1 be a p-dimension complex subspace transverse to L. Let $\pi:\mathbb{C}^n\to L_1$ be the natural projection along L. Recall that

$$\mathcal{H}_{2p}(E)=\sup_{\delta>0}\inf\Big\{\sum_s(diam(E_s))^{2p}:E\subset\cup_{s=1}^{\infty}E_s,\ diam(E_s)\le\delta\Big\}.$$

Since π is Lipschitz on compact subsets in $\mathbb{C}^n$, using the above formula, we infer that $\mathcal{H}_{2p}(\pi(E))=0$. Hence for almost everywhere $x\in L_1$, the $(n-p)$-dimensional complex subspace $\pi^{-1}(x)$ doesn't intersect E.

Now fix $x_0\in E$. We can assume $x_0=0$. Let $\mathcal{G}_{n-p}$ be the Grassmanian space of $(n-p)$-dimensional complex subspaces of $\mathbb{C}^n$. Note that $\dim\mathcal{G}_{n-p}=p(n-p)$. Let $\mathcal{G}'_{n-p}$ be the space of (x,L) where $L\in\mathcal{G}_{n-p}$ and $x\in L$. Observe that $\mathcal{G}'_{n-p}$ is a submanifold of $\mathbb{C}^n\times\mathcal{G}_{n-p}$ of dimension $(n-p)p+n-p$. Let π_1,π_2 be the natural projections from $\mathcal{G}'_{n-p}$ to the first and second components. For $x\neq 0$, the fiber $\pi_1^{-1}(x)$ is of codimension n. This combined with the fact that $\mathcal{H}_{2p}(E\backslash\{0\})=0$ implies that

$$\mathcal{H}_{2(n-p)p}(\pi_1^{-1}(E\backslash\{0\}))=0.$$

Arguing as in the first part of the proof, we deduce that

$$\mathcal{H}_{2(n-p)p}\big(\pi_2(\pi_1^{-1}(E\backslash\{0\}))\big)=0.$$

Since $\pi_2(\pi_1^{-1}(E\backslash\{0\}))\subset\mathcal{G}_{n-p}$ which is of real dimension $2(n-p)p$, we obtain that $\pi_2(\pi_1^{-1}(E\backslash\{0\}))$ is of zero Lebesgue measure in $\mathcal{G}_{n-p}$. It follows that almost every affine $(n-p)$-dimensional complex subspace L passing through x_0 doesn't intersect E except at x_0. Let L be such a subspace.

Let $L^{\perp}$ be the p-dimensional complex subspace passing through x_0 and orthogonal to L. Let (z,w) be coordinates on $\mathbb{C}^n$ such that $\{z=0\}=L^{\perp}$ and $\{w=0\}=L$. For every constant $r>0$, let $\mathbb{D}^p(x_0,r)\subset L^{\perp}$ and $\mathbb{D}^{n-p}(x_0,r)\subset L$ be polydisks. Since $E\cap L=\{x_0\}$ and E is closed, we can choose r' small enough such that $B:=\partial\mathbb{D}^{n-p}(x_0,r)\times\mathbb{D}_p(x_0,r')$ doesn't intersect E.

Since most of subspaces parallel to L doesn't intersect E, we choose a sequence of $(n-p)$-dimensional subspaces $(L_j)_j$ parallel to L such that $L_j\to L$ as $j\to\infty$ and $L_j\cap E=\varnothing$. Let x_j be the intersection point of L_j and $L^{\perp}$. We have $x_j\to x_0$ as $j\to\infty$. Put

$$B_j:=\partial\mathbb{D}^{n-p}(x_j,r)\times\mathbb{D}_p(x_j,r').$$

Put $D_j:=\mathbb{D}^{n-p}(x_j,r)\times\mathbb{D}_p(x_j,r)$ which is a polydisk containing x_0 if j is big enough. Since B_j is compact and $B_j\to B$ which doesn't intersect E, we infer that $B_j\cap E=\varnothing$ as j big enough. Fix such a j. Hence by thickening a bit the set $B_j\cup(\mathbb{D}^{n-p}(x_j,r)\times\{0\})$ inside D_j (where $\mathbb{D}^{n-p}(x_j,r)$ is the polydisk in $L_j^{\perp}$ centered at x_j), we can find a small constant $0<r''<r$ such that

$$H:=\{(z,w)\in D_j:r-r''<\|z\|'<r\}\cup\{(z,w)\in D_j:\|w\|'<r''\}$$

doesn't intersect E. Clearly H is a Hartogs' figure and $\widehat{H} = D_j$ containing x_0. This finishes the proof. □

Corollary 3.7.16. *Let E be a closed subset in Ω such that $\mathcal{H}_{2p}(E) = 0$. Let K be a compact subset in Ω. Then there exists a compact K_1 in $\Omega \backslash E$ and a constant C such that for every closed positive current T of bi-dimension (p,p) on Ω and every negative psh function u on Ω, we have*

$$\|uT\|_K \le C\|uT\|_{K_1}.$$

Proof. This is a direct consequence of Lemma 3.7.15 and Theorem 3.7.13. □

Theorem 3.7.17. *([19]) Let Ω be a domain in $\mathbb{C}^n$. Let T be a closed positive current of bi-dimension (p,p) on Ω with $p < n$. Let $L_1, \dots, L_m$ be closed subsets in Ω such that for every subset $J \subset \{1, \dots, m\}$ we have*

$$\mathcal{H}_{2p-2|J|+2}\big(\cap_{j\in J} L_j \cap \mathrm{Supp}T\big) = 0.$$

Let $u_1, \dots, u_m$ ($m \le p$) be psh functions on Ω such that $L(u_j) \subset L_j$ for every j. Then u_j is locally integrable with respect to the trace measure of $dd^c u_{j+1} \wedge \dots \wedge dd^c u_m \wedge T$ for every $1 \le j \le m$, and for every compact K in Ω there exist a constant C and compact subsets $K_1, \dots, K_m$ all independent of $T, u_1, \dots, u_m$ such that $K_j \cap L_j = \varnothing$ for every $1 \le j \le m$, and

$$\|u_1 dd^c u_2 \wedge \dots \wedge dd^c u_m \wedge T\|_K \le C\|u_1\|_{L^\infty(K_1)} \cdots \|u_m\|_{L^\infty(K_m)}\|T\|_\Omega. \tag{3.7.16}$$

Furthermore, let $(u_{jk})_k$ be a sequence of psh functions on Ω converging to u_j in L^1_{loc} and $u_{jk} \ge u_j$ for every j, k. Then u_{jk} is locally integrable with respect to the trace measure of $dd^c u_{(j+1)k} \wedge \dots \wedge dd^c u_{mk} \wedge T$ for every $1 \le j \le m$, and we have

$$u_{1k} dd^c u_{2k} \wedge \dots \wedge dd^c u_{mk} \wedge T \to u_1 dd^c u_2 \wedge \dots \wedge dd^c u_m \wedge T$$

weakly as $k \to \infty$.

Here $|J|$ denotes the cardinality of J. We refer to [13] for a weaker version of this result where $\mathcal{H}_{2k-2|J|+2}$ is replaced by $\mathcal{H}_{2k-2|J|+1}$.

Proof. We prove by induction the desired assertions on m. Assume that they are true for $m-1$. We check them for m. Note that by induction hypothesis the operator $dd^c u_{jk} \wedge \dots \wedge dd^c u_{mk} \wedge T$ is symmetric in $u_{jk}, \dots, u_{mk}$ for $2 \le j \le m$, and they satisfy the Condition $(*)$ by Proposition 3.7.11. We will need this observation at the end of the proof.

Let Q_{jk}, Q_j be as in the proof of Theorem 3.7.9. We will prove that Q_1 and Q_{1k} are well-defined (Q_{jk}, Q_j are well-defined for $j \ge 2$ by induction hypothesis). Let

$$E := \cap_{s=1}^m L_s \cap \mathrm{Supp}T.$$

Let $K \Subset \Omega$. By hypothesis $\mathcal{H}_{2p-2m+2}(E) = 0$. By Lemma 3.7.15, there exists a compact $K_1 \Subset \Omega \backslash E$ such that

$$\|Q_1\|_K \lesssim \|Q_1\|_{K_1}.$$

Let $x \in K_1$. We have either $x \notin \cap_{s=1}^m L_s$ or $x \notin \mathrm{Supp}T$. If the latter case occurs, it is clear that Q_1 is zero on some small open neighborhood of x. We consider $x \notin \cap_{s=1}^m L_s$. Then u_j is locally bounded near x for some $1 \le j \le m$. If u_1 is locally bounded, then we are done. If u_j is bounded for some $j \ge 2$, then by symmetry, we can assume that u_2 is locally bounded. Thus

$$Q_1 = u_1 dd^c u_2 \wedge dd^c Q_3.$$

Since u_1 is locally integrable with respect to Q_3 by induction hypothesis, using Theorem 3.2.5 or Corollary 3.2.7, we infer that Q_1 is of finite mass on compact subsets, and

$$\|Q_1\|_{\mathbb{B}(x,\epsilon)} \lesssim \|u_2\|_{L^\infty(\mathbb{B}(x,2\epsilon))} \|u_1 dd^c Q_3\|_{\mathbb{B}(x,2\epsilon)},$$

where $\epsilon > 0$ is a constant small enough so that u_2 is still bounded on $\mathbb{B}(x, 2\epsilon)$. By induction hypothesis, we get

$$\|u_1 dd^c Q_3\|_{\mathbb{B}(x,2\epsilon)} \lesssim \|u_1\|_{L^\infty(K_{1,x})} \|u_3\|_{L^\infty(K_{1,x})} \cdots \|u_m\|_{L^\infty(K_{1,x})} \|T\|_\Omega,$$

where $K_{j,x}$ is some compact subset having empty intersection with L_j. Hence letting x run over K and using the compactness of K, we deduce (3.7.16). Similar arguments also show that Q_{1k} is of mass bounded uniformly on compact subsets.

Let Q_1' be a limit current of $(Q_{1k})_k$ as $k \to \infty$. As usual we note that $Q_1' \le Q_1$. Let $x_0 \in \Omega$. If $x_0 \notin \mathrm{Supp}T$, then both Q_1', Q_1 are zero in a small open neighborhood of x. Consider $x_0 \in \mathrm{Supp}T$. By the hypothesis and Lemma 3.7.15, we can find a $(n-(p-m+1), p-m+1)$ Hartogs' figure H such that $\overline{H} \cap (\cap_{j=1}^m L_j \cap \mathrm{Supp}T) = \varnothing$ and $x_0 \in \widehat{H}$. Hence there is $1 \le j_0 \le m$ such that u_{j_0} is locally bounded in H. If $j_0 = 1$, then we get $Q_1' = Q_1$ locally near x_0 thanks to Proposition 3.7.11 and the induction hypothesis (the intersection of $dd^c u_2, \ldots, dd^c u_m, T$ is classically well-defined and u_1 is bounded on a small neighborhood of x_0).

Consider $j_0 \ge 2$. Since $dd^c Q_2$ is symmetric in $u_2, \ldots, u_m$ by induction hypothesis. Without loss of generality we can assume $j_0 = 2$. We are now being exactly in the situation in (ii) of Lemma 3.7.14 with $dd^c Q_3, dd^c Q_{3k}, u_1, u_2$ in place of T, T_k, u, v. Using that lemma we obtain the desired convergence. This finishes the proof. □

In the last part of this section, we will define Lelong numbers of closed positive currents.

Proposition 3.7.18. *Let u, u' be psh functions which are locally integrable with respect to the trace measure of T and $u \ge u'$. Then we have*

$$\mathbf{1}_{\{u=-\infty\}} dd^c u \wedge T \le \mathbf{1}_{\{u'=-\infty\}} dd^c u \wedge T \le \mathbf{1}_{\{u'=-\infty\}} dd^c u' \wedge T.$$

Proof. Since $u \ge u'$, we get $\{u = -\infty\} \subset \{u' = -\infty\}$. Let $\epsilon > 0$ be a constant. Put $w_j := \max\{(1-\epsilon)u - j, u'\}$. We have $w_j = (1-\epsilon)u - j$ on $\{u' < -j/\epsilon\}$ because $u \ge u'$. Using this and Theorem 3.3.9, we obtain

$$\begin{aligned} dd^c w_j \wedge T &\ge \mathbf{1}_{\{u'<-j/\epsilon\}} dd^c w_j \wedge T \\ &= (1-\epsilon)\mathbf{1}_{\{u'<-j/\epsilon\}} dd^c u \wedge T \ge (1-\epsilon)\mathbf{1}_{\{u'=-\infty\}} dd^c u \wedge T. \end{aligned}$$

Letting $j \to \infty$, and then $\epsilon \to 0$ gives

$$dd^c u' \wedge T \geq \mathbf{1}_{\{u'=-\infty\}} dd^c u \wedge T.$$

This finishes the proof. $\square$

The last result was proved in ([1] and also [13]) when T is a Monge-Ampère of closed positive $(1,1)$- currents. We define *the Lelong number of T at $a \in \Omega$* to be

$$\nu(T,a) := \int_{\{a\}} (dd^c \log \|x-a\|)^{n-p} \wedge T.$$

The last expression is well-defined thanks to Lemma 3.7.7. The following is a direct consequence of Proposition 3.7.18.

Theorem 3.7.19. *(comparison of Lelong numbers) Let $u_1, u_1', \ldots, u_m, u_m'$ be psh functions on X such that the intersections of $dd^c u_1, \ldots, dd^c u_m, T$ and of $dd^c u_1, \ldots, dd^c u_m, T$ are classically well-defined. Assume that $u_j \geq u_j'$ for $1 \leq j \leq m$. Then we have*

$$\mathbf{1}_{\cap_{j=1}^m \{u_j=-\infty\}} \bigwedge_{j=1}^m dd^c u_1 \wedge T \leq \mathbf{1}_{\cap_{j=1}^m \{u_j'=-\infty\}} \bigwedge_{j=1}^m dd^c u_j' \wedge T \tag{3.7.17}$$

and

$$\nu(dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T, x) \leq \nu(dd^c u_1' \wedge \cdots \wedge dd^c u_m' \wedge T, x) \tag{3.7.18}$$

for every $x \in \Omega$. In particular for every $a \in \Omega$ and every psh function φ on an open neighborhood of a in Ω such that $\varphi(x) - \log\|x-a\| = O(1)$ near a then we have

$$\nu(T,a) = \int_{\{a\}} (dd^c \varphi)^{n-p} \wedge T.$$

The second assertion in Theorem 3.7.19 is due to Demailly [13]. The above result implies that the notion of Lelong number is independent of the local coordinates ([13, 36]). Hence for every closed positive current T on a complex manifold X, for every $x \in X$, we can define the Lelong number $\nu(T,x)$ of T at x is that of T at x in any local chart around x. Here is another way to calculate the Lelong number. We refer to [13, Chapter 3] for proofs.

Lemma 3.7.20. *We have*

$$\nu(T,x) = \left(\epsilon^{2(n-p)} \pi^{n-p}/(n-p)!\right)^{-1} \int_{\mathbb{B}(a,\epsilon)} T \wedge \omega^{n-p}.$$

Note that $\epsilon^{2(n-p)}\pi^{n-p}/(n-p)!$ is the volume of a ball of radius ϵ in $\mathbb{C}^{n-p}$. So the Lelong number is the infinitesimal quantity measuring the $(n-p)$ dimension of the trace measure of T.

Proof. ???? $\square$

Lemma 3.7.21. *Let u be a psh function on Ω. Then for every $x \in \Omega$, we have $\nu(u,x) = \nu(dd^c u, x)$.*

Proof. ??? □

We admit the following fundamental result.

Theorem 3.7.22. *(Siu [36], and [13, Chapter 3] for generalizations) (i) For every constant c, the set $\{x \in \Omega : \nu(T,x) \geq c\}$ is an analytic subset in Ω.*

(ii) There are irreducible analytic subsets $(V_j)_{j\in\mathbb{N}}$ of dimension $n-p$ on Ω and a closed positive (p,p)-current R on Ω such that for every constant $\delta > 0$ the set $\{x : \nu(R,x) \geq \delta\}$ is of dimension $< n-p$, and

$$T = \sum_{j=1}^{\infty} \lambda_j [V_j] + R,$$

for some constant $\lambda_j > 0$.

(iii) For every irreducible analytic subset V in Ω, there exist a proper analytic subset W in V and a constant $\lambda > 0$ such that $\nu(T,x) = \lambda$ for every $x \in V \backslash W$, and $\nu(V,x) \geq \lambda$ for every $x \in V$. We call λ the generic Lelong number of T along V.

The above result can be deduced from a deep theorem due to Demailly saying that one can approximate psh functions locally by those with analytic singularities. We refer to [14, Section 14] for details.

Further notes. Lemma 3.2.1 is from [19]. Section 3.3 generalizing some results from [6, 7] is taken from [38]. Theorem 3.4.1 and Corollary 3.4.2 and Section 3.6 are essentially from [6]. The proof of Theorem 3.4.1 is based on arguments from [21]. The case where $X = \mathbb{C}^n$ of Theorem 3.5.8 was proved in [26]. The proof was then simplified in [6] and [4]. The case where X is projective or Kähler manifolds were proved in [16] and [20]. Finally Theorem 3.5.8 in the current form was proved in [37]. The capacity cap_{BTK} was introduced in [29] as an analogue to the local capacity cap. The capacity cap_{ADS} was introduced in [16], see [3, 35, 22] for related notions and more information. Theorem 3.7.4 generalizes a result from [38].

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