An invitation to pluripotential theory

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August 5, 2023

Around 1940s Georges de Rham (Swiss mathematician, 1903-1990) coined the notion of currents as a higher-dimensional generalization of the notion of distributions by Laurent Schwartz (French mathematician, 1915-2002). Distributions have been nowadays an indispensable part of partial differential equations. We will see that currents also play an important role in developments of various fields in Mathematics.

De Rham had used currents to prove his famous theorem that the de Rham cohomology of a smooth compact manifold (defined using differential forms) is isomorphic to the singular cohomology (a purely topological objects). Another non-trivial application of currents by de Rham is his proof of Hodge theory. At the present time there are other standard proofs of these just-mentioned fundamental results by tools which were not available at the time of proofs of these theorem by de Rham: the first one is proved by sheaf theory, and the second one is proved by using elliptic operators. These examples are, however, to mention an already wide range of applicability of currents at time of its birth.

The notion of currents generalize both differential forms and oriented submanifolds. They can be viewed as differential forms with distribution coefficients. As de Rham explained in his book [1] (see Footnote in Page 34), the term "current" has a physical interpretation as an electrical current in the Euclidean 3-dimensional space. Currents have played an vital role in geometric measure theory. A milestone is the solution of Herbert Federer and Wendell Fleming of the 200 years old Plateau's problem in the framework of integral currents (see [6]). Much of the theory of currents developed by Federer-Fleming and Federer himself later is essential for what forming the main objects of pluripotential theory: closed positive currents.

Subharmonic functions are classical objects in analysis. Around 1940s again, Pierre Lelong [9, 11] and Kiyoshi Oka [12] independently introduced the notion of plurisubharmonic functions which are higher-dimensional generalization of subharmonic functions on complex plane. Plurisubharmonic functions are now one of the most spoken terms in complex geometry. The notion of positivity for currents is due to Pierre Lelong [10]. One thus can speak of closed positive currents generalizing effective analytic cycles (i.e., linear combinations of complex submanifolds with positive coefficients) and closed positive forms. Complex Hessians of plurisubharmonic functions are basic examples of closed positive currents.

Humans are fascinated by geometry in general: counting objects, detecting intersection of patterns, etc. In algebraic geometry, one is interested in studying intersection of algebraic varieties (see [7]). Correspondingly a fundamental question in pluripotential theory is to understand how closed positive currents intersect. In terms of analysis, it means one wants to study the wedge-product of closed positive currents. This question has occupied a central part in this theory since its birth. Fundamental contributions are made by Chern-Levine-Nirenberg, Bedford-Taylor, Demailly, Fornaess-Sibony, etc., for the question of intersection of currents of bi-degree (1,1) (see [2]) and by Dinh-Sibony [5] for intesection of currents of higher bi-degree. In the next paragraphs, we discuss a few applications of intersection of currents which most pertain to my research.

Complex geometry. Holomorphic line bundles over compact complex manifolds are fundamental objects in complex geometry. Their sections are good replacements for "holomorphic functions" on compact complex manifolds. For example, homogeneous polynomials can be interpreted as sections of suitable line bundles on complex projective space. Closed positive $(1,1)$-currents already appear in complex geometry as singular positive Hermitian metrics on holomorphic line bundles. Many important quantities associated to line bundles can be described in terms of intersection of currents such as volume of line bundles or more generally restricted volumes. Such descriptions are crucial to study extension of classical theory on line bundles to transcendental cohomology classes. A standard reference is Demailly's book [3].

Another main topic for which the pluripotential theory has an essential role is the study of special Kähler metrics on Kähler manifolds. The quest of finding special Kähler metrics on compact complex manifolds has been of central importance in complex geometry. A classical example is the uniformization theory of Riemann surfaces where these surfaces are classified by using the existence of Kähler metrics with constant Ricci curvature. Analogues of this uniformization for higher dimensional compact Kähler manifolds (or to be more precise, the existence and uniqueness of special Kähler metrics) have occupied much of attention in complex geometry since more than fifty years. Since the pioneering work of Yau [13] and others, it is well-known that finding such metrics can be boiled down to solving a so-called complex Monge-Ampère equation: such an equation is highly non-linear and of order 2. With the fundamental paper of Kołodziej [8], the pluripotential theory has entered the field as a very important tool.

Complex Dynamics. In dynamics, one is interested in how systems evolve in time (past or future). A main tool is the measure theory. In this case one speaks of measurable dynamical systems or ergodic theory. In ergodic theory, constructing meaningful invariant measures of the dynamical system in consideration is fundamental. In the context of higher dimensional complex dynamics, such measures are usually obtained as the intersection of so-called Green currents which are closed positive currents (of higher bi-degree). Pluripotential theory is hence indispensable for studying ergodic properties of these measures such as equidistribution of periodic points or of forward or backward orbits of subvarieties. A standard reference is the lecture note [4] by Dinh-Sibony.

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