# An invitation to pluripotential theory

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In the 1940s, the Swiss mathematician Georges de Rham (1903-1990) introduced the notion of currents as higher-dimensional generalisations of distributions. The theory of currents has become an indispensable part of the development of various fields of mathematics, in a manner analogous to the fundamental role of distributions in partial differential equations.

The notion of currents generalizes both differential forms and oriented submanifolds. They can be thought of as differential forms with distribution coefficients. As de Rham explained in his book [1] (see footnote on page 34), the term "current" has a physical interpretation as an electric current in the Euclidean 3-dimensional space. Currents have played a vital role in geometric measure theory. A milestone is the solution of the Plateau problem by Herbert Federer and Wendell Fleming within the framework of integral currents (see [4]). A substantial portion of the theory of currents developed by Federer-Fleming, and later by Federer himself, is essential for what are the main objects of pluripotential theory: closed positive currents.

Subharmonic functions are classical objects of analysis. Again around the 1940s, Pierre Lelong [7, 8] and Kiyoshi Oka [9] independently introduced the notion of plurisubharmonic functions which are higher-dimensional generalizations of subharmonic functions in the complex plane. Plurisubharmonic functions are now one of the most widely used notions in complex geometry. Complex Hessians of plurisubharmonic functions are basic examples of closed positive currents.

People are fascinated by geometry in general, e.g., counting objects, detecting intersections of patterns, etc. Within the realm of algebraic geometry, the intersection of algebraic varieties has been a focal point of interest (see [5]). Correspondingly, a fundamental question in pluripotential theory pertains to the intersection of closed positive currents. From an analytic perspective, this entails the study of the wedge-product of closed positive currents. This question has played a central role in the development of this theory since its inception. Major contributions have been made by Chern-Levine-Nirenberg, Bedford-Taylor, Demailly, Boucksom-Eyssidieux-Guedj-Zeriahi for the intersection of $(1,1)$-currents, and by Dinh-Sibony for currents of higher bi-degree (see [3]). In the following paragraphs, we discuss a number of applications of intersection of currents that are most relevant to my research.

**Complex geometry.** Holomorphic line bundles over compact complex manifolds are basic objects in complex geometry. Their sections are natural substitutes for holomorphic functions on compact complex manifolds. For example, homogeneous polynomials can be interpreted as sections of suitable line bundles on complex projective spaces. Closed positive (1, 1)-currents already appear in complex geometry as singular positive Hermitian metrics on holomorphic line bundles. Many important quantities associated with line bundles can be described in terms of intersections of currents such as the volumes of line bundles or, more generally, restricted volumes. Such descriptions are crucial for studying extensions of the classical theory on line bundles to transcendental cohomology classes. A standard reference is Demailly's book [2].

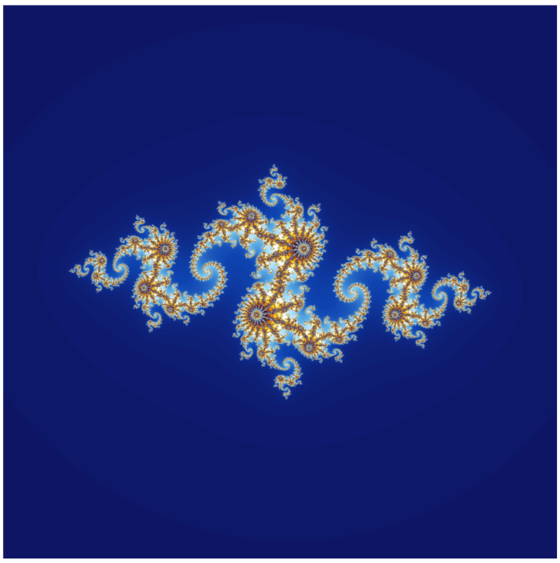
Another main subfield of complex geometry in which the pluripotential theory plays an essential role is the study of special Kähler metrics on Kähler manifolds. The search for special Kähler metrics on compact complex manifolds has been of central importance in complex geometry. A classical example is the uniformization theory of Riemann surfaces where these surfaces are classified by using the existence of Kähler metrics with constant Ricci curvature.


(The Ricci curvature of each surface, from left to right, is +1, 0, -1, respectively)

Analogues of this uniformization for higher dimensional compact Kähler manifolds (or, more precisely, the existence and uniqueness of special Kähler metrics) have attracted much of attention in complex geometry for more than fifty years. Since the pioneering work of Yau [10] and others, it is well-known that finding such metrics boils down to solving a so-called complex Monge-Ampère equation. With the fundamental work of Kołodziej [6], the pluripotential theory has entered the field as a very important tool.

**Complex dynamics and probability.** In dynamics, one is interested in understanding how systems evolve in time (past or future). A main tool is the measure theory. In this regard, one speaks of measurable dynamical systems or ergodic theory. In ergodic theory, constructing meaningful invariant measures of the dynamical system in consideration is fundamental. In the context of higher dimensional complex dynamics, such measures are usually obtained as the intersection of so-called Green currents which are closed positive currents (of higher bi-degree). Pluripotential theory is hence indispensable for studying ergodic properties of these measures.

In the following picture, we see the image of the Julia set of the map $f(z) = z^2+c$, $z \in \mathbb{C}$ with $c = -0.8+0.156i$ (the Julia set is the set where the dynamical system $f$ behaves most wildly). Since the Julia set is usually very difficult to describe geometrically, analytic methods are crucial here: one can depict the Julia set as the support of (or of self-intersection of) the Green current, which is a canonical closed positive current associated with the holomorphic dynamical system in consideration.



One of my main interests in dynamics is the equidistribution problem, in particular, the equidistribution of periodic points or of forward or backward orbits of subvarieties. In complex geometry and probability, one encounters very often similar questions. In my research, I study the equidistribution of Fekete points or zeros of random polynomials as the degree tends to infinity. The pluripotential theory methods, which I employ, can be applied to approach other problems from approximation theory and mathematical physics such as the distribution of beta-ensembles and physical gases (e.g. Coulomb gas). We refer to this Wolfram Demonstrations Project for a graphic illustration of the distribution of zeros of Kac polynomials.

# References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