EXERCISES FOR 05.07

Exercise 1

For $f \in \mathcal{D}^0(\mathbb{R}^3)$ consider $[f]$ as an element of $\mathcal{D}_3(R^3)$ by setting $[f](\omega) = \int_{R^3} f\omega$. Express the boundary current $\partial[f] \in \mathcal{D}_2(\mathbb{R}^3)$ and its mass in terms of the partial derivatives of f.

Exercise 2

Let $1 = r_1 > r_2.... > r_n > ... > 0$ be a decreasing sequence of numbers converging to 0. Let B be unit ball in $\mathbb{R}^2$ and let $A_i \subset B$ denote the annulus of all points z of norm $r_i \geq |z| > r_{i+1}$. Denote by τ the measurable orientation on the unit ball, for which all "odd" A_{2n+1} are equipped with the usual orientation coming from $\mathbb{R}^2$ and all "even" A_{2n} are equipped with the orientation opposite to the canonical one. By definition $T = [B \wedge \tau]$ is a rectifiable current and we have (more or less) proven that is has mass $\pi = \mathcal{H}^2(B)$, see Chapter 4 in the book of Morgan for a picture.

Show by examples (with proofs!) that depending on the sequence r_i the boundary $\partial T \in \mathcal{D}_1(\mathbb{R}^2)$ can be of finite mass (and then a rectifaible current) or of infinite mass.

Exercise 3

For a current $T \in \mathcal{D}_m(\mathbb{R}^n)$ the support $spt(T)$ is defined as the complement $\mathbb{R}^n \setminus U$ of the (well-defined) largest open set U for which $spt(\omega) \subset U$ implies $T(\omega) = 0$ for $\omega \in \mathcal{D}^m(\mathbb{R}^n)$. Using a partition of unity (recall from Ana III), one can see that $x \in \mathbb{R}^n$ is not in the support of T if and only if there is a small ball B around x such that for any $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ with support contained in B one has $T(\omega) = 0$.

Let $U \subset \mathbb{R}^n$ be open. Consider $T = [U] \in \mathcal{D}_n(\mathbb{R}^n)$ defined as usual by integration along U and let ∂T be the boundary current of T.

The task comes only now:

a) Prove that $spt(\partial T)$ is contained in the topological boundary $\partial U = \bar{U} \setminus U$.

b) Show by an example (already discussed in the lecture) that $spt(\partial T)$ may be strictly smaller than the topological boundary ∂U.