EXERCISES FOR 15.06

Exercise 1

Check the correctness of the co-area formula for $A = B(0,1) \subset \mathbb{R}^2$ and $f : A \to \mathbb{R}$ given by $f(x,y) = x^2 + y^2$.

Exercise 2

a) Prove that $e_1^* \wedge e_2^* + e_3^* \wedge e_4^* \in \Lambda^2\mathbb{R}^4$ cannot be written as $\phi \wedge \psi$ for 1-forms $\phi, \psi \in \Lambda^1\mathbb{R}^4$.

b) For any n consider the map $L : \mathbb{R}^n \to \Lambda^{n-1}\mathbb{R}^n$ given as

$$L(w)(v_1, v_2, ..., v_{n-1}) = \det(w, v_1, v_2, ..., v_{n-1}).$$

Prove that the map L is a linear isomorphism, which sends e_i to $(-1)^{i-1}e_1^* \wedge .... \wedge e_{i-1}^* \wedge e_{i+1}^* \wedge ... \wedge e_n^*$. Prove that for any positively oriented ON-basis $(f_1, ..., f_n)$ of $\mathbb{R}^n$ we have $L(w)(f_2, ..., f_n) =< w, f_1 >$.

Exercise 3

Let $U \subset \mathbb{R}^n$ be open. Deduce from the previous exercise that any differential form $\omega \in \mathcal{E}^{n-1}(U)$ has the form $\omega(x) = L(F(x))$ for a smooth vector field $F : U \to \mathbb{R}^n$. Show that $d\omega = div(F)\, dx_1 \wedge ... \wedge dx_n$, where $div(F) = \sum_i \frac{\delta F_i}{\delta x_i}$ is the divergence of F.

Deduce that the theorem of Stokes for domain $(V, \partial V) \subset U$ is exactly the divergence theorem of Gauss (look up the meaning). In the case $n = 2$ the theorem takes the form (after changing a sign) of the theorem of Green (recall what it means).

Exercise 4

Consider the manifold with boundary $M \subset \mathbb{R}^3$ given as the set of all (x_1, x_2, x_3) which satisfy $\sum x_i = 0$ and $\sum x_i^2 \leq 1$. Give M an orientation and compute the integral $\int_M \omega$, where $\omega = dx_1 \wedge dx_2 + 2dx_1 \wedge dx_3 + 3dx_2 \wedge dx_3$.