EXERCISES 1

1. Basic level

Problem 1. *Let X be a metric space, $B(X)$ be the Borel sigma algebra and $\mu : 2^X \to \mathbb{R}$ be a Borel outer measure. Show that an outer measure $\nu : 2^X \to \mathbb{R}$ given by*

$$\nu(A) = \inf_{B \in B(X), A \subset B} \mu(B)$$

is Borel regular.

Problem 2. *Let U be an open subset of $\mathbb{R}^k$ and $f : U \to \mathbb{R}^n$ be a C^1-map (continuously differentiable). Show that*

(1) *$dim_H(f(U)) \leq k$,*
(2) *there are examples of f such that $H^k(f(U)) = \infty$,*
(3) *there are examples of f such that $H^k(f(U)) = 0$,*
(4) *if there exists $x \in U$, s.t. $D_x(f)$ is non-degenerate then $H^k(f(u)) > 0$.*

2. Intermediate level

Problem 3. *Construct Cantor sets with*

(1) *positive 1 dimensional Hausdorff measure,*
(2) *Hausdorff dimension zero.*

Hint: Vary the size of intervals taken out at each step.

3. Advanced level

Problem 4. *Provide a lower bound on the Hausdorff $\log_3 2$-measure of the standard Cantor set.*

Hint for solution 1: Investigate connections between notions of Holder map and Hausdorff dimension. Construct a Holder map from Cantor set to the segment of the real line.

Hint for solution 2: Start with an arbitrary open covering and improve it. Use the notion of Lebesgue's number.

4. Unrated bonus

Problem 5. *Prove that symmetrization doesn't increase the diameter of a set.*