EXERCISES 1

1. Basic level

Problem 1. *Let $\bar{B}(0,1)$ be the unit disk in $\mathbb{R}^2$. Calculate $\Theta_2(\bar{B}(0,1),x)$ the 2-dimensional density of $\bar{B}(0,1)$.*

Here we describe a version of Hausdorff measure where we use coverings by balls instead of coverings by arbitrary sets.

Definition 1. For a metric space X, $s \in [0,\infty)$ and $A \subset X$ we set

$$B^s_\delta(A) = \inf\{\sum_1^\infty \omega_s\big(\frac{diam A_i}{2}\big)^s \;:\; A \subset \cup_i A_i, \text{where } A_i \text{ are balls of diam} < \delta\},$$

$$B^s(A) = \lim_{\delta\to 0} B^s_\delta(A).$$

The following simple facts were discussed in the lecture

(1) $B^s(A) \geq H^s(A)$,
(2) $B^s(A) \leq 2^s H^s(A)$.

Problem 2. *Using the theorem of Besicovitch show that in the case of Euclidean space $X = \mathbb{R}^n$,*

$$B^n(A) = H^n(A),$$

for every set A.

2. Intermediate level

Problem 3. *Let S^2 be the unit sphere in $\mathbb{R}^3$. Calculate $\Theta_2(S^2,x)$ the 2-dimensional density of S^2.*

3. Advanced level

Problem 4. *Let $A = \cup_{k=0}^\infty[\frac{1}{2}10^{-k}, 10^{-k}] \subset \mathbb{R}$. Does A have a 1-dimensional density at 0?*