EXERCISES 1

Problem 1. *Let μ be a Borel regular measure on $\mathbb{R}^n$ s.t.,*

1. $\mu(A) = \mathcal{H}^m(A)$, *for every subset A of an m-dimensional affine subspace,*
2. *For every $A \subset \mathbb{R}^n$ and a 1-Lipschitz map $f : A \to \mathbb{R}^n$ we have* $\mu(f(A)) \leq \mu(A)$.

Show that $\mu(A) = \mathcal{H}^m(A)$, for every subset A of an m-dimensional (compact) C^1-submanifold.

Problem 2. *Let μ be a Radon measure on $\mathbb{R}^n$, $g : \mathbb{R}^n \to [0, \infty)$ be a locally L^1 function with respect to μ. We define*

1. *an outer measure $g\mu$ given by $g\mu(A) = \inf_{A \subset B, B \in B(X)} \int_B g d\mu$, where $B(X)$ is a Borel sigma algebra,*
2. *a functional $L : C_c(\mathbb{R}^n) \to \mathbb{R}$ given by $L(f) = \int_{\mathbb{R}^n} fgd\mu$, where $C_c(R^n)$ denotes the set of continuous functions with bounded supports.*

Show that $g\mu$ is a Radon measure and equality $L(f) = \int_{\mathbb{R}^n} fdg\mu$ holds for every $f \in C_c(\mathbb{R}^n)$.

Problem 3. *For a point $x \in \mathbb{R}$ we denote by δ_x a Dirac measure with a support x. Find a limit of a sequence of measures $\{\mu_k\}$ given by*

1. $\mu_k = \frac{1}{k}(\delta_0 + \delta_{\frac{1}{k}} + \delta_{\frac{2}{k}} + \dots \delta_{\frac{k-1}{k}})$,
2. $\mu_k = \frac{1}{k}(\delta_0 + \delta_{\frac{1}{k}} + \delta_{\frac{2}{k}} + \dots \delta_{\frac{k-1}{k}} + \delta_{\frac{2k}{2k}} + \delta_{\frac{2k+1}{2k}} + \delta_{\frac{2k+2}{2k}} + \dots \delta_{\frac{4k-1}{k}})$,

Problem 4. *Let $g_j : \mathbb{R}^n \to [0, \infty)$ be continuous functions such that $spt(g_j) \subset B(0, \frac{1}{j})$ and $\int_{\mathbb{R}^n} g_j d\mathcal{H}^n = 1$. Show that $g_j \mathcal{H}^n \underset{j \to \infty}{\to} \delta_0$.*