EXERCISES 4

Problem 1. *Let X be a metric space, I be a set and $f_\alpha : X \to \mathbb{R}$ be a family of 1-Lipschitz functions, $\alpha \in I$. Suppose that there exists $x \in X$ s.t., $\inf_{\alpha\in I} f_\alpha(x) > -\infty$. Show that for every $y \in X$ we have $\inf_{\alpha\in I} f_\alpha(y) > -\infty$ and a function g given by $g(y) = \inf_{\alpha\in I} f_\alpha(y)$ is 1-Lipschitz.*

Problem 2. *Let $A \subset \mathbb{R}^n$ be a compact set. Denote*

$$f(x) := d(A, x) = min\{d(x, a)|a \in A\}.$$

Prove that f is differentiable at $x \in \mathbb{R}^n \setminus A$ iff there exists unique point $a \in A$ s.t., $d(x, a) = d(x, A)$. What is $Df(x)$ in terms of x and a?

Hint: Consult the proof of Theorem 3.14 in the book Evans/Gariepy.

Problem 3. *We recall the definition of k-Jacobian from the lecture. For a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ and $k \in \mathbb{Z}_+$ the k-Jacobian of L is given by*

$$J_k(L) := \begin{cases} \sup_{e_1,\dots,e_k\in\mathbb{R}^n,|e_i|=1,\langle e_i,e_j\rangle=0,i\neq j} \mathcal{H}^k(P_{L(e_1),\dots,L(e_n)}), k \leq n, \\ 0, k > n, \end{cases}$$

where $P_{L(e_1),\dots,L(e_k)}$ denotes a k-dimensional parallelotope spanned by $L(e_1), \dots, L(e_k)$. Let $V = (KerL)^\perp$ and let $P : \mathbb{R}^n \to V$ be an orthogonal projection from $\mathbb{R}^n$ to V. A linear map $K : V \to \mathbb{R}^m$ is given by $K = L \circ P$. Show that

(1) $J_k(L) = J_k(K)$, for every k,
(2) $J_n(L) = \sqrt{\det(L \circ L^*)}$,
(3) $J_m(L) = \sqrt{\det(L^* \circ L)}$.

Problem 4. *Write $C = \{(x, y, z) \in \mathbb{R}^n|x^2 + y^2 = z^2\}$ as a union of 2 graphs of a functions. Compute $\mathcal{H}^2(C \cap B(0, r))$. Calculate $\Theta_2(C, x)$ the 2-dimensional density of C.*