3.4.4 Applications

A. Integrals over balls.

THEOREM 3.12 (Polar coordinates). *Let $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-summable. Then*

$$\int_{\mathbb{R}^n} g\,dx = \int_0^\infty \left(\int_{\partial B(r)} g\,d\mathcal{H}^{n-1} \right) dr.$$

In particular,

$$\frac{d}{dr}\left(\int_{B(r)} g\,dx \right) = \int_{\partial B(r)} g\,d\mathcal{H}^{n-1}$$

for $\mathcal{L}^1$-a.e. $r > 0$.

Proof. Set $f(x) = |x|$; then for $x \neq 0$ we have

$$Df(x) = \frac{x}{|x|}, \quad Jf(x) = 1.$$

□

B. Integration over level sets.

THEOREM 3.13 (Integration over level sets). *Assume $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous.*

(i) *Then*

$$\int_{\mathbb{R}^n} |Df|\,dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{f = t\})\,dt.$$

(ii) *Assume also*

$$\operatorname{ess\,inf} |Df| > 0,$$

and suppose $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}^n$-summable. Then

$$\int_{\{f>t\}} g\,dx = \int_t^\infty \left(\int_{\{f=s\}} \frac{g}{|Df|}\,d\mathcal{H}^{n-1} \right) ds.$$

(iii) *In particular,*

$$\frac{d}{dt}\left(\int_{\{f>t\}} g\,dx \right) = -\int_{\{f=t\}} \frac{g}{|Df|}\,d\mathcal{H}^{n-1}$$

Remark. Compare (i) with the coarea formula for BV functions, proved later in Theorem 5.9. □

Proof. 1. To prove (i), observe that $Jf = |Df|$.

2. Write $E_t := \{f > t\}$ and use Theorem 3.11 to calculate

$$\begin{aligned}\int_{\{f>t\}} g\,dx &= \int_{\mathbb{R}^n} \chi_{E_t} \frac{g}{|Df|} Jf\,dx \\ &= \int_{-\infty}^{\infty} \left(\int_{\partial E_s} \frac{g}{|Df|} \chi_{E_t}\,d\mathcal{H}^{n-1} \right) ds \\ &= \int_t^\infty \left(\int_{\partial E_s} \frac{g}{|Df|}\,d\mathcal{H}^{n-1} \right) ds.\end{aligned}$$

This gives (ii), and (iii) follows. □

C. Distance functions.

THEOREM 3.14 (Level sets of distance functions). *Assume $K \subset \mathbb{R}^n$ is a nonempty compact set and write*

$$d(x) := \operatorname{dist}(x, K) \quad (x \in \mathbb{R}^n).$$

Then for each $0 < a < b$ we have

$$\int_a^b \mathcal{H}^{n-1}(\{d = t\})\,dt = \mathcal{L}^n(\{a \leq d \leq b\}).$$

Proof. 1. Given $x \in \mathbb{R}^n$, select $c \in K$ so that $d(x) = |x - c|$. Then for any other point $y \in \mathbb{R}^n$, we have

$$d(y) - d(x) \leq |y - c| - |x - c| \leq |x - y|.$$

Interchanging x and y, we see that $|d(y) - d(x)| \leq |x - y|$; consequently,

$$\operatorname{Lip}(d) \leq 1.$$

Rademacher's Theorem therefore implies that the distance function is

2. Select any point $x \in \mathbb{R}^n - K$ at which $Dd(x)$ exists. Then $|Dd(x)| \leq 1$, since $\operatorname{Lip}(d) \leq 1$. As above, select $c \in K$ so that $d(x) = |x - c|$. Then

$$d(tx + (1-t)c) = t|x - c|$$

for all $0 \leq t \leq 1$; and therefore

$$|x - c| = Dd(x) \cdot (x - c) \leq |Dd(x)||x - c|.$$

Thus $|Dd(x)| \geq 1$.

3. It follows that

$$|Dd| = 1 \qquad \mathcal{L}^n\text{-a.e. in } \mathbb{R}^n - K.$$

We may consequently invoke Theorem 3.13 to finish the proof. □

3.5 References and notes

The primary reference is again Federer [F, Chapters 1 and 3]. Theorem 3.1 is from Simon [S, Section 5.1]. The proof of Rademacher's Theorem, which we took from [S, Section 5.2], is due to Morrey (cf. [My, p. 65]). Theorem 3.3 in Section 3.1 is [F, Section 3.2.8]. See Clarke [C] for more on calculus for Lipschitz continuous functions.

The discussion of linear maps and Jacobians in Section 3.2 is strongly based on Hardt [H]. S. Antman helped us with the proof of the Polar Decomposition Theorem, and A. Damlamian provided the calculations for the Binet–Cauchy formula. See also Gantmacher [Ga, pages 9–12, 276–278].

The proof of the area formula in Section 3.3, originating with [F, Sections 3.2.2–3.2.5], follows Hardt's exposition in [H]. Our proof in Section 3.4 of the coarea formula also closely follows [H], and is in turn based on [F, Sections 3.2.8–3.2.13]. Theorem 3.14 is from [F, Section 3.2.34].

Chapter 4

Sobolev Functions

In this chapter we study *Sobolev functions* on $\mathbb{R}^n$, functions w
eak first partial derivatives belonging to some L^p space. The v
ous Sobolev spaces have good completeness and compactness pro
ties and consequently are often the proper settings for application
functional analysis to, for instance, linear and nonlinear PDE theo

Now, as we will see, by definition, integration-by-parts is valid
Sobolev functions. It is, however, far less obvious to what extent
other rules of calculus are valid. We intend to investigate this gen
question, with particular emphasis on pointwise properties of Sob
functions.

Section 4.1 provides basic definitions. In Section 4.2 we derive
ous ways of approximating Sobolev functions by smooth functions.
tion 4.3 interprets boundary values of Sobolev functions using tr
and Section 4.4 discusses extending such functions off Lipschitz
tinuous domains. We prove the fundamental Sobolev-type inequal
in Section 4.5, an immediate application of which is the compact
theorem in Section 4.6. The key to understanding the fine prope
of Sobolev functions is capacity, introduced in Section 4.7 and util
in Sections 4.8 and 4.9.

4.1 Definitions and elementary properties

Throughout this chapter, U denotes an open subset of $\mathbb{R}^n$.

DEFINITION 4.1. *Assume $f \in L^1_{\text{loc}}(U)$ and $i \in \{1, \ldots, n\}$. W
$g_i \in L^1_{\text{loc}}(U)$ is the* ***weak partial derivative*** *of f with respect to
U if*

$$\int_U f\phi_{x_i}\,dx = -\int_U g_i\phi\,dx$$