Efficient estimators for functionals of Markov chains with parametric marginals

Spiridon Penev ∗
University of New South Wales

Hanxiang Peng
University of Mississippi

Anton Schick †
Binghamton University

Wolfgang Wefelmeyer
Universität Siegen

Abstract

Suppose we observe a geometrically ergodic Markov chain with a parametric model for the marginal, but no (further) information about the transition distribution. Then the empirical estimator for a linear functional of the joint law of two successive observations is no longer efficient. We construct an improved estimator and show that it is efficient. The construction is similar to a recent one for bivariate models with parametric marginals. The result applies to discretely observed parametric continuous-time processes.


Key words and Phrases. Least squares estimator, series estimator, orthonormal basis, efficient influence function, reversible Markov chain, discretely observed diffusion, constrained model.

1. Introduction

Let \( X_0, \ldots, X_n \) be observations from a geometrically ergodic Markov chain with arbitrary state space. We want to estimate a linear functional \( E[h(X_0, X_1)] \) of the joint stationary law of two successive observations. If nothing is known about the distribution of the chain, then the empirical estimator \( \hat{H} = \frac{1}{n} \sum_{k=1}^{n} h(X_{k-1}, X_k) \) is efficient; see Penev (1990, 1991), Bickel (1993), and Greenwood and Wefelmeyer (1995). Suppose now that we have a finite-dimensional parametric model \( F_\theta, \theta \in \Theta \), for the marginal stationary law of the chain, but that we cannot or do not want to specify anything (else) about the transition distribution. Then we can construct better estimators for \( E[h(X_0, X_1)] \). This includes the case where the transition distribution follows a parametric model involving the parameter \( \theta \) and perhaps further parameters, but

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that we do not know this model or are not sure that it is correct. Our model is nonparametric, with a constraint that involves the unknown parameter \( \vartheta \).

Our results apply in particular to parametric continuous-time Markov processes that are discretely observed at fixed time intervals. Under such an observation scheme, estimators for the parameter \( \vartheta \) were constructed in parametric diffusion processes by Pedersen (1995a,b), Bibby and Sørensen (1995, 1996, 1997, 2001), Kessler and Sørensen (1999) and Kessler (2000), and in general parametric continuous-time processes by Kessler and Sørensen (2002). These estimators could be used to estimate the coefficients of the diffusion and then linear functionals \( E[h(X_0, X_1)] \) as considered here. If the diffusion model is correctly specified, and if the estimators for \( \vartheta \) are efficient (or nearly so), this would lead to better estimators for \( E[h(X_0, X_1)] \) than ours. However, the marginals of a discretely observed process can be modeled much better than the dynamics. Estimators of \( E[h(X_0, X_1)] \) based on a misspecified continuous-time model will usually be inconsistent. In contrast, our estimator uses only the information in the parametric model for the marginal law and is always \( \sqrt{n}/2 \)-consistent and asymptotically normal unless the marginals are misspecified.

Our results are closely related to results for bivariate models, which we recall first. Let \((Y_1, Z_1), \ldots, (Y_n, Z_n)\) be i.i.d. bivariate random variables with joint law \( Q \). We want to estimate a linear functional \( E[h(Y, Z)] = \int h \, dQ \) for a fixed function \( h \in L_2(Q) \). A natural estimator is the empirical estimator \( \hat{H}_{biv} = \frac{1}{n} \sum_{k=1}^{n} h(Y_k, Z_k) \). If additional structural assumptions on the joint law hold, this estimator can be improved.

Assume first that the marginals \( F \) and \( G \) of \( Q \) are known. In this case there is a large class of unbiased estimators. Indeed,

\[
\hat{H}_{biv}(a, b) = \frac{1}{n} \sum_{k=1}^{n} \left( h(Y_k, Z_k) - a(Y_k) - b(Z_k) \right)
\]

is unbiased for each \( a \in L_{2,0}(F) \) and \( b \in L_{2,0}(G) \). Here, for any measure \( \mu \),

\[
L_{2,0}(\mu) = \{ h \in L_2(\mu) : \int h \, d\mu = 0 \}.
\]

The smallest variance is achieved by \( \hat{H}_{biv}(a_Q, b_Q) \), where \( a_Q \) and \( b_Q \) minimize \( E[(h(Y, Z) - a(Y) - b(Z))^2] \) over \( a \in L_{2,0}(F) \) and \( b \in L_{2,0}(G) \). Bickel, Ritov and Wellner (1991) have shown that any estimator equivalent to \( \hat{H}_{biv}(a_Q, b_Q) \) is efficient, and have obtained such an estimator using the modified minimum chi-square estimator of Deming and Stephan (1940). Peng and Schick (2002) give a more direct construction, estimating \( a_Q \) and \( b_Q \) by a series estimator in terms of orthonormal bases \( v_1, v_2, \ldots \) of \( L_{2,0}(F) \) and \( w_1, w_2, \ldots \) of \( L_{2,0}(G) \). Their estimator is of the form

\[
\frac{1}{n} \sum_{k=1}^{n} \left( h(Y_k, Z_k) - \sum_{i=1}^{M_n} \hat{\alpha}_i v_i(Y_k) - \sum_{j=1}^{N_n} \hat{\beta}_j w_j(Z_k) \right),
\]
where $M_n$ and $N_n$ are integers that tend slowly to infinity with the sample size $n$, and $\hat{\alpha}_{n1}, \ldots, \hat{\alpha}_{nM_n}, \hat{\beta}_{n1}, \ldots, \hat{\beta}_{nN_n}$ are chosen to minimize

$$\sum_{k=1}^{n} \left( h(Y_k, Z_k) - \sum_{i=1}^{M_n} \alpha_i v_i(Y_k) - \sum_{j=1}^{N_n} \beta_j w_j(Z_k) \right)^2.$$  

Of course, $\hat{\alpha}_{n1}, \ldots, \hat{\alpha}_{nM_n}, \hat{\beta}_{n1}, \ldots, \hat{\beta}_{nN_n}$ are simply least squares estimators for the response vector $H = (h(Y_1, Z_1), \ldots, h(Y_n, Z_n))^T$ and the design matrix with $k$-th row formed by $(v_1(Y_k), \ldots, v_{M_n}(Y_k), w_1(Z_k), \ldots, w_{N_n}(Z_k))$.

The assumption of known marginals is not always justifiable. A more realistic assumption is that the marginals depend on some unknown parameter $\vartheta$, i.e., $F = F_\vartheta$ and $G = G_\vartheta$. This model is considered by Peng and Schick (2003). They replace, in the above construction, $v_i$ by $v_i(\cdot, \vartheta)$ and $w_i$ by $w_i(\cdot, \vartheta)$, where $v_1(\cdot, \vartheta), v_2(\cdot, \vartheta), \ldots$ is a basis for $L_{2,0}(F_\vartheta)$; $w_1(\cdot, \vartheta), w_2(\cdot, \vartheta), \ldots$ is a basis for $L_{2,0}(G_\vartheta)$; and $\vartheta$ is a $n^{1/2}$-consistent estimator of $\vartheta$. They show under mild assumptions on the bases that the resulting estimator $\hat{H}_{biv}^*$ has an expansion

$$\hat{H}_{biv}^* = \frac{1}{n} \sum_{k=1}^{n} \left( h(Y_k, Z_k) - a_Q(Y_k) - b_Q(Z_k) \right) + D_{biv}(\hat{\vartheta} - \vartheta) + o_p(n^{-1/2})$$

if the parametric models for the marginals are Hellinger differentiable at $\vartheta$ with derivatives $\phi_\vartheta$ and $\gamma_\vartheta$, say. Here

$$D_{biv} = E[a_Q(Y)\phi_\vartheta(Y)] + E[b_Q(Z)\gamma_\vartheta(Z)].$$

Bickel and Kwon (2001) have suggested that results on efficient estimation for bivariate models carry over to geometrically ergodic Markov chains. They point out that the calculation of efficient influence functions is identical if one parametrizes the Markov chain by the joint law of two successive observations, which corresponds to the description of the bivariate model by the joint law of $(Y, Z)$. See also the discussion of Greenwood, Schick and Wiegelmeier (2001). Bickel and Kwon also suggest that the construction of efficient estimators for bivariate models should carry over to corresponding Markov chain models. In this paper we carry out this program for Markov chains with a parametric model $F_\vartheta$, $\vartheta \in \Theta$, for the marginal stationary law. For the corresponding bivariate model we have $G_\vartheta = F_\vartheta$. Recall that the observations for the Markov chain are $X_0, \ldots, X_n$. The Markov chain analog $\hat{H}^*$ of $H_{biv}^*$ is obtained by replacing the pairs $(Y_k, Z_k)$ by pairs $(X_{k-1}, X_k)$ of successive observations. We show in Section 2 that the analog of (1.1) is

$$\hat{H}^* = \frac{1}{n} \sum_{k=1}^{n} \left( h(X_{k-1}, X_k) - a_Q(X_{k-1}) - b_Q(X_k) \right) + D^\top(\hat{\vartheta} - \vartheta) + o_p(n^{-1/2})$$

under the assumption that the parametric model for the marginal stationary law is Hellinger differentiable at $\vartheta$ with derivative $\phi_\vartheta$. Now $a_Q$ and $b_Q$ are minimizers of $E[(h(X_0, X_1) - a(\vartheta) - b_{Q}(\vartheta))]$. 

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$b(X_1))^2$] over $a$ and $b$ in $L_{2,0}(F_\vartheta)$, and

$$D = E[(aQ(X_0) + bQ(X_0))\phi_\vartheta(X_0)].$$

Kessler, Schick and Wefelmeyer (2001) have constructed an efficient estimator $\hat{\vartheta}$ of $\vartheta$. If such an estimator is used, $\hat{H}^*$ is also efficient, as shown in Section 3.

We note that the results of this paper can be adapted to the case of a reversible chain. If the chain is known to be reversible, then $Q$ is symmetric, $Q(dx,dy) = Q(dy,dx)$, and we can improve the empirical estimator $\hat{H} = \frac{1}{n}\sum_{k=1}^n h(X_{k-1},X_k)$ by symmetrization,

$$\hat{H}_{sym} = \frac{1}{2n}\sum_{k=1}^n (h(X_{k-1},X_k) + h(X_k,X_{k-1})).$$

If $Q$ is completely unknown, $\hat{H}_{sym}$ is efficient; see Greenwood and Wefelmeyer (1999) and, for a simpler argument, Greenwood, Schick and Wefelmeyer (2001). If we have a parametric model $F_\vartheta$ for the marginal, it is natural to consider the symmetric improvement

$$\hat{H}^*_{sym} = \hat{H}_{sym} - \frac{1}{2n}\sum_{k=1}^n \sum_{i=1}^{M_n} \alpha_{ni}(v_i(X_{k-1},\hat{\vartheta}) + v_i(X_k,\hat{\vartheta})),$$

where $\alpha_{n1}, \ldots, \alpha_{nM_n}$ are chosen to minimize

$$\sum_{k=1}^n \left( h(X_{k-1},X_k) + h(X_k,X_{k-1}) - \sum_{i=1}^{M_n} \alpha_i(v_i(X_{k-1},\hat{\vartheta}) + v_i(X_k,\hat{\vartheta})) \right)^2.$$

If $\hat{\vartheta}$ is efficient, so is $\hat{H}^*_{sym}$. Efficient estimators for $\vartheta$ in reversible Markov chain models with parametric marginals are constructed in Kessler, Schick and Wefelmeyer (2001). We note that the diffusion models referred to above are reversible.

2. Stochastic expansion of the estimator

Let $X_0, \ldots, X_n$ be observations from a stationary Markov chain on an arbitrary state space $S$ with countably generated $\sigma$-field, transition distribution $K(x,dy)$, and marginal law $F_\vartheta(dx)$, with $\vartheta$ in an open subset of $\mathbb{R}^r$. Let $Q(dx,dy)$ denote the law of two successive observations. We want to estimate an expectation $E[h(X_0,X_1)] = \int h\,dQ$ for a fixed $Q$-square-integrable function $h$.

Let $v_1(\cdot,\vartheta), v_2(\cdot,\vartheta), \ldots$ be an orthonormal basis for $L_{2,0}(F_\vartheta)$, and let $\hat{\vartheta}$ be a $n^{1/2}$-consistent estimator of $\vartheta$. Our estimator for $\int h\,dQ$ is

$$\hat{H}^* = \frac{1}{n}\sum_{k=1}^n \left( h(X_{k-1},X_k) - \sum_{i=1}^{M_n} \alpha_{ni}v_i(X_{k-1},\hat{\vartheta}) - \sum_{j=1}^{N_n} \beta_{nj}v_j(X_k,\hat{\vartheta}) \right),$$
where $M_n$ and $N_n$ are integers, and $\hat{\alpha}_{n1}, \ldots, \hat{\alpha}_{nM_n}, \hat{\beta}_{n1}, \ldots, \hat{\beta}_{nN_n}$ are chosen to minimize

$$
\sum_{k=1}^{n} \left( h(X_{k-1}, X_k) - \sum_{i=1}^{M_n} \alpha_i v_i(X_{k-1}, \hat{\vartheta}) - \sum_{j=1}^{N_n} \beta_j v_j(X_k, \hat{\vartheta}) \right)^2.
$$

We prove a stochastic expansion for $\hat{H}^*$ for a fixed parameter $\vartheta_0$ under the following assumptions on the Markov chain, the parametric model, and the basis.

**Assumption 1.** The chain is geometrically ergodic in the $L_2$ sense: There is a $\lambda < 1$ such that for all $f \in L_{2,0}(F_{\vartheta_0})$,

$$
\int \left( \int K(x, dy) f(y) \right)^2 F_{\vartheta_0}(dx) \leq \lambda \int f^2 dF_{\vartheta_0}.
$$

**Assumption 2.** The chain fulfills the following minorization criterion: There is an $\eta > 0$ such that for $F_{\vartheta_0}$-a.a. $x$ and all measurable $B$,

$$
K(x, B) \geq \eta F_{\vartheta_0}(B).
$$

**Assumption 3.** The parametric model is Hellinger differentiable at $\vartheta_0$: There is a function $\varphi \in L_{2,0}(F_{\vartheta_0})^*$ such that

$$
\int \left( \sqrt{dF_{\vartheta_0} + t} - \sqrt{dF_{\vartheta_0} - \frac{1}{2} t^\top \varphi \sqrt{dF_{\vartheta_0}}} \right)^2 = o(\|t\|^2).
$$

Moreover, the Fisher information matrix $\int \varphi^\top dF_{\vartheta_0}$ is positive definite.

**Assumption 4.** The basis elements are bounded: For each $i = 1, 2, \ldots$ and each $\vartheta \in \Theta$,

$$
\sup_{x \in S} |v_i(x, \vartheta)| < \infty.
$$

**Assumption 5.** The basis elements are locally Lipschitz: There are a neighborhood of $\vartheta_0$ and constants $L_1, L_2, \ldots$ such that, for all $s$ and $t$ in the neighborhood,

$$
\sup_{x \in S} |v_i(x, t) - v_i(x, s)| \leq L_i \|t - s\|.
$$

Assumptions 1 and 3 were used in Kessler, Schick and Wefelmeyer (2001). Assumption 2 is equivalent to

$$
Q(A \times B) \geq \eta F_{\vartheta_0}(A) F_{\vartheta_0}(B)
$$

for all measurable $A$ and $B$. This version was used for corresponding bivariate models in Bickel, Ritov and Wellner (1991) and Peng and Schick (2002, 2003). Assumption 2 is used by Glynn
and Ormoneit (2002) to prove a Hoeffding inequality for Markov chains that will be applied in the proof of our result. Assumptions 4 to 5 are as in Peng and Schick (2003).

To state our result, set \( m_n = M_n \lor N_n \), and let

\[
\Delta_n = \sum_{i=1}^{m_n} \sup_{x \in S} |v_i(x, \vartheta_0)|^2 \quad \text{and} \quad \Gamma_n = \sum_{i=1}^{m_n} L_i^2.
\]

**Theorem 1.** Let Assumptions 1 to 5 hold, and let \( \hat{\vartheta} \) be a \( n^{1/2} \)-consistent estimator for \( \vartheta_0 \). Assume that \( M_n \) and \( N_n \) tend to infinity, and

\[
\frac{m_n^2 (\Delta_n + \Gamma_n)}{n} \to 0 \quad \text{and} \quad \frac{\Gamma_n \log(1 + \Gamma_n)}{n} \to 0.
\]

Then \( \hat{H}^* \) has the stochastic expansion

\[
\hat{H}^* = \frac{1}{n} \sum_{k=1}^{n} \left( h(X_{k-1}, X_k) - a_Q(X_{k-1}) - b_Q(X_k) \right) + D^\top (\hat{\vartheta} - \vartheta_0) + o_p(n^{-1/2}),
\]

where \( a_Q \) and \( b_Q \) minimize

\[
\int (h(x, y) - a(x) - b(y))^2 Q(dx, dy)
\]

over \( a, b \in L_2(F_{\vartheta_0}) \), and

\[
D = \int (a_Q + b_Q) \varphi dF_{\vartheta_0}.
\]

A specific basis with these properties in the case of real state space and continuous distribution functions \( F_{\vartheta} \) is given in Peng and Schick (2003). It is of the form \( v_i(x, \vartheta) = \sqrt{2} \cos(i\pi F_{\vartheta}(x)) \). For this basis, Assumption 4 holds, and Assumption 5, with \( L_i = c_i \), follows from Assumption 3. In this case the rate conditions (2.1) are equivalent to \( m_n^2 / n \to 0 \).

Suppose now that \( \hat{\vartheta} \) is asymptotically linear, i.e.,

\[
n^{1/2}(\hat{\vartheta} - \vartheta_0) = n^{-1/2} \sum_{k=1}^{n} J(X_{k-1}, X_k) + o_p(1)
\]

for some \( J \in L_2^r(Q) \) with \( E(J(X_0, X_1) \mid X_0) = 0 \). Then \( \hat{H}^* \) is asymptotically normal. We show in Section 3 that \( \hat{H}^* \) is also efficient if \( \hat{\vartheta} \) is efficient.

Our proof is similar to that for the bivariate model in Peng and Schick (2003). Their exponential inequality, Lemma 2, must be replaced by an appropriate version for Markov chains, which we state first.

**Lemma 1.** Let \( B = \{t \in \mathbb{R}^q : \|t\| \leq C\} \) be the closed ball of radius \( C \) in \( \mathbb{R}^q \). Let \( u_t, t \in B, \) be a family of functions on \( S \) such that \( u_0 = 0 \) and, for some \( L > 0 \),

\[
|u_t(x) - u_s(x)| \leq L\|t - s\|, \quad x \in S; \; s, t \in B.
\]

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Suppose Assumption 2 holds. Then
\[
U_n(t) = \frac{1}{n} \sum_{k=1}^{n} (u_t(X_k) - \int u_t \, dF_{\hat{\vartheta}_0}), \quad t \in B,
\]
fulfills, for each \( \varepsilon > 0 \) and \( n \varepsilon > 2LC/\eta \),
\[
P(\sup_{t \in B} |U_n(t)| > 3\varepsilon) \leq 2 \left( 1 + \frac{2q^{1/2}LC}{\varepsilon} \right)^{q} \exp \left( - \frac{\eta^2(n \varepsilon - 2LC/\eta)^2}{2nL^2C^2} \right).
\]

The proof of this result is identical to that of Lemma 2 in Peng and Schick (2003) for the case of independent observations. Instead of the classical Hoeffding inequality we now use the Markovian extension given by Glynn and Ormoneit (2002).

**Proof of Theorem 1.** It suffices to show
\[\tag{2.2} \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{M_n} (\hat{a}_n v_i(X_{k-1}, \hat{\vartheta}) - a_Q(X_{k-1})) + \left( \int a_Q \varphi^\top \, dF_{\hat{\vartheta}_0} \right) (\hat{\vartheta} - \vartheta_0) = o_p(n^{-1/2}), \]
\[\tag{2.3} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{N_n} (\hat{b}_n v_j(X_k, \hat{\vartheta}) - b_Q(X_k)) + \left( \int b_Q \varphi^\top \, dF_{\hat{\vartheta}_0} \right) (\hat{\vartheta} - \vartheta_0) = o_p(n^{-1/2}). \]

We only show (2.2); (2.3) is similar. Let \( V_m \) denote the linear span of \( v_1(\cdot, \vartheta_0), \ldots, v_m(\cdot, \vartheta_0) \). Let \( a_n = \sum_{i=1}^{M_n} a_{ni} v_i(\cdot, \vartheta_0) \) and \( b_n = \sum_{j=1}^{N_n} b_{nj} v_j(\cdot, \vartheta_0) \) be chosen to minimize \( \int (h(x, y) - a(x) - b(x))^2 Q(dx, dy) \) over \( a \in V_{M_n} \) and \( b \in V_{N_n} \). As shown in Peng and Schick (2002), \( a_n \) and \( b_n \) are uniquely determined, and \( a_n \to a_Q \) and \( b_n \to b_Q \) in \( L_2(F_{\hat{\vartheta}_0}) \). Assumption 1 and the Cauchy-Schwarz inequality imply that for \( k = 1, 2, \ldots \) and \( f \in L_2(Q) \),
\[
|E[f(X_0, X_1)f(X_{k-1}, X_k)]| \leq \lambda^{(k-1)/2} E[f^2(X_0, X_1)].
\]
Thus we obtain for \( C = 2/(1 - \lambda^{1/2}) \) that
\[\tag{2.4} E \left[ \left( \frac{1}{n} \sum_{k=1}^{n} f(X_{k-1}, X_k) \right)^2 \right] \leq \frac{C}{n} E[f^2(X_0, X_1)] \quad \text{for} \ f \in L_{2,0}(Q). \]

This immediately gives
\[
\frac{1}{n} \sum_{k=1}^{n} a_n(X_{k-1}) = \frac{1}{n} \sum_{k=1}^{n} a_Q(X_{k-1}) + o_p(1).
\]
As in Peng and Schick (2003) we have
\[\sum_{i=1}^{M_n} a_{ni} \int v_i(x, \hat{\vartheta}) \, F_{\hat{\vartheta}_0} (dx) + \left( \int a_Q \varphi^\top \, dF_{\hat{\vartheta}_0} \right) (\hat{\vartheta} - \vartheta_0) = o_p(1). \]
Thus it suffices to show

\[(2.5) \quad \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{M_n} (\hat{\alpha}_{n_i} - \alpha_{n_i}) v_i(X_{k-1}, \hat{\theta}) = o_p(n^{-1/2}),\]

\[(2.6) \quad \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{M_n} \alpha_{ni} \left( v_i(X_{k-1}, \hat{\theta}) - v_i(X_{k-1}, \vartheta_0) - \int v_i(x, \hat{\theta}) F_{\vartheta_0}(dx) \right) = o_p(n^{-1/2}).\]

As in Peng and Schick (2003) one can show

\[(2.7) \quad \sum_{i=1}^{M_n} (\hat{\alpha}_{n_i} - \alpha_{n_i})^2 = O_p\left( \frac{M_n (\Gamma_n + \Delta_n)}{n} \right).\]

The proof is essentially the same, but now using (2.4) to deal with the appropriate averages. It is shown in Peng and Schick (2003) that

\[(2.8) \quad n \sum_{i=1}^{M_n} \left( \int v_i(x, \hat{\theta}) F_{\vartheta_0}(dx) \right)^2 = O_p(M_n).\]

It follows from (2.4) that

\[(2.9) \quad n \sum_{i=1}^{M_n} \left( \frac{1}{n} \sum_{k=1}^{n} v_i(X_{k-1}, \vartheta_0) \right)^2 = O_p(M_n).\]

In view of (2.7), (2.8) and (2.9), statement (2.5) is equivalent to

\[(2.10) \quad \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{M_n} (\hat{\alpha}_{n_i} - \alpha_{n_i}) \left( v_i(X_{k-1}, \hat{\theta}) - v_i(X_{k-1}, \vartheta_0) - \int v_i(x, \vartheta_0) F_{\vartheta_0}(dx) \right) = o_p(n^{-1/2}).\]

Relations (2.5) and (2.6) are verified as in Peng and Schick (2003), but now using the above Lemma 1 instead of their Lemma 2.

3. Efficiency of the estimator

Let us now prove that \( \hat{H}^* \) is efficient if an efficient estimator \( \hat{\vartheta} \) for \( \vartheta_0 \) is used. We need the following notation. Let \( \hat{K}(y, dx) \) denote the transition distribution of the reversed chain, defined by \( F_{\vartheta_0}(dy) K(x, dy) = \hat{K}(y, dx) F_{\vartheta_0}(dy) \). For a function \( g \in L_{2,0}(Q) \) write \( \hat{K}g(x) = \int \hat{K}(x, dy) g(x, y) \) and \( \hat{\vartheta}g(y) = \int \hat{\vartheta}(y, dx) g(x, y) \). Let \( K^j \) and \( \hat{K}^j \) be the operators on \( L_{2,0}(F_{\vartheta_0}) \) defined by \( K^j f(X_0) = E(f(X_j) \mid X_0) \) and \( \hat{K}^j f(X_j) = E(f(X_0) \mid X_j), j = 1, 2, \ldots \). Let \( U = \sum_{j=0}^{\infty} K^j \) and \( \hat{U} = \sum_{j=0}^{\infty} \hat{K}^j \) be the corresponding potentials. Let now

\[ T = \{ t \in L_2(Q) : Kt = 0 \}, \]

and let \( A \) be the operator from \( L_{2,0}(F_{\vartheta_0}) \) into \( T \) defined by \( Af(x, y) = Uf(y) - KUf(x). \)
We can now recall the characterization of efficient estimators in Kessler, Schick and Wefelmeyer (2001). Consider (Hellinger differentiable) perturbations $K_{nt}(x, dy) = K(x, dy)(1 + n^{-1/2}t(x, y))$ consistent with the parametric model for the stationary law. The space $T_s$ of all such functions $t$ is called the tangent space for our model. It is a subset of $T$. An $r$-dimensional functional $\chi$ of $K$ is called differentiable with gradient $g$ if $g \in T^r$ and

$$n^{1/2}(\chi(K_{nt}) - \chi(K)) \to \int gt \, dQ \quad \text{for all } t \in T_s.$$  

The canonical gradient is the (componentwise) projection $g_*$ of $g$ onto $T^r_\ast$. An estimator $\hat{\chi}$ for $\chi$ is called regular if there is a random vector $L$ such that

$$n^{1/2}(\hat{\chi} - \chi(K_{nt})) \Rightarrow L \quad \text{under } K_{nt} \text{ for all } t \in T_s.$$  

An estimator $\hat{\chi}$ for $\chi$ is called asymptotically linear with influence function $h$ if $h \in T^r$ and

$$n^{1/2}(\hat{\chi} - \chi(K)) = n^{-1/2} \sum_{k=1}^n h(X_{k-1}, X_k) + o_p(1).$$  

An estimator is regular and efficient if and only if it is asymptotically linear with influence function equal to the canonical gradient. Moreover, an asymptotically linear estimator is regular if and only if its influence function is a gradient. In particular, the canonical gradient can be obtained as the projection onto $T^r_\ast$ of the influence function of an arbitrary regular and asymptotically linear estimator.

As shown in Kessler, Schick and Wefelmeyer (2001), the tangent space for our model is

$$T_s = \{ t \in T : \bar{U}Kt \in [\varphi] \},$$

where $[\varphi]$ is the linear span of the Hellinger derivative $\varphi$. Moreover, the influence function of an efficient estimator $\hat{\vartheta}$ of $\vartheta_0$ is

$$g_*(x, y) = \left( \int e_\ast \varphi^\top dF_{\vartheta_0} \right)^{-1} A e_\ast \quad \text{with} \quad e_\ast = (\bar{U}KA)^{-1} \varphi.$$  

Note that $\bar{U}K$ corresponds to $\bar{V}$ in Kessler, Schick and Wefelmeyer (2001). If an efficient estimator $\hat{\vartheta}$ is used, then by Theorem 1 the influence function of our estimator $\hat{H}^\ast$ is

$$h_*(x, y) = h_0(x, y) - aQ(x) - bQ(y) + D^\top g_*(x, y),$$

where $h_0 = h - \int h \, dQ$. Efficiency of $\hat{H}^\ast$ follows if we show that $h_*$ is in $T_s$ and equals the projection of the influence function of the empirical estimator $\hat{H}$, which is

$$\tilde{h}(x, y) = h_0(x, y) - Kh_0(x) + AKh_0(x, y)$$

by Greenwood and Wefelmeyer (1995). Showing these two properties amounts to showing that $Kh_* = 0$ and $\int \tilde{h} t \, dQ = \int h_* t \, dQ$ for all $t \in T_s$. 


By definition of $a_Q$ and $b_Q$ we have that $h_Q(X_0, X_1) = h_0(X_0, X_1) - a_Q(X_0) - b_Q(X_1)$ is orthogonal to $a(X_0) + b(X_1)$ for all $a, b \in L_{\mathbb{L}^2}(F_{\vartheta_0})$. Thus $E(h_Q(X_0, X_1) \mid X_0) = 0$ and $E(h_Q(X_0, X_1) \mid X_1) = 0$. The former shows that $Kh_\ast = Kh_Q + D^\top Kg_\ast = 0$. It also gives $Kh_0 - a_Q - Kb_Q = 0$, which implies

(3.1) \[ a_Q + b_Q = (I - K)b_Q + Kh_0. \]

Now fix $t \in T_\ast$. Then $\bar{U}Kt = \varphi^\top u$ for some $u \in \mathbb{R}$. We have

\[
\int \tilde{h}t \, dQ = \int h_0 t \, dQ + \int AKh_0 \cdot t \, dQ = \int h_Q t \, dQ + \int (A(I - K)b_Q + AKh_0)t \, dQ.
\]

Here we have used that $b = U(I - K)b$ and that $Kt = 0$. It was shown in Kessler, Schick and Wefelmeyer (2001) that $\int g_\ast t \, dQ = u$ and

\[
\int tA f \, dQ = \int \bar{U}Kt \cdot f \, dF_{\vartheta_0} = \int f\varphi^\top u \, dF_{\vartheta_0}.
\]

In particular, if $f = a_Q + b_Q$, we get from (3.1) that

\[
\int (A(I - K)b_Q + AKh_0)t \, dQ = D^\top u = D^\top \int g_\ast t \, dQ.
\]

Hence we get \( \int \tilde{h}t \, dQ = \int h_\ast t \, dQ \).

For the case $r = 1$, Kessler, Schick and Wefelmeyer (2001) construct an efficient estimator $\hat{\vartheta}_0$ of $\vartheta_0$ under the additional assumption of continuous Hellinger differentiability of $F_{\vartheta}$. The construction carries over to $r$-dimensional $\vartheta$.

References


