Efficient Density Estimation in an AR(1) Model.

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Abstract. ccc This paper studies a class of estimators of the stationary density of an autoregressive model with autoregression parameter $0 < \varrho < 1$. These estimators use two types of estimators of the innovation density, a standard kernel estimator and a weighted kernel estimator with weights chosen to mimic the condition that the innovation density has mean zero. Bahadur expansions are obtained for this class of estimators in L_1, the space of integrable functions. It is shown that the density estimators based on the weighted kernel estimators are efficient if an efficient estimator of the autoregression parameter is used.

1. Introduction

Consider observations $X_0, \dots, X_n$ from a stationary autoregressive process, AR(1),

$$X_t = \varrho X_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z},$$

with unknown autoregression parameter ϱ in the open interval $(0, 1)$. The innovations ε_t, $t \in \mathbb{Z}$, are i.i.d. with a common density f, mean zero and finite variance σ^2, and $\{X_s, s \le t\}$ and $\{\varepsilon_r, r > t\}$ are independent. Then X_t has the infinite series representation

$$X_t = \varepsilon_t + \sum_{j=1}^{\infty} \varrho^j \varepsilon_{t-j}.$$

We are interested in estimating the stationary density g of the process. The usual estimators are density estimators based on the observations $X_0, \dots, X_n$. They do not use the autoregressive structure of the model and work for ergodic nonparametric Markov chains or more general time series. See, for example, Chanda (1983), Yakowitz (1989), Hart and Vieu (1990), Tran (1992), Chan and Tran (1992), Hallin and Tran (1996), Honda (2000), Wu and Mielniczuk (2002), Bryk and Mielniczuk (2005) and Schick and Wefelmeyer (2008a).

For the autoregressive process, the density satisfies the equation

$$g(x) = \int_{-\infty}^{\infty} f(x - \varrho y) g(y)\, dy, \quad x \in \mathbb{R}.$$

Thus a natural estimator of g is given by the plug-in estimator

$$\hat{g}_0(x) = \int_{-\infty}^{\infty} \hat{f}(x - \hat{\varrho} y) \hat{g}(y)\, dy, \quad x \in \mathbb{R},$$

with $\hat\varrho$ a root-n consistent estimator of ϱ, $\hat f$ a kernel estimator of f based on the residuals $\hat\varepsilon_j = X_j - \hat\varrho X_{j-1}$, $j = 1, \dots, n$, and $\hat g$ a kernel estimator of g based on the observations $X_0, \dots, X_n$. It can be deduced from Schick and Wefelmeyer (2008b) that the plug-in estimator $\hat g_0$ is root-n consistent in L_1 under mild assumptions. Similar results for moving average processes are in Schick and Wefelmeyer (2004a, 2004b).

We can repeat the above plug-in procedure with $\hat g_0$ replacing $\hat g$. This leads to the estimator

$$\hat g_1(x) = \int_{-\infty}^{\infty} \hat f(x - \hat\varrho y)\hat g_0(y)\, dy, \quad x \in \mathbb{R}.$$

One expects the estimator $\hat g_1$ to be better than $\hat g_0$ as it uses a better initial estimator of g. Proceeding in this way one recursively defines new estimators

$$\hat g_{k+1}(x) = \int_{-\infty}^{\infty} \hat f(x - \hat\varrho y)\hat g_k(y)\, dy, \quad x \in \mathbb{R},$$

for positive integers k. It is easy to check that $\hat g_k$ has the representation

$$\hat g_k(x) = \int_{\mathbb{R}^{k+1}} \hat f\Big(x - \sum_{i=1}^k \hat\varrho^i y_i - \hat\varrho^{k+1} z\Big) \prod_{j=1}^k \hat f(y_j)\, dy_j\ \hat g(z)\, dz$$

for nonnegative k.

Our goal in this paper is to study the estimator $\hat g_{k_n}$ where k_n is a sequence of integers that grow to infinity slowly. We shall do so under the following assumptions.

(A1) The density f has finite Fisher information for location.
(A2) The estimator $\hat\varrho$ satisfies the stochastic expansion

$$\hat\varrho = \varrho + \frac{1}{n}\sum_{j=1}^n \psi(X_{j-1}, \varepsilon_j) + o_P(n^{-1/2})$$

for a function ψ satisfying $\int_{-\infty}^{\infty} \psi(x,y) f(y)\, dy = 0$ and

$$\Psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^2(x,y) f(y)\, dy\ g(x)\, dx < \infty.$$

Recall that the density f has finite Fisher information for location if f is absolutely continuous and the integral

$$J_f = \int_{-\infty}^{\infty} \frac{(f'(x))^2}{f(x)}\, dx$$

is finite, where f' denotes the almost everywhere derivative of f. In this case we let $\ell_f = -f'/f$ denote the score function for location. Assumption (A1) implies that f' is integrable with L_1-norm $\|f'\|_1 = \|\ell_f f\|_1 \le J_f^{1/2}$. This allows the representation

$$f(x) = \int_{-\infty}^{x} f'(t)\, dt, \quad x \in \mathbb{R},$$

and shows that f is bounded by $\|f'\|_1$. Furthermore, the moment assumptions on f, assumption (A1) and an application of the Cauchy–Schwarz inequality show that the integral

$$\int_{-\infty}^{\infty} (1+|x|)|f'(x)|\, dx = \int_{-\infty}^{\infty} |\ell_f(x)|(1+|x|) f(x)\, dx$$

is finite.

It follows from (A2) that $n^{1/2}(\hat\varrho - \varrho)$ converges in distribution to a normal random variable with mean zero and variance Ψ. The sample autocorrelation coefficient

$$\frac{\frac{1}{n}\sum_{i=1}^{n} X_{i-1}X_i}{\frac{1}{n}\sum_{i=1}^{n} X_{i-1}^2}$$

meets this requirement with $\psi(x,y) = xy/E(X_0^2) = xy(1-\varrho^2)/\sigma^2$. An efficient estimator of ρ is characterized by (A2) with

$$\psi(x,y) = \frac{x\ell_f(y)}{E(X_0^2)J_f} = \frac{x\ell_f(y)(1-\varrho^2)}{\sigma^2 J_f}.$$

For AR(p) and ARMA(p,q) models see Kreiss (1987a,b) and Drost et al. (1997). For nonlinear autoregression see Koul and Schick (1997) and Schick and Wefelmeyer (2002a).

We shall work with two estimators of f. The first one is the usual kernel density estimator

$$\hat f_1(x) = \frac{1}{n}\sum_{j=1}^{n} K_b(x-\hat\varepsilon_j), \quad x \in \mathbb{R},$$

based on the residuals. Here $K_b(x) = (1/b)K(x/b)$ for a density K and a bandwidth b. The second one is the weighted kernel density estimator

$$\hat f_2(x) = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{1+\hat\lambda\hat\varepsilon_j} K_b(x-\hat\varepsilon_j), \quad x \in \mathbb{R},$$

where $\hat\lambda$ is chosen such that $1-\hat\lambda\hat\varepsilon_1, \dots, 1-\hat\lambda\hat\varepsilon_n$ are positive and

$$\frac{1}{n}\sum_{j=1}^{n} \frac{\hat\varepsilon_j}{1+\hat\lambda\hat\varepsilon_j} = 0$$

on the event $\{\min_{1\le j\le n}\hat\varepsilon_j < 0 < \max_{1\le j\le n}\hat\varepsilon_j\}$ and is taken to be zero otherwise. The second estimator satisfies

$$\int_{-\infty}^{\infty} y\hat f_2(y)\,dy = 0$$

on this event and thus mimics that f has mean zero. Rates of convergence in the L_1-norm of these two estimators were derived by Müller et al. (2005) in the more general setting of nonlinear autoregressive models. We shall improve their results for the present model in later sections. Both density estimators have the same rates of convergence in the L_1-norm, but the estimator $\hat f_2$ performs better as plug-in estimator for linear functionals of f. This was observed in Müller et all. (2005) in the context of estimating the innovation distribution function and further exploited in Müller et al. (2006) in the prediction for autoregressive models.

To state our first result we introduce some notation. We start with the random variables

$$\dot X_0 = \sum_{j=1}^{\infty} j\varrho^{j-1}\varepsilon_{-j} \quad \text{and} \quad Y_j = X_0 - \varrho^j\varepsilon_{-j} = \sum_{i=0}^{\infty} \mathbf{1}[i\ne j]\varrho^i\varepsilon_{-i}, \quad j \ge 0.$$

Let $\dot g$ denote the function defined by

$$\dot g(x) = E[-\dot X_0 f'(x-\rho X_{-1})], \quad x \in \mathbb{R}.$$

This function is integrable with L_1-norm

$$\|\dot g\|_1 \le \|f'\|_1 E[|\dot X_0|] \le \|f'\|_1 \frac{E[|\varepsilon_0|]}{(1-\varrho)^2} = \frac{\|f'\|_1 \|\iota_{\mathbb{R}} f\|_1}{(1-\varrho)^2}, \tag{1.1}$$

where $\iota_{\mathbb{R}}$ denotes the identity map on $\mathbb{R}$. For $j = 0, 1, 2, \dots,$ let γ_j denote the density of Y_j. Then we have the following representation of the stationary density,

$$g(x) = \int_{-\infty}^{\infty} \gamma_j(x - \varrho^j y) f(y)\, dy, \quad x \in \mathbb{R}, \tag{1.2}$$

for each such j. Now introduce functions γ and γ^* by

$$\gamma(x, y) = \sum_{j=0}^{\infty} \big(\gamma_j(x - \varrho^j y) - g(x)\big), \quad x, y \in \mathbb{R},$$

and

$$\gamma^*(x, y) = \gamma(x, y) - \int_{-\infty}^{\infty} \gamma(x, z) z f(z)\, dz \frac{y}{\sigma^2}, \quad x, y \in \mathbb{R}.$$

These functions satisfy the integrability conditions

$$\Big(\int_{\mathbb{R}^2} |\gamma(x, y)|\, dx f(y)\, dy\Big)^2 \le \pi \int_{\mathbb{R}^2} (1 + x^2)|\gamma(x, y)|^2\, dx f(y)\, dy < \infty \tag{1.3}$$

and

$$\Big(\int_{\mathbb{R}^2} |\gamma^*(x, y)|\, dx f(y)\, dy\Big)^2 \le \pi \int_{\mathbb{R}^2} (1 + x^2)|\gamma^*(x, y)|^2\, dx f(y)\, dy < \infty \tag{1.4}$$

as will be shown in Section 2. Finally we introduce the average

$$\bar\Psi_n = \frac{1}{n} \sum_{i=1}^{n} \psi(X_{i-1}, \varepsilon_i)$$

and assume that the kernel estimator $\hat g$ also uses the kernel K and the bandwidth b,

$$\hat g(x) = \frac{1}{n+1} \sum_{j=0}^{n} K_b(x - X_j), \quad x \in \mathbb{R}.$$

THEOREM 1. *Suppose* (A1) *and* (A2) *are met, the kernel K is a symmetric density with finite variance and is twice continuously differentiable with $\|(1 + \iota_{\mathbb{R}}^2)K'\|_1$ and $\|(1 + \iota_{\mathbb{R}}^2)(K'')^2\|_1$ finite, and the sequence k_n and the bandwidth $b = b_n$ are chosen to satisfy*

$$\frac{k_n}{\log(n)} \to \infty, \quad k_n^4 b_n^4 n \to 0 \quad \text{and} \quad \frac{k_n^2}{n b_n^3} \to 0.$$

Then, for the choice $\hat f = \hat f_1$, the estimator $\hat g_{k_n}$ satisfies the L_1-Bahadur expansion

$$\int_{-\infty}^{\infty} \Big|\hat g_{k_n}(x) - g(x) - \bar\Psi_n \dot g(x) - \frac{1}{n} \sum_{i=1}^{n} \gamma(x, \varepsilon_i)\Big|\, dx = o_P(n^{-1/2})$$

while, for the choice $\hat f = \hat f_2$, it satisfies the L_1-Bahadur expansion

$$\int_{-\infty}^{\infty} \Big|\hat g_{k_n}(x) - g(x) - \bar\Psi_n \dot g(x) - \frac{1}{n} \sum_{j=1}^{n} \gamma^*(x, \varepsilon_i)\Big|\, dx = o_P(n^{-1/2}).$$

The proof of Theorem 1 is in Section 5. The assumptions on k_n and b_n are met by taking $k_n = (\log n)^\alpha$ for some $\alpha > 1$ and $b_n = n^{-\beta}$ for some β in the open interval $(1/4, 1/3)$. The standard normal density is a possible choice for K.

Inspecting the proof of Theorem 1 reveals that the theorem remains valid if we omit integration with respect to z resulting in the estimator

$$\hat{p}(x) = \int_{\mathbb{R}^{k_n}} \hat{f}\Big(x - \sum_{i=1}^{k_n} \hat{\varrho}^i y_i\Big) \prod_{j=1}^{k_n} \hat{f}(y_j)\, dy_j, \quad x \in \mathbb{R}.$$

In view of the identity

$$\int_{-\infty}^{\infty} h(y)\hat{f}_1(y)\, dy = \frac{1}{n}\sum_{j=1}^{n} \int_{-\infty}^{\infty} h(\hat{\varepsilon}_j - u) K_b(u)\, du$$

this estimator with $\hat{f} = \hat{f}_1$ can be written as a V-statistic

$$\hat{p}(x) = \frac{1}{n^{k_n+1}} \sum_{j_0=1}^{n} \cdots \sum_{j_{k_n}=1}^{n} \mathbb{K}_n(x - \sum_{i=0}^{k_n} \hat{\varrho}^i \hat{\varepsilon}_{j_i}), \quad x \in \mathbb{R},$$

with $\mathbb{K}_n$ the convolution of the densities K_b, $K_{\hat{\varrho}b}, \dots, K_{\hat{\varrho}^{k_n}b}$. For the estimator $\hat{f} = \hat{f}_2$ we can write it as a weighted V-statistic

$$\hat{p}(x) = \frac{1}{n^{k_n+1}} \sum_{j_0=1}^{n} \cdots \sum_{j_{k_n}=1}^{n} \frac{\mathbb{K}_n(x - \sum_{i=0}^{k_n} \hat{\varrho}^i \hat{\varepsilon}_{j_i})}{\prod_{l=0}^{k_n}(1 + \hat{\lambda}\hat{\varepsilon}_{j_l})}, \quad x \in \mathbb{R}.$$

If we take K to be the standard normal density, then $\mathbb{K}_n$ equals the normal density with mean zero and variance $b^2 \sum_{i=0}^{k_n} \hat{\varrho}^{2i}$. This allows for a straighforward computation of the estimator $\hat{p}$ for both, $\hat{f}_1$ and $\hat{f}_2$.

It follows from the integrability conditions of γ and γ^* that the CLT in L_1 applies to the L_1-valued random variables $Z_j = \gamma(\cdot, \varepsilon_j)$, $j = 1, 2, \dots$, and $Z_j^* = \gamma^*(\cdot, \varepsilon_j)$, $j = 1, 2 \dots$, and yields that

$$n^{-1/2} \sum_{j=1}^{n} Z_j = n^{-1/2} \sum_{j=1}^{n} \gamma(\cdot, \varepsilon_j) \quad \text{and} \quad n^{-1/2} \sum_{j=1}^{n} Z_j^* = n^{-1/2} \sum_{j=1}^{n} \gamma^*(\cdot, \varepsilon_j)$$

converge in distribution to centered Gaussian processes. Indeed, as shown in Lemma 3 of Schick and Wefelmeyer (2007) the integrability conditions imply the necessary and sufficient conditions of the CLT in L_1; see Ledoux and Talagrand (1991), Theorem 10.10 or van der Vaart and Wellner (1996, p. 92).

Our next result gives Hadamard differentiability of the stationary density which will be crucial in the characterization of efficient estimators of g in L_1. For this we write $g_{\varrho,f}$ for g to stress the dependence of g on the parameters ϱ and f. Let $\mathcal{H}$ denote the set of all measurable function h which satisfy

$$\int_{-\infty}^{\infty} h(y)f(y)\, dy = 0, \quad \int_{-\infty}^{\infty} yh(y)f(y)\, dy = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} h^2(y)f(y)\, dy < \infty.$$

This set is the tangent space at f of the set $\mathcal{F}$ of all densities with mean zero, finite variance and finite Fisher information. Indeed, one can show that for each h in $\mathcal{H}$ there is a sequence f_n of densities with finite Fisher information satisfying

$$\int_{-\infty}^{\infty} \Big(n^{1/2}\big(\sqrt{f_n(x)} - \sqrt{f(x)}\big) - \frac{1}{2}h(x)\sqrt{f(x)}\Big)^2\, dx \to 0, \tag{1.5}$$

$$\int_{-\infty}^{\infty} x f_n(x)\,dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 \big|f_n(x) - f(x)\big|\,dx \to 0. \tag{1.6}$$

The proof of the next theorem which establishes Hadamard differentiability is in Section 6.

THEOREM 2. *Suppose f has finite Fisher information for location. Let h belong to $\mathcal{H}$. Let f_n be a sequence of densities satisfying (1.5) and (1.6) and let ϱ_n be a sequence in $(0,1)$ satisfying $n^{1/2}(\varrho_n - \varrho) \to t$ for some real t. Then g_{ϱ_n,f_n} satisfies*

$$\int_{-\infty}^{\infty} \big|n^{1/2}\big(g_{\varrho_n,f_n}(x) - g_{\varrho,f}(x)\big) - Ah(x) - \dot g(x)t\big|\,dx = o_P(n^{-1/2})$$

with

$$Ah(x) = \int \gamma_*(x,y)h(y)f(y)\,dy, \quad x \in \mathbb{R}.$$

We will now show that the density estimator $\hat g_{k_n}(x)$ is efficient for $g(x)$ if we use $\hat f_2$ and an efficient estimator $\hat\varrho$ for ϱ. More specifically we consider linear functionals of the density g of the form $\Phi(g) = \int_{-\infty}^{\infty} \phi(y)g(y)\,dy$ with ϕ bounded and measurable, i.e., $\phi \in L_\infty$, and show that $\Phi(\hat g_{k_n})$ is efficient for the functional $\Phi(g)$. We follow efficiency proofs for other models and functionals and will be brief. See Kreiss (1987), Drost et al. (1997), Koul and Schick (1997), and Schick and Wefelmeyer (2002b). Write $P_{n\varrho f}$ for the joint distribution of $(X_0, \dots, X_n)$ when ϱ and f are true. Choose ϱ_n and f_n close to ϱ and f as in Theorem 2. Under the above assumptions it follows from Koul and Schick (1997) that the *local log-likelihood ratio* admits the stochastic expansion

$$\log \frac{dP_{n\varrho_n f_n}}{dP_{n\varrho f}} = n^{-1/2}\sum_{j=1}^n tX_{j-1}\ell_f(\varepsilon_j) + n^{-1/2}\sum_{j=1}^n h(\varepsilon_j) + o_P(1).$$

In other words, the model is *locally asymptotically normal* (LAN) with *central sequence* $n^{-1/2}\sum_{j=1}^n tX_{j-1}\ell_f(\varepsilon_j) + n^{-1/2}\sum_{j=1}^n h(\varepsilon_j)$. It follows from Theorem 1 and the above characterization of efficient estimators for ϱ that $n^{1/2}(\Phi(\hat g_{k_n}) - \Phi(g))$ is approximated stochastically by an expression of the form of a central sequence. This implies that $\Phi(\hat g_{k_n})$ is efficient for $\Phi(g)$ (and also regular and asymptotically linear).

This efficiency result is an instance of the plug-in principle formulated by Klaassen and Putter (2002) in the i.i.d. case. In order to see this, fix ϱ, write $\varepsilon_j(\varrho) = \varepsilon_j = X_j - \varrho X_{j-1}$, and set

$$\tilde f_2(x,\varrho) = \frac{1}{n}\sum_{j=1}^n \frac{1}{1+\tilde\lambda(\varrho)\varepsilon_j(\varrho)} K_b(x - \varepsilon_j(\varrho)), \quad x \in \mathbb{R},$$

where $\tilde\lambda(\varrho)$ is chosen such that $1 - \tilde\lambda(\varrho)\varepsilon_1(\varrho), \dots, 1 - \tilde\lambda(\varrho)\varepsilon_n(\varrho)$ are positive and

$$\frac{1}{n}\sum_{j=1}^n \frac{\varepsilon_j(\varrho)}{1+\tilde\lambda(\varrho)\varepsilon_j(\varrho)} = 0$$

on the event $\{\min_{1\le j\le n} \varepsilon_j(\varrho) < 0 < \max_{1\le j\le n} \varepsilon_j(\varrho)\}$ and is taken to be zero otherwise. Set

$$\tilde{g}_k(x,\varrho) = \int_{\mathbb{R}^{k+1}} \tilde{f}_2\Big(x - \sum_{i=1}^k \varrho^i y_i - \varrho^{k+1} z, \varrho\Big) \prod_{j=1}^k \tilde{f}_2(y_j,\varrho)\, dy_j\ \hat{g}(z)\, dz, \quad x \in \mathbb{R}.$$

Then for k_n as in Theorem 1 the estimator $\Phi(\tilde{g}_{k_n}(\cdot,\varrho)) = \int_{-\infty}^{\infty} \phi(x)\tilde{g}_{k_n}(x,\varrho)\, dx$ is efficient for $\Phi(g) = \int_{-\infty}^{\infty} \phi(y)g(y)\, dy$ when ϱ is fixed. Plugging in an efficient estimator $\hat{\varrho}$ for ϱ, we obtain an efficient estimator $\Phi(\tilde{g}_{k_n}(\cdot,\hat{\varrho})) = \int_{-\infty}^{\infty} \phi(x)\tilde{g}_{k_n}(x,\hat{\varrho})\, dx = \int_{-\infty}^{\infty} \phi(x)\hat{g}_{k_n}(x)\, dx$ when ϱ is unknown.

2. Some Auxiliary Lemmas

In this section we collect some lemmas that will be used in the proofs of our theorems. We start with three inequalities.

Lemma 1. *For numbers r and s in the interval $(0,1)$, we have the inequalities*

$$\sum_{j=1}^{\infty} |r^j - s^j| \le \frac{|r-s|}{(1-\max\{r,s\})^2}, \tag{2.1}$$

$$\sum_{j=1}^{\infty} |r^j - s^j|^2 \le \frac{|r-s|^2}{(1-\max\{r,s\})^3}, \tag{2.2}$$

$$\sum_{j=1}^{\infty} |r^j - s^j - js^{j-1}(r-s)| \le \frac{|r-s|^2}{(1-\max\{r,s\})^3}. \tag{2.3}$$

Proof. Recall the infinite series

$$\sum_{j=1}^{\infty} jt^{j-1} = \frac{1}{(1-t)^2} \quad \text{and} \quad \sum_{j=1}^{\infty} j(j-1)t^{j-2} = \frac{2}{(1-t)^3}, \qquad |t|<1.$$

Using the inequality $|r^j - s^j| \le |r-s| j \max\{r,s\}^{j-1}$ and the first infinite series, we obtain (2.1). Using $|r^j - s^j|^2 \le \frac{1}{2}(r-s)^2 2j(2j-1)\max\{r,s\}^{2j-2}$ and the second infinite series, we obtain (2.3). Using

$$|r^j - s^j - js^{j-1}(r-s)| \le \frac{1}{2}(r-s)^2 j(j-1)\max\{r,s\}^{j-2}$$

and the second infinite series, we obtain (2.3). □

Lemma 2. *Let h be a measurable function. Then we have the inequality*

$$\|h\|_1^2 \le \int_{-\infty}^{\infty} \pi(1+x^2)h^2(x)\, dx.$$

Proof. Let us set $w(x) = \pi(1+x^2)$, $x \in \mathbb{R}$. Then $1/w$ is the Cauchy density. We calculate

$$\|h\|_1^2 = \|\sqrt{w}h/\sqrt{w}\|_1^2 \le \|wh^2\|_1 \|1/w\|_1 = \|wh^2\|_1$$

which is the desired result. □

Lemma 3. *Let p and q be two integrable functions with $\|\iota_{\mathbb{R}}^2 p\|_1$ and $\|\iota_{\mathbb{R}}^2 q\|_1$ finite. Then the inequality $\|\iota_{\mathbb{R}} p - \iota_{\mathbb{R}} q\|_1^2 \le (\|\iota_{\mathbb{R}}^2 p\|_1 + \|\iota_{\mathbb{R}}^2 q\|_1)\|p-q\|_1$ holds.*

PROOF. Bound $|\iota_{\mathbb{R}}p-\iota_{\mathbb{R}}q|$ by $|p-q|^{1/2}(|p|+|q|)^{1/2}|\iota_{\mathbb{R}}|$ and then use the Cauchy–Schwarz inequality. □

Let h be an integrable function. Then the L_1-*modulus of continuity* of h is the map w_h defined by

$$w_h(t)=\sup_{|u|\le t}\int_{-\infty}^{\infty}|h(x-u)-h(x)|\,dx,\quad t\ge 0.$$

The map w_h is bounded by $2\|h\|_1$ and continuous at 0, see Rudin (1974), Theorem 9.5 for the latter. We say h is L_1-*Lipschitz* if there is a constant Λ such that

$$\int_{-\infty}^{\infty}|h(x-t)-h(x)|\,dx\le\Lambda|t|,\quad t\in\mathbb{R}.$$

In this case the inequality $w_h(t)\le\Lambda t$ holds for all $t\ge 0$.

LEMMA 4. *Let h be an integrable function and $T,T_1,T_2,\dots$ be random variables such that $E[|T_n-T|]\to 0$. Then*

$$\int_{-\infty}^{\infty}\big|E[h(x-T_n)]-E[h(x-T)]\big|\,dx\to 0.$$

PROOF. In view of the inequality $|E(X)|\le E(|X|)$ and Fubini's theorem, the integral is bounded by $E[w_h(|T_n-T|)]$, and the desired result follows from the dominated convergence theorem. □

Let $\mathbb{H}_1$ denote the set of all integrable functions of the form

$$h(x)=\int_{-\infty}^{x}h'(x)\,dx$$

for some integrable function h' and let $\mathbb{H}_2$ denote set of all h in $\mathbb{H}_1$ with h' in $\mathbb{H}_1$. We write h'' for $(h')'$. If h belongs to $\mathbb{H}_1$, then h is bounded by $\|h'\|_1$ and uniformly continuous. More precisely, we have

$$|h(y)-h(x)|=\Big|\int_{\infty}^{y}h'(s)\,ds-\int_{-\infty}^{y}h'(s-(y-x))\,ds\Big|\le w_{h'}(|y-x|)$$

for all real x and y. As an integrable and uniformly continuous function, $h(x)\to 0$ as $x\to\infty$. This implies that h' integrates to zero,

$$\int_{-\infty}^{\infty}h'(x)\,dx=0,$$

and this gives the alternative representation

$$h(x)=-\int_{x}^{\infty}h'(x)\,dx.$$

Using this we can show that $\|h\|_1\le\|\iota_{\mathbb{R}}h'\|_1$. Indeed, the left-hand side is bounded by

$$\int_{-\infty}^{0}\int_{-\infty}^{x}|h'(t)|\,dt\,dx+\int_{0}^{\infty}\int_{x}^{\infty}|h'(t)|\,dt\,dx\le\int_{-\infty}^{0}|th'(t)|\,dt+\int_{0}^{\infty}|th'(t)|\,dt.$$

In view of this inequality, we conclude that a continuously differentiable function h belongs to $\mathbb{H}_1$ if $\lim_{|x|\to\infty}h(x)=0$ and $\|(1+|\iota_{\mathbb{R}}|)h'\|_1$ is finite.

Assumption (A1) implies that the density f belongs to $\mathbb{H}_1$. The next two results are easily verified.

LEMMA 5. *Let h belong to $\mathbb{H}_1$ and let t be a positive number. Then the function h_t defined by*

$$h_t(x) = h(x/t)/t, \quad x \in \mathbb{R},$$

belongs to $\mathbb{H}_1$, and we can take

$$h_t'(x) = h'(x/t)/t^2, \quad x \in \mathbb{R}.$$

Thus $\|h_t'\|_1 = \|h'\|_1/t$.

LEMMA 6. *Let $h = h_1 * h_2$ denote the convolution of the integrable functions h_1 and h_2. Then the following are true.*

(1) *If h_1 is L_1-Lipschitz with constant Λ, then h is L_1-Lipschitz with constant $\Lambda\|h_2\|_1$.*
(2) *If h_1 belongs to $\mathbb{H}_1$, then h belongs to $\mathbb{H}_1$ with $h' = h_1' * h_2$.*
(3) *If h_1 and h_2 belong to $\mathbb{H}_1$, then h belongs to $\mathbb{H}_2$ with $h'' = h_1' * h_2'$.*

LEMMA 7. *Let h belong to $\mathbb{H}_1$. Then h is L_1-Lipschitz with constant $\|h'\|_1$. Moreover, we have the inequality*

$$\int_{-\infty}^{\infty} \big|h(x-t) - h(x) + th'(x)\big|\,dx \le |t| w_{h'}(|t|), \quad t \in \mathbb{R}.$$

In particular, if h' is L_1-Lipschitz with constant Λ, then we have

$$\int_{-\infty}^{\infty} \big|h(x-t) - h(x) + th'(x)\big|\,dx \le t^2\Lambda, \quad t \in \mathbb{R}.$$

PROOF. Fix $t \in \mathbb{R}$. Then we have the identity

$$h(x-t) - h(x) = -t\int_0^1 h'(x-st)\,ds$$

and consequently the bounds

$$\int_{-\infty}^{\infty} |h(x-t) - h(x)|\,dx \le |t| \int_{-\infty}^{\infty}\int_0^1 |h'(x-st)|\,ds\,dx = |t|\|h'\|_1$$

and

$$\int_{-\infty}^{\infty} \big|h(x-t) - h(x) + th'(x)\big|\,dx \le \int_{-\infty}^{\infty} |t| \int_0^1 |h'(x-st) - h'(x)|\,ds\,dx \le |t| w_{h'}(|t|).$$

If h' is L_1-Lipschitz with constant Λ, then $|t| w_{h'}(|t|) \le \Lambda t^2$. □

LEMMA 8. *Let h belong to $\mathbb{H}_1$ and let T, U and V be random variables. If T and U have finite means, then we have the inequalities*

$$\int_{-\infty}^{\infty} \big|E(h(x-V-T)) - E(h(x-V))\big|\,dx \le \|h'\|_1 E(|T|)$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \big|E(h(x-V-T)) - E(h(x-V)) + E(Uh'(x-V))\big|\,dx \\ \le E(|T| w_{h'}(|T|)) + \|h'\|_1 E(|T-U|) \\ \le E(|U| w_{h'}(|T|)) + 3\|h'\|_1 E(|T-U|).\end{aligned}$$

Proof. Using the formula $|E(X)| \le E(|X|)$, Fubini's theorem and then the substitution $u = x - V$ we obtain that the left-hand side of the first inequality is bounded by

$$E\int_{-\infty}^{\infty} |h(u-T) - h(u)|\,du,$$

and of the second inequality by

$$E\int_{-\infty}^{\infty} \big|h(u-T) - h(u) + Th'(u) - (T-U)h'(u)\big|\,du.$$

The desired result then follows from the previous lemma and the fact that $w_{h'}$ is bounded by $2\|h'\|_1$. □

Corollary 1. *Let f be in $\mathbb{H}_1$. Then, as $r \to \varrho$,*

$$\big\|g_{r,f} - g_{\varrho,f} - (r-\varrho)\dot{g}\big\|_1 = o(|r-\varrho|). \tag{2.4}$$

Proof. For $0 < r < 1$ let $S_r = \sum_{j=1}^{\infty} r^j \varepsilon_{-j}$. Recall that $\dot{X}_0$ is defined to be $\sum_{j=1}^{\infty} j\varrho^{j-1}\varepsilon_{-j}$. It follows from Lemma 1 that

$$E(|S_r - S_\varrho|) \le \frac{|r-\varrho|\|\iota_{\mathbb{R}} f\|_1}{(1-\max\{r,\varrho\})^2}$$

and

$$E\big(|S_r - S_\varrho - (r-\varrho)\dot{X}_0|\big) \le \frac{|r-\varrho|^2\|\iota_{\mathbb{R}} f\|_1}{(1-\max\{r,\varrho\})^3}.$$

Note the identity

$$g_{r,f}(x) = E(f(x - S_r)).$$

Applying Lemma 8 with $h = f$, $V = S_\varrho = \varrho X_{-1}$, $T = S_r - S_\varrho$ and $U = (r-\varrho)\dot{X}_0$ shows that the left-hand side of (2.4) is bounded by

$$|r-\varrho| E\big(|\dot{X}_0| w_{f'}(|S_r - S_\varrho|)\big) + 3\|f'\|_1 \|\iota_{\mathbb{R}} f\|_1 \frac{|r-\varrho|^2}{(1-\max\{r,\varrho\})^3}.$$

The desired result now follows from the dominated convergence theorem. □

For integrable functions p and q and $t > 0$, we denote by $B_t(p,q)$ the integrable function defined by

$$B_t(p,q)(x) = \int_{-\infty}^{\infty} p(x-ty)q(y)\,dy, \quad x \in \mathbb{R}.$$

The integrability of $B_t(p,q)$ follows from the inequality

$$\|B_t(p,q)\|_1 \le \|p\|_1\|q\|_1.$$

We can view B_t as a bilinear operator from $L_1 \times L_1$ to L_1. Since $B_t(p,q)$ is the convolution of p and q_t, where $q_t(x) = q(x/t)/t$, $x \in \mathbb{R}$, the following lemma is an immediate consequence of Lemmas 5 and 6.

Lemma 9. *Let p and q be integrable functions and t be a positive number. Then the following hold.*

1. *If p is L_1-Lipschitz with constant Λ, then $B_t(p,q)$ is L_1-Lipschitz with constant $\Lambda\|q\|_1$.*
2. *If p belongs to $\mathbb{H}_1$, then $B_t(p,q)$ belongs to $\mathbb{H}_1$ with $B_t(p,q)' = B_t(p',q)$.*
3. *If q belongs to $\mathbb{H}_1$, then $B_t(p,q)$ belongs to $\mathbb{H}_1$ with $B_t(p,q)' = B_t(p,q')/t$.*

(4) *If p and q belong to $\mathbb{H}_1$, then $B_t(p,q)$ belongs to $\mathbb{H}_2$ with $B_t(p,q)'' = B_t(p',q')/t$.*

The next two results are consequences of Lemmas 7 and 8.

LEMMA 10. *Let p and q be an integrable function with $\|\iota_{\mathbb{R}} q\|_1$ finite and p being L_1-Lipschitz with constant Λ. Then we have the inequality*

$$\Big\| B_t(p,q) - p\int_{-\infty}^{\infty} q(y)\,dy \Big\|_1 \le \Lambda \|\iota_{\mathbb{R}} q\|_1 t, \quad t > 0.$$

In particular, if the integral of q is zero, we have

$$\|B_t(p,q)\|_1 \le \Lambda \|\iota_{\mathbb{R}} q\|_1 t, \quad t > 0.$$

LEMMA 11. *Let p belong to $\mathbb{H}_1$ and let q be a density. If q has finite mean, then we have the inequality*

$$\|B_t(p,q) - p\|_1 \le \|p'\|_1 \|\iota_{\mathbb{R}} q\|_1 t, \quad t > 0.$$

If p' is L_1-Lipschitz with constant Λ and q has mean zero and finite variance, then we have the inequality

$$\|B_t(p,q) - p\|_1 \le \Lambda \|\iota_{\mathbb{R}}^2 q\|_1 t^2, \quad t > 0.$$

Let v be the function defined by

$$v(x) = (1 + |x|), \quad x \in \mathbb{R}.$$

This function satisfies the inequality

$$v(x+y) \le v(x)v(y), \quad x, y \in \mathbb{R}.$$

If vh is integrable, then we have

$$\int_{-\infty}^{\infty} v(x)|h(x-t)|\,dx = \int_{-\infty}^{\infty} v(x+t)|h(x)|\,dx \le v(t)\|vh\|_1, \quad t \in \mathbb{R}.$$

¿From this we immediately derive the following result.

LEMMA 12. *Let vp and vq be integrable. Then, for every $0 < t \le 1$, $vB_t(p,q)$ is integrable with*

$$\|vB_t(p,v)\|_1 \le \|vp\|_1 \int_{-\infty}^{\infty} v(ty)|q(y)|\,dy \le \|vp\|_1 \|vq\|_1.$$

LEMMA 13. *Let h belong to $\mathbb{H}_1$ with vh' integrable. Then $\|vh\|_\infty \le \|vh'\|_1$.*

PROOF. For negative x we have the bound

$$v(x)|h(x)| \le v(x)\int_{-\infty}^{x} |h'(u)|\,du \le \int_{-\infty}^{x} v(u)|h'(u)|\,du$$

while for positive x we have the inequality

$$v(x)|h(x)| \le v(x)\int_{x}^{\infty} |h'(u)|\,du \le \int_{x}^{\infty} v(u)|h'(u)|\,du.$$

These inequalities imply $\|vh\|_\infty \le \|vh'\|_1$. □

We have seen that (A1) implies that $\|vf'\|_1$ is finite. The stationary density g equals $B_\varrho(f,g)$ and therefore belongs to $\mathbb{H}_2$ with $g' = B_\varrho(f',g)$ and $g'' = B_\varrho(f',g')/\varrho$ yielding

$$\|g'\|_1 \le \|f'\|_1, \quad \|vg'\|_1 \le \|vf'\|_1\|vg\|_1 \quad \text{and} \quad \|g''\|_1 \le \|f'\|_1^2/\varrho.$$

Recall that γ_j denotes the density of $Y_j = X_0 - \varrho^j\varepsilon_{-j} = \sum_{i=0}^\infty \mathbf{1}[i \ne j]\varrho^i\varepsilon_{-i}$ for nonnegative integers j. Since Y_0 equals ϱX_{-1}, we have $\gamma_0(x) = g(x/\varrho)/\varrho$. Thus the density γ_0 belongs to $\mathbb{H}_2$ with $\gamma_0' = g'(x/\varrho)/\varrho^2$ and $\gamma_0''(x) = g''(x/\varrho)/\varrho^3$ yielding the bounds

$$\|\gamma_0'\|_1 \le \|f'\|_1/\varrho, \quad \|v\gamma_0'\|_1 \le \|vf'\|_1\|vg\|_1/\varrho \quad \text{and} \quad \|\gamma_0''\|_1 \le \|f'\|_1^2/\varrho^3.$$

For $j \ge 1$, the density γ_j equals $B_\varrho(f,\gamma_{j-1})$ as is easily checked and thus belongs to $\mathbb{H}_2$ with $\gamma_j' = B_\varrho(f',\gamma_{j-1})$ and $\gamma_j'' = B_\varrho(f',\gamma_{j-1}')/\varrho$ giving the bounds

$$\|\gamma_j'\|_1 \le \|f'\|_1, \quad \|v\gamma_j'\|_1 \le \|vf'\|_1\|v\gamma_{j-1}\|_1 \quad \text{and} \quad \|\gamma_j''\|_1 \le \|f'\|_1\|\gamma_{j-1}'\|_1/\varrho.$$

Note also the bounds $\|v\gamma_j\|_1 = E(1+|Y_j|) \le 1 + \sum_{i=0}^\infty \varrho^i\|\iota_\mathbb{R} f\| \le \|vf\|_1/(1-\rho)$ and similarly $\|vg\|_1 \le \|vf\|_1/(1-\varrho)$. Consequently, we have the following result.

Corollary 2. *Suppose (A1) holds. Then there are constants C_1, C_2 and C_3 such that the inequalities $\|\gamma_j'\|_1 \le C_1$, $\|v\gamma_j'\|_1 \le C_2$ and $\|\gamma_j''\|_1 \le C_3$ hold for all $j \ge 0$.*

Let us now verify the integrability condition (1.3). The first inequality follows from the moment inequality and Lemma 2. The finiteness of the integral

$$I_2 = \int_{\mathbb{R}^2} (1+x^2)\gamma^2(x,y)f(y)\,dy\,dx$$

follows if we verify the inequality

$$\tau_j = \int_{\mathbb{R}^2} (1+x^2)(\gamma_j(x-\varrho^j y) - g(x))^2 f(y)\,dx\,dy \le M\varrho^j, \quad j \ge 0,$$

for some finite constant M. Indeed, we first bound I_2 by

$$\sum_{i=0}^\infty\sum_{j=0}^\infty \int_{\mathbb{R}^2} (1+x^2)|\gamma_i(x-\varrho^i y) - g(x)||\gamma_j(x-\varrho^j y) - g(x)|f(y)\,dy\,dx$$

and then use the Cauchy–Schwarz inequality to obtain

$$I_2 \le \sum_{i=0}^\infty\sum_{j=0}^\infty \sqrt{\tau_i\tau_j} = \Big(\sum_{i=0}^\infty \sqrt{\tau_i}\Big)^2 \le \frac{M}{(1-\sqrt{\varrho})^2}.$$

The formula (1.2) yields the identity

$$\int_{-\infty}^\infty (\gamma_j(x-\varrho^j y) - g(x))^2 f(y)\,dy = \int_{-\infty}^\infty \gamma_j^2(x-\varrho^j y)f(y)\,dy - g^2(x), \quad x \in \mathbb{R}.$$

Using the substitution $x = u + \varrho^j y$ and the fact that f has mean zero and finite variance σ^2, we calculate

$$\int_{\mathbb{R}^2} (1+x^2)\gamma_j^2(x-\varrho^j y)f(y)\,dy\,dx = \int_{-\infty}^\infty (1+u^2+\varrho^{2j}\sigma^2)\gamma_j^2(u)\,du$$

and then, utilizing the inequality $1 + x^2 \le (1 + |x|)^2 = v^2(x)$,

$$\begin{aligned}\tau_j &= \int_{-\infty}^{\infty} (1 + x^2)(\gamma_j^2(x) - g^2(x))\, dx + \varrho^{2j}\sigma^2 \int \gamma_j^2(x)\, dx \\ &\le (\|v\gamma_j\|_\infty + \|vg\|_\infty)\|v(\gamma_j - g)\|_1 + \varrho^{2j}\sigma^2\|\gamma_j\|_\infty.\end{aligned}$$

Next we use the inequalities $\|\gamma_j\|_\infty \le \|v\gamma_j\|_\infty \le \|v\gamma_j'\|_1 \le C_2$ and

$$\begin{aligned}\|v(g - \gamma_j)\|_1 &= \int_{-\infty}^{\infty} v(x)\Big|\int_{-\infty}^{\infty}\int_0^1 |\gamma_j'(x - s\varrho^j y)|\varrho^j y\, ds f(y)\, dy\Big|\, dx \\ &\le \int_0^1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v(x - s\varrho^j y)|\gamma_j'(x - s\varrho^j y)|\, dx v(y)\varrho^j|y|f(y)\, dy\, ds \\ &\le \|v\gamma_j'\|_1\varrho^j\|v^2 f\|_1\end{aligned}$$

to conclude the exponential decay $\tau_j \le M\varrho^j$.

Let us now verify the integrability condition (1.4). The first inequality follows from the moment inequality and Lemma 2. The finiteness of the second integral follows from (1.3) and the inequality

$$\begin{aligned}\int (\gamma^*(x,y))^2 f(y)\, dy &= \int \gamma^2(x,y)f(y)\, dy - \Big(\int \gamma(x,y)yf(y)\, dy\Big)^2/\sigma^2 \\ &\le \int \gamma^2(x,y)f(y)\, dy.\end{aligned}$$

3. Behavior of the Density Estimators

Let $\tilde f$ denote the kernel density estimate

$$\tilde f(x) = \frac{1}{n}\sum_{j=1}^{n} K_b(x - \varepsilon_j), \quad x \in \mathbb{R},$$

based on the actual innovations, and let

$$\bar f(x) = \int_{-\infty}^{\infty} K_b(x - y)f(y)\, dy = \int_{-\infty}^{\infty} f(x - bu)K(u)\, du, \quad x \in \mathbb{R},$$

denotes its expectation. We have

$$\|\tilde f - \bar f\|_1 = O_P((nb)^{-1/2}). \tag{3.1}$$

Indeed, we calculate, using Lemma 2 and the substitution $x = y + bu$,

$$\begin{aligned}nbE[\|\tilde f - \bar f\|_1^2] &\le \pi\int_{-\infty}^{\infty}(1 + x^2)nbE[(\tilde f(x) - \bar f(x))^2]\, dx \\ &\le \pi\int_{-\infty}^{\infty}(1 + x^2)\int_{-\infty}^{\infty} bK_b^2(x - y)f(y)\, dy\, dx \\ &= \pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1 + (y + bu)^2)f(y)K^2(u)\, dy\, du \\ &= \pi(1 + \sigma^2)\|K^2\|_1 + \pi b^2\|(\iota_{\mathbb{R}}K)^2\|_1.\end{aligned}$$

The following result was proved in Müller et al. (2005) under stronger assumptions. In particular, we remove the assumption that the error density has a finite moment of order 5/2.

THEOREM 3. *Suppose the bandwidth $b = b_n$ satisfies $nb_n^4 \to 0$ and $nb_n^3 \to \infty$, the kernel K is a symmetric density with finite variance and is continuously differentiable with a bounded derivative K' with $\int_{-\infty}^{\infty}(1+u^2)|K'(u)|\,du$ finite, the estimator $\hat\varrho$ is root-n consistent, i.e., $\sqrt{n}(\hat\varrho - \varrho) = O_P(1)$, and the density f is L_1-Lipschitz with constant Λ. Then we have the stochastic rates*

$$\int_{-\infty}^{\infty} \Big|\hat f_1(x) - \tilde f(x)\Big|\,dx = O_P\Big(\frac{1}{nb^{3/2}}\Big)$$

and

$$\int_{-\infty}^{\infty} \Big|\hat f_2(x) - \tilde f(x) + xf(x)\hat\lambda\Big|\,dx = O_P\Big(\frac{1}{nb^{3/2}}\Big).$$

PROOF. The assumptions on K imply that K belongs to $\mathbb{H}_1$. Thus Lemma 9 implies that $\bar f = B_b(f,K)$ belongs to $\mathbb{H}_1$ and $\bar f' = B_b(f,K')/b$ is L_1-Lipschitz with constant $\Lambda\|K'\|_1/b$. It follows from Lemma 10 and $\int_{-\infty}^{\infty} K'(u)\,du = 0$ that $\|\bar f'\|_1 \le \Lambda\|\iota_{\mathbb{R}} K'\|_1$.

The residuals are of the form

$$\hat\varepsilon_j = X_j - \hat\varrho X_{j-1} = \varepsilon_j - (\hat\varrho - \varrho)X_{j-1}, \quad j = 1,\dots,n.$$

This representation, the root-n consistency of $\hat\varrho$ and the stochastic rates

$$\frac{1}{n}\sum_{j=1}^n X_{j-1} = O_P(n^{-1/2}) \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^n \varepsilon_j X_{j-1} = O_P(n^{-1/2})$$

yield

$$\frac{1}{n}\sum_{j=1}^n \hat\varepsilon_j = \frac{1}{n}\sum_{j=1}^n \varepsilon_j + O_P(1/n)$$

and

$$\frac{1}{n}\sum_{j=1}^n \hat\varepsilon_j^2 = \frac{1}{n}\sum_{j=1}^n \varepsilon_j^2 + O_P(1/n).$$

We also need the following results already established in Müller et al. (2005),

$$\hat\lambda = \frac{1}{n}\sum_{j=1}^n \frac{\varepsilon_j}{\sigma^2} + o_P(n^{-1/2}) = O_P(n^{-1/2}), \tag{3.2}$$

$$\max_{1\le j\le n} |\hat\varepsilon_j| = o_P(n^{1/2}). \tag{3.3}$$

A consequence of the above is the stochastic rate

$$\frac{1}{n}\sum_{j=1}^n \frac{\hat\lambda^2\hat\varepsilon_j^2}{1+\hat\lambda\hat\varepsilon_j} = O_P(n^{-1}) \tag{3.4}$$

which plays a role in comparing $\hat f_2$ with $\hat f_1$.

We shall establish the following stochastic rates.

$$\int_{-\infty}^{\infty} |\hat f_2(x) - \hat f_1(x) + \hat\lambda x \hat f_1(x)|\,dx = O_P(bn^{-1/2}), \tag{3.5}$$

$$\|\hat f_1 - \tilde f\|_1 = O_P(1/(nb^{3/2})), \tag{3.6}$$

$$\int_{-\infty}^{\infty} |x||\hat{f}_1(x) - f(x)|\,dx = O_P(b^{1/2}). \tag{3.7}$$

These stochastic rates together with $\hat{\lambda} = O_P(n^{-1/2})$ and $nb^4 \to 0$ imply the desired results.

To verify (3.5), we write

$$\begin{aligned}\hat{f}_2(x) - \hat{f}_1(x) + \hat{\lambda} x \hat{f}_1(x) &= \frac{1}{n}\sum_{j=1}^{n} K_b(x - \hat{\varepsilon}_j)\Big(\frac{1}{1+\hat{\lambda}\hat{\varepsilon}_j} - 1 + \hat{\lambda}\hat{\varepsilon}_j + \hat{\lambda}(x - \hat{\varepsilon}_j)\Big)\\ &= \frac{1}{n}\sum_{j=1}^{n} K_b(x - \hat{\varepsilon}_j)\Big(\frac{\hat{\lambda}^2\hat{\varepsilon}_j^2}{1+\hat{\lambda}\hat{\varepsilon}_j} + \hat{\lambda}(x - \hat{\varepsilon}_j)\Big)\end{aligned}$$

and then find that the left-hand side of (3.5) is bounded by

$$\frac{1}{n}\sum_{j=1}^{n} \frac{\hat{\lambda}^2\hat{\varepsilon}_j^2}{1+\hat{\lambda}\hat{\varepsilon}_j} + |\hat{\lambda}| b \int_{-\infty}^{\infty} |u| K(u)\,du = O_P(1/n) + O_P(bn^{-1/2}),$$

where we used (3.2) and (3.4) in the last step. This proves (3.5).

Next we prove (3.6). For this we write

$$\begin{aligned}\hat{f}_1(x) - \tilde{f}(x) &= \frac{1}{n}\sum_{j=1}^{n}\big(K_b\big(x - \varepsilon_j + (\hat{\varrho} - \varrho)X_{j-1}\big) - K_b(x - \varepsilon_j)\big)\\ &= H_{\sqrt{n}(\hat{\varrho}-\varrho)}(x) + D(x) + (\hat{\varrho} - \varrho)\frac{1}{n}\sum_{j=1}^{n} X_{j-1}\bar{f}'(x)\end{aligned}$$

with

$$H_t(x) = \frac{1}{n}\sum_{j=1}^{n}\Big(K_b\Big(x - \varepsilon_j + \frac{tX_{j-1}}{\sqrt{n}}\Big) - K_b(x - \varepsilon_j) - \bar{f}\Big(x + \frac{tX_{j-1}}{\sqrt{n}}\Big) + \bar{f}(x)\Big),$$

$$D(x) = \frac{1}{n}\sum_{j=1}^{n}\big(\bar{f}(x + (\hat{\varrho} - \varrho)X_{j-1}) - \bar{f}(x) - (\hat{\varrho} - \varrho)X_{j-1}\bar{f}'(x)\big).$$

Since $\bar{f}'$ has norm $\|\bar{f}'\|_1 \le \Lambda\|\iota_{\mathbb{R}} K'\|_1$ and is L_1-Lipschitz with constant $\Lambda\|K'\|_1/b$, we derive, utilizing Lemma 7 and setting $B = \Lambda\max\{\|K'\|_1, \|\iota_{\mathbb{R}} K'\|_1\}$,

$$\begin{aligned}\|\hat{f}_1 - \tilde{f}\|_1 &\le \|H_{\sqrt{n}(\hat{\varrho}-\varrho)}\|_1 + B\Big(\frac{1}{b}(\hat{\varrho} - \varrho)^2\frac{1}{n}\sum_{j=1}^{n} X_{j-1}^2 + |\hat{\varrho} - \varrho|\Big|\frac{1}{n}\sum_{j=1}^{n} X_{j-1}\Big|\Big)\\ &= \|H_{\sqrt{n}(\hat{\varrho}-\varrho)}\|_1 + O_P(1/(nb)).\end{aligned}$$

Thus the desired (3.6) follows if we verify that the stochastic rate

$$\sup_{|t|\le C} \|H_t\|_1 = O_P(1/(nb^{3/2}) \tag{3.8}$$

holds for every large constant C. Fix such a C. Since $\max_{1\le j\le n}|X_{j-1}|$ is of stochastic order $o_P(n^{1/2})$, we may replace H_t by $\bar{H}_t$ where

$$\bar{H}_t(x) = \frac{1}{n}\sum_{j=1}^{n}\Big(K_b\Big(x - \varepsilon_j + \frac{tX_{j-1}}{\sqrt{n}}\Big) - K_b(x - \varepsilon_j) - \bar{f}\Big(x + \frac{tX_{j-1}}{\sqrt{n}}\Big) + \bar{f}(x)\Big)W_{j-1}$$

with

$$W_{j-1} = \mathbf{1}[C|X_{j-1}| \le \sqrt{n}], \quad j = 1, \dots, n.$$

We have $\bar{H}_0(x) = 0$ for all x, and for s and t in $[-C, C]$ we have

$$\bar{H}_t(x) - \bar{H}_s(x) = \frac{1}{n}\sum_{j=1}^{n}(t-s)\frac{X_{j-1}}{\sqrt{n}}W_{j-1}V_j(x)$$

with

$$V_j(x) = \int_0^1 \Big(K_b'\Big(x - \varepsilon_j + (s+v(t-s))\frac{X_{j-1}}{\sqrt{n}}\Big) - \bar{f}'\Big(x + (s+v(t-s))\frac{X_{j-1}}{\sqrt{n}}\Big)\Big)\,dv.$$

A martingale argument yields

$$\begin{aligned} E[(\bar{H}_t(x) - \bar{H}_s(x))^2] &\le \frac{(t-s)^2}{n^2}E[X_0^2W_0V_1(x)^2] \\ &\le \frac{(t-s)^2}{n^2b^3}\int_0^1 E\Big[X_0^2W_0M_b(x-\varepsilon_1+(s+v(t-s))\frac{X_0}{\sqrt{n}})\Big]\,dv \end{aligned}$$

with $M_b(x) = (1/b)M(x/b)$ and $M = (K')^2$. Lemma 2 yields the bound

$$|\|\bar{H}_t\|_1 - \|\bar{H}_s\|_1|^2 \le \|\bar{H}_t - \bar{H}_s\|_1^2 \le \pi\int_{-\infty}^{\infty}(1+x^2)(\bar{H}_t(x) - \bar{H}_s(x))^2\,dx.$$

Combining the above yields the inequality

$$\begin{aligned} n^2b^3E[|\|\bar{H}_t\|_1 - \|\bar{H}_s\|_1|^2] &\le \pi\int_{-\infty}^{\infty}(1+x^2)n^2b^3E[(\bar{H}_t(x) - \bar{H}_s(x))^2]\,dx \\ &\le \pi(t-s)^2\int_0^1 I(v)\,dv \end{aligned}$$

with

$$\begin{aligned} I(v) &= \int_{-\infty}^{\infty}(1+x^2)E\Big[X_0^2\mathbf{1}[C|X_0| \le \sqrt{n}]M_b\Big(x-\varepsilon_1+(s+v(t-s))\frac{X_0}{\sqrt{n}}\Big)\Big]\,dx \\ &= \int_{-\infty}^{\infty}E\big[X_0^2\mathbf{1}[C|X_0| \le \sqrt{n}]\big(1+\big(\varepsilon_1-(s+v(t-s))X_0/\sqrt{n}+bu\big)^2\big)\big]M(u)\,du \\ &\le \int_{-\infty}^{\infty}E[X_0^2\mathbf{1}[C|X_0| \le \sqrt{n}](1+3\varepsilon_1^2+3b^2u^2+3)M(u)\,du \\ &\le E[X_0^2]\int_{-\infty}^{\infty}4(1+\sigma^2+b^2u^2)M(u)\,du \\ &\le 4(1+\sigma^2+b^2)E[X_0^2]\int_{-\infty}^{\infty}(1+u^2)M(u)\,du, \quad 0 < v < 1. \end{aligned}$$

In the first inequality we used the Cauchy–Schwarz inequality and the fact that $s + v(t-s)$ belongs to the interval $[-C, C]$ and is hence bounded by C. Note that $\|(1+\iota_{\mathbb{R}}^2)M\|_1$ is finite by the assumptions on K'. In view of Theorem 12.3 in Billingsley (1968), the process $\{\mathbb{X}_n(t) = nb^{3/2}\|\bar{H}_t\|_1, |t| \le C\}$ is tight and this implies (3.8).

We are left to verify (3.7). Since f is L_1-Lipschitz, the identity $\bar{f} = B_b(f, K)$ and Lemma 10 yield $\|\bar{f} - f\|_1 \le \Lambda b\|\iota_{\mathbb{R}}K\|_1$. This, the rate $nb^3 \to \infty$, (3.1) and (3.6) establish the stochastic rate

$$\|\hat{f}_1 - f\|_1 \le \|\hat{f}_1 - \tilde{f}\|_1 + \|\tilde{f} - \bar{f}\|_1 + \|\bar{f} - f\|_1 = O_P\Big(\frac{b^{3/2}}{nb^3} + \frac{b}{(nb^3)^{1/2}} + b\Big) = O_P(b).$$

In view of Lemma 3 the desired result follows from this, the stochastic rate

$$\int_{-\infty}^{\infty} x^2 \hat{f}_1(x)\,dx = \frac{1}{n}\sum_{j=1}^{n} \hat{\varepsilon}_j^2 + \int_{-\infty}^{\infty} b^2 u^2 K(u)\,du = O_P(1)$$

and the fact that f has a finite second moment. □

COROLLARY 3. *Under the assumptions of Theorem 3 the estimator $\hat{f}_1$ satisfies*

$$\|\hat{f}_1 - f\|_1 = O_P(b) \quad \text{and} \quad \|\iota_{\mathbb{R}}(\hat{f}_1 - f)\|_1 = O_P(b^{1/2}),$$

and the estimator $\hat{f}_2$ satisfies

$$\|\hat{f}_2 - f\|_1 = O_P(b) \quad \text{and} \quad \|\iota_{\mathbb{R}}(\hat{f}_2 - f)\|_1 = O_P(b^{1/2}).$$

PROOF. The first two rates were established in the proof of (3.7). The third rate follows from the first one and $\|\hat{f}_2 - \hat{f}_1\|_1 = O_P(n^{-1}b^{-3/2}) + O_P(n^{-1/2}) = o_p(b^{3/2})$ which is a consequence of Theorem 3. ¿From (3.2) and (3.3) we derive the bound

$$\int x^2 \hat{f}_2(x)\,d \le \max_{1\le j\le n} \frac{1}{1+\hat{\lambda}\varepsilon_j} \int x^2 \hat{f}_1(x)\,dx = O_p(1).$$

The argument used to prove (3.7) now yields $\|\iota_{\mathbb{R}}(\hat{f}_2 - f)\|_1 = O_P(b^{1/2})$. □

4. Behavior of the derivative of the density estimators

In this section we assume that the density f belongs to $\mathbb{H}_1$ and show that the derivatives of our kernel estimators estimate f' consistently in the L_1 norm.

THEOREM 4. *Suppose the bandwidth $b = b_n$ satisfies $nb_n^4 \to 0$ and $nb_n^3 \to \infty$, the kernel K is a symmetric density with finite variance and is twice continuously differentiable with $\|(1+\iota_{\mathbb{R}}^2)K'\|_1$ and $\|(1+\iota_{\mathbb{R}}^2)(K'')^2\|_1$ finite, the estimator $\hat{\varrho}$ is root-n consistent, and f belongs to $\mathbb{H}_1$. Then we have the stochastic rates*

$$\|\hat{f}_1' - f'\|_1 = o_P(1) \quad \text{and} \quad \|\hat{f}_2' - f'\|_1 = o_P(1).$$

PROOF. The assumptions on K imply that K belongs to $\mathbb{H}_2$ and meets the assumptions of Theorem 3. Since f' is integrable and $\bar{f}'$ equals $B_b(f', K)$, we have

$$\|\bar{f}' - f'\|_1 \le \int_{-\infty}^{\infty} w_{f'}(|bu|)K(u)\,du \to 0. \tag{4.1}$$

The desired result thus follows from the following stochastic rates:

$$\|\hat{f}_2' - \hat{f}_1'\|_1 = O_P(1/(bn^{1/2})), \tag{4.2}$$

$$\|\hat{f}_1' - \tilde{f}'\|_1 = O_P(1/(nb^{5/2})), \tag{4.3}$$

$$\|\tilde{f}' - \bar{f}'\|_1 = O_P(1/(nb^3)^{1/2}). \tag{4.4}$$

Let us first establish (4.2). In view of $\hat{\lambda} = O_P(n^{-1/2})$, it suffices to show the rates $\|\iota_{\mathbb{R}}\hat{f}_1'\|_1 = O_P(1/b)$ and $\|\hat{f}_2' - \hat{f}_1' - \hat{\lambda}\iota_{\mathbb{R}}\hat{f}_1'\|_1 = O_P(n^{-1/2})$. The former follows from the inequality

$$\begin{aligned}\|\iota_{\mathbb{R}}\hat f_1'\|_1 &\le \frac{1}{n}\sum_{j=1}^n \int_{-\infty}^{\infty} |x||K_b'(x-\hat\varepsilon_j)|\,dx\\ &\le \frac{1}{n}\sum_{j=1}^n \int_{-\infty}^{\infty} |\hat\varepsilon_j + bu|\frac{|K'(u)|}{b}\,du\\ &\le \frac{\|K'\|_1}{b}\Big(\frac{1}{n}\sum_{j=1}^n|\varepsilon_j| + |\hat\varrho-\varrho|\frac{1}{n}\sum_{j=1}^n |X_{j-1}|\Big) + \|\iota_{\mathbb{R}}K'\|_1.\end{aligned}$$

For the latter we use the identity

$$\begin{aligned}\hat f_2'(x) - \hat f_1'(x) + \hat\lambda x \hat f_1'(x) &= \frac{1}{n}\sum_{j=1}^n K_b'(x-\hat\varepsilon_j)\Big(\frac{1}{1+\hat\lambda\hat\varepsilon_j} - 1 + \hat\lambda\hat\varepsilon_j + \hat\lambda(x-\hat\varepsilon_j)\Big)\\ &= \frac{1}{n}\sum_{j=1}^n K_b'(x-\hat\varepsilon_j)\Big(\frac{\hat\lambda^2\hat\varepsilon_j^2}{1+\hat\lambda\hat\varepsilon_j} + \hat\lambda(x-\hat\varepsilon_j)\Big)\end{aligned}$$

to obtain the inequality

$$\|\hat f_2' - \hat f_1' - \hat\lambda\iota_{\mathbb{R}}\hat f_1'\|_1 \le \frac{1}{n}\sum_{j=1}^n \frac{\hat\lambda^2\hat\varepsilon_j^2}{1+\hat\lambda\hat\varepsilon_j}\frac{\|K'\|_1}{b} + |\hat\lambda|\|\iota_{\mathbb{R}}K'\|_1 = O_P\Big(\frac{1}{nb}\Big) + O_P\Big(\frac{1}{\sqrt{n}}\Big)$$

where we used (3.2) and (3.4) in the last step. This proves (4.2).

Let us now prove (4.3). We write

$$\begin{aligned}\hat f_1'(x) - \tilde f'(x) &= \frac{1}{n}\sum_{j=1}^n \big(K_b'(x-\varepsilon_j + (\hat\varrho-\varrho)X_{j-1}) - K_b'(x-\varepsilon_j)\big)\\ &= H'_{\sqrt{n}(\hat\varrho-\varrho)}(x) + D'(x) + (\hat\varrho-\varrho)\frac{1}{n}\sum_{j=1}^n X_{j-1}\bar f''(x)\end{aligned}$$

with

$$H_t'(x) = \frac{1}{n}\sum_{j=1}^n \Big(K_b'\Big(x-\varepsilon_j+\frac{tX_{j-1}}{\sqrt{n}}\Big) - K_b'(x-\varepsilon_j) - \bar f'\Big(x+\frac{tX_{j-1}}{\sqrt{n}}\Big) + \bar f'(x)\Big),$$

$$D'(x) = \frac{1}{n}\sum_{j=1}^n \big(\bar f'(x+(\hat\varrho-\varrho)X_{j-1}) - \bar f'(x) - (\hat\varrho-\varrho)X_{j-1}\bar f''(x)\big).$$

It follows from Lemma 9 that $\bar f$ belongs to $\mathbb{H}_2$ and $\bar f'' = B_b(f',K')/b$ has norm $\|\bar f''\|_1 \le \|f'\|_1\|K'\|_1/b$ and is L_1-Lipschitz with Lipschitz constant $\|f'\|_1\|K''\|_1/b^2$. Using this and Lemma 7 we obtain with $B = \|f'\|_1 \max\{\|K'\|_1, \|K''\|_1\}$,

$$\begin{aligned}\|\hat f_1' - \tilde f'\|_1 &\le \|H'_{\sqrt{n}(\hat\varrho-\varrho)}\|_1 + B\Big(\frac{1}{b^2}(\hat\varrho-\varrho)^2\frac{1}{n}\sum_{j=1}^n X_{j-1}^2 + \frac{1}{b}|\hat\varrho-\varrho|\Big|\frac{1}{n}\sum_{j=1}^n X_{j-1}\Big|\Big)\\ &= \|H'_{\sqrt{n}(\hat\varrho-\varrho)}\|_1 + O_P(1/(nb^2)).\end{aligned}$$

Thus the desired (4.3) follows if we verify that the stochastic rate

$$\sup_{|t|\le C}\|H_t'\|_1 = O_P(1/(nb^{5/2})) \tag{4.5}$$

holds for every large constant C. Fix such a C. Since $\max_{1\le j\le n}|X_{j-1}|$ is of stochastic order $o_P(n^{1/2})$, we may replace H'_t by $\bar{H}'_t$ where

$$\bar{H}'_t(x) = \frac{1}{n}\sum_{j=1}^{n}\Big(K'_b\Big(x-\varepsilon_j+\frac{tX_{j-1}}{\sqrt{n}}\Big)-K'_b(x-\varepsilon_j)-\bar{f}'\Big(x+\frac{tX_{j-1}}{\sqrt{n}}\Big)+\bar{f}'(x)\Big)W_{j-1}$$

with $W_{j-1} = \mathbf{1}[C|X_{j-1}|\le\sqrt{n}]$, $j=1,\dots,n$, as in the proof of Theorem 3. We have $\bar{H}'_0(x)=0$ for all x, and for s and t in $[-C,C]$ we have

$$\bar{H}'_t(x)-\bar{H}'_s(x) = \frac{1}{n}\sum_{j=1}^{n}(t-s)\frac{X_{j-1}}{\sqrt{n}}W_{j-1}V_j(x)$$

with

$$V_j(x) = \int_0^1\Big(K''_b\Big(x-\varepsilon_j+(s+v(t-s))\frac{X_{j-1}}{\sqrt{n}}\Big)-\bar{f}''\Big(x+(s+v(t-s))\frac{X_{j-1}}{\sqrt{n}}\Big)\Big)\,dv.$$

A martingale argument yields

$$\begin{aligned}E[(\bar{H}'_t(x)-\bar{H}'_s(x))^2] &\le \frac{(t-s)^2}{n^2}E[X_0^2W_0V_1(x)^2]\\ &\le \frac{(t-s)^2}{n^2b^5}\int_0^1 E\Big[X_0^2W_0M_b\Big(x-\varepsilon_1+(s+v(t-s))\frac{X_0}{\sqrt{n}}\Big)\Big]\,dv\end{aligned}$$

with $M_b(x)=(1/b)M(x/b)$ and $M=(K'')^2$. Lemma 2 yields the bound

$$|\|\bar{H}'_t\|_1-\|\bar{H}'_s\|_1|^2 \le \|\bar{H}'_t-\bar{H}'_s\|_1^2 \le \pi\int_{-\infty}^{\infty}(1+x^2)(\bar{H}'_t(x)-\bar{H}'_s(x))^2\,dx.$$

Combining the above yields the inequality

$$\begin{aligned}n^2b^5E[|\|\bar{H}'_t\|_1-\|\bar{H}'_s\|_1|^2] &\le \pi\int_{-\infty}^{\infty}(1+x^2)n^3b^5E[(\bar{H}'_t(x)-\bar{H}'_s(x))^2]\,dx\\ &\le \pi(t-s)^2\int_0^1 I(v)\,dv\end{aligned}$$

with

$$\begin{aligned}I(v) &= \int_{-\infty}^{\infty}(1+x^2)E[X_0^2\mathbf{1}[C|X_0|\le\sqrt{n}]M_b\Big(x-\varepsilon_1+(s+v(t-s))\frac{X_0}{\sqrt{n}}\Big)\,dx\\ &\le 4(1+\sigma^2+b^2)E[X_0^2]\int_{-\infty}^{\infty}(1+u^2)M(u)\,du, \quad 0<v<1,\end{aligned}$$

where the inequality is obtained as in the proof of Theorem 3. In view of Theorem 12.3 in Billingsley (1968), the process $\{\mathbb{X}'_n(t)=nb^{5/2}\|\bar{H}'_t\|_1, |t|\le C\}$ is tight and this implies (4.5).

We are left to verify (4.4). Using Lemma 2 we calculate

$$\begin{aligned}nb^3E[\|\tilde{f}'-\bar{f}'\|_1^2] &\le \pi\int_{-\infty}^{\infty}(1+x^2)nb^3E[(\tilde{f}'(x)-\bar{f}'(x))^2]\,dx\\ &\le \pi\int_{-\infty}^{\infty}(1+x^2)\int_{-\infty}^{\infty}b^3(K'_b(x-y))^2f(y)\,dy\,dx\\ &= \pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+(y+bu)^2)f(y)(K'(u))^2\,dy\,du\\ &= \pi(1+\sigma^2)\|(K')^2\|_1+\pi b^2\|(\iota_{\mathbb{R}}K')^2\|_1.\end{aligned}$$

This shows $\|\tilde{f}' - \bar{f}'\|_1 = O_P(1/\sqrt{nb^3})$.

5. Proof of Theorem 1

Let $\hat{f}$ denote either the estimator $\hat{f}_1$ or the estimator $\hat{f}_2$. In view of the properties of k_n and $b = b_n$, Corollary 3 implies the stochastic rates

$$(k_n + 1)\|\hat{f} - f\|_1 = o_P(n^{-1/4}) \tag{5.1}$$

and

$$\|\iota_{\mathbb{R}}(\hat{f} - f)\|_1 = O_p(b^{1/2}), \tag{5.2}$$

while Theorem 4 yields

$$\|\hat{f}' - f'\|_1 = o_P(1). \tag{5.3}$$

Recall that $\hat{g}_{k_n}$ can be expressed as

$$\hat{g}_{k_n}(x) = \int_{\mathbb{R}^{k_n+1}} \hat{f}\Big(x - \sum_{i=1}^{k_n} \hat{\varrho}^i y_i - \hat{\varrho}^{k_n+1} z\Big) \prod_{j=1}^{k_n} \hat{f}(y_j)\, dy_j\ \hat{g}(z)\, dz, \quad x \in \mathbb{R}. \tag{5.4}$$

Corollary 1, assumption (A2) and the bound (1.1) yield

$$\Big\| g_{\hat{\varrho},f} - g - \dot{g}\frac{1}{n}\sum_{j=1}^{n} \psi(X_{j-1}, \varepsilon_j)\Big\|_1 = o_P(n^{-1/2}).$$

Thus we need to compare $\hat{g}_{k_n}$ with $g_{\hat{\varrho},f}$. For this we represent $g_{\hat{\varrho},f}$ as

$$g_{\hat{\varrho},f}(x) = \int_{\mathbb{R}^{k_n+1}} f\Big(x - \sum_{i=1}^{k_n} \hat{\varrho}^i y_i - \hat{\varrho}^{k_n+1} z\Big) \prod_{j=1}^{k_n} f(y_j)\, dy_j\ g_{\hat{\varrho},f}(z)\, dz, \quad x \in \mathbb{R}. \tag{5.5}$$

Replacing $\hat{\varrho}^{k_n+1}$ by ϱ^{k_n+1} and $g_{\hat{\varrho},f}(z)$ by $g(z)$ yields

$$g^*_{\hat{\varrho},f}(z) = \int_{\mathbb{R}^{k_n+1}} f\Big(x - \sum_{i=1}^{k_n} \hat{\varrho}^i y_i - \varrho^{k_n+1} z\Big) \prod_{j=1}^{k_n} f(y_j)\, dy_j\ g(z)\, dz, \quad x \in \mathbb{R}.$$

Replacing $\hat{\varrho}^{k_n+1}$ by ϱ^{k_n+1} and $\hat{g}(z)$ by $g(z)$ in (5.4) yields

$$\hat{g}^*_{k_n}(x) = \int_{\mathbb{R}^{k_n+1}} \hat{f}\Big(x - \sum_{i=1}^{k_n} \hat{\varrho}^i y_i - \varrho^{k_n+1} z\Big) \prod_{j=1}^{k_n} \hat{f}(y_j)\, dy_j\ g(z)\, dz, \quad x \in \mathbb{R}.$$

We begin by establishing the rates

$$\|\hat{g}_{k_n} - \hat{g}^*_{k_n}\|_1 = o_P(n^{-1/2}) \tag{5.6}$$

and

$$\|g_{\hat{\varrho},f} - g^*_{\hat{\varrho},f}\|_1 = o_P(n^{-1/2}). \tag{5.7}$$

We have the identities $\hat{g}_{k_n} = B_{\hat{\varrho}^{k_n+1}}(\hat{p}, \hat{g})$ and $\hat{g}^*_{k_n} = B_{\varrho^{k_n+1}}(\hat{p}, g)$ with

$$\hat{p}(x) = \int_{\mathbb{R}^{k_n}} \hat{f}\Big(x - \sum_{i=1}^{k_n} \hat{\varrho}^i y_i\Big) \prod_{j=1}^{k_n} \hat{f}(y_j)\, dy_j, \quad x \in \mathbb{R}.$$

It is easy to check that $\hat{p}$ is $\mathbb{H}_1$-valued with

$$\hat{p}'(x) = \int_{\mathbb{R}^{k_n}} \hat{f}'\Big(x - \sum_{i=1}^{k_n} \hat{\varrho}^i y_i\Big) \prod_{j=1}^{k} \hat{f}(y_j)\, dy_j, \quad x \in \mathbb{R},$$

and that $\|\hat{p}'\|_1 \le \|\hat{f}'\|_1 = O_P(1)$ holds in view of (5.3). Using Lemma 11 we obtain

$$\|\hat{g}_{k_n} - \hat{g}^*_{k_n}\|_1 \le \|\hat{g}_{k_n} - \hat{p}\|_1 + \|\hat{g}^*_{k_n} - \hat{p}\|_1 \le \|\hat{p}'\|_1(\hat{\varrho}^{k_n+1}\|\iota_{\mathbb{R}}\hat{g}\|_1 + \varrho^{k_n+1}\|\iota_{\mathbb{R}} g\|_1).$$

The desired (5.6) now follows from $\hat{\varrho}^{k_n+1} = o_P(n^{-1/2})$ and $\|\iota_{\mathbb{R}}\hat{g}\|_1 = O_P(1)$. Indeed, the former follows from

$$n^{1/2}\hat{\varrho}^{k_n+1} = \exp\Big(-(k_n+1)\Big(\log(1/\hat{\varrho}) - \frac{\log(n)}{2(k_n+1)}\Big)\Big) = o_P(1)$$

and the latter from

$$E(\|\iota_{\mathbb{R}}\hat{g}\|_1) = \|\iota_{\mathbb{R}}E(\hat{g})\|_1 = \|\iota_{\mathbb{R}} g * K_b\|_1 \le \|\iota_{\mathbb{R}} g\|_1 + \|\iota_{\mathbb{R}} K_b\|_1 = \|\iota_{\mathbb{R}} g\|_1 + O(b).$$

A similar argument yields (5.7).

We are left to compare $\hat{g}^*_{k_n}$ and $g^*_{\hat{\varrho},f}$. For this we express $\hat{g}^*_{k_n}$ as $L_{\hat{\varrho}}(\hat{f},\dots,\hat{f})$ and $g^*_{\hat{\varrho},f}$ as $L_{\hat{\varrho}}(f,\dots,f)$ where

$$L_r(h_0,\dots,h_{k_n})(x) = \int_{\mathbb{R}^{k_n+1}} h_0\Big(x - \sum_{i=1}^{k_n} r^i y_i - \varrho^{k_n+1} z\Big)\prod_{j=1}^{k_n} h_j(y_j)\,dy_j g(z)\,dz$$

for integrable functions $h_0,\dots,h_{k_n}$ and positive numbers r. One checks

$$\|L_r(h_0,\dots,h_{k_n})\|_1 \le \prod_{j=0}^{k_n}\|h_j\|_1. \tag{5.8}$$

To simplify notation, we set

$$\bar{L}_{r,h} = L_r(h,\dots,h)$$

and, for a subset A of $\{0,\dots,k_n\}$,

$$L_{r,A}(p,q) = L_r(h_0,\dots,h_{k_n}) \quad \text{with } h_i = \begin{cases} p, & i\in A,\\ q, & i\notin A,\end{cases}, \quad i = 0,\dots,k_n.$$

We use the identity

$$\prod_{j=0}^{k_n}(a_j+b_j) = \sum_{A\subset\{0,1,\dots,k_n\}}\prod_{j\in A} a_j \prod_{j\in A^c} b_j$$

to conclude

$$\bar{L}_{r,p+q} = \sum_{A\in\{0,1,\dots,k_n\}} L_{r,A}(p,q).$$

Applying this with $r=\hat{\varrho}$, $p = \hat{f} - f$ and $q = f$, and singling out the terms with A of cardinality at most one, we obtain

$$\bar{L}_{\hat{\varrho},\hat{f}} = \bar{L}_{\hat{\varrho},f} + \sum_{j=0}^{k_n} L_{\hat{\varrho},\{j\}}(\hat{f}-f,f) + R_1$$

with

$$R_1 = \sum_{\substack{A\subset\{0,\dots,k_n\}\\ \mathrm{card}(A)\ge 2}} L_{\hat{\varrho},A}(\hat{f}-f,f)$$

By (5.8), $\|L_{r,A}(p,q)\|_1$ is bounded by $\|p\|_1^a\|q\|_1^{k_n+1-a}$ with a the cardinality of A. Since there are $\binom{k_n+1}{a} \le (k_n+1)^a/a!$ subsets of cardinality a, we obtain

$$\|R_1\|_1 \le \sum_{a=2}^{k_n+1} \frac{(k_n+1)^a}{a!}\|\hat f - f\|_1^a \le ((k_n+1)\|\hat f - f\|_1)^2 e^{(k_n+1)\|\hat f - f\|_1}.$$

In view of the rate $(k_n+1)\|\hat f - f\|_1 = o_P(n^{-1/4})$ given in (5.1), this establishes

$$\Big\|\hat g^*_{k_n} - g^*_{\hat\varrho,f} - \sum_{j=0}^{k_n} L_{\hat\varrho,\{j\}}(\hat f - f, f)\Big\|_1 = o_P(n^{-1/2}). \tag{5.9}$$

Our next goal is to verify

$$\Big\|\sum_{j=0}^{k_n} L_{\hat\varrho,\{j\}}(\hat f - f, f) - \sum_{j=0}^{k_n} L_{\varrho,\{j\}}(\hat f - f, f)\Big\|_1 = o_P(n^{-1/2}). \tag{5.10}$$

To achieve this we derive bounds on the terms

$$D_i(h) = \|L_{\hat\varrho,\{i\}}(h,f) - L_{\varrho,\{i\}}(h,f)\|_1, \quad i = 0,\dots,k_n,$$

for $h \in \mathbb{H}_1$ with $\|\iota_{\mathbb{R}} h\|_1$ finite. Using Lemma 7 we obtain

$$D_0(h) \le \|h'\|_1 \sum_{j=1}^{k_n} |\hat\varrho^j - \varrho^j|\|\iota_{\mathbb{R}} f\|_1$$

and

$$\begin{aligned} D_i(h) &\le \|f'\|_1 \sum_{j=1}^{k_n} |\hat\varrho^j - \varrho^j|\big(\mathbf{1}[j\ne i]\|h\|_1\|\iota_{\mathbb{R}} f\|_1 + \mathbf{1}[j=i]\|\iota_{\mathbb{R}} h\|_1\big) \\ &\le \|f'\|_1\|h\|_1\|\iota_{\mathbb{R}} f\|_1 \sum_{j=1}^{k_n} |\hat\varrho^j - \varrho^j| + \|f'\|_1\|\iota_{\mathbb{R}} h\|_1|\hat\varrho^i - \varrho^i|, \quad i = 1,\dots,k_n. \end{aligned}$$

Using the inequality (2.1) with $r = \hat\varrho$ and $s = \varrho$ and taking $h = \hat f - f$ we obtain that the left-hand side of (5.10) is bounded by

$$\frac{\|\hat f' - f'\|_1\|\iota_{\mathbb{R}} f\|_1 + \|f'\|_1\|\iota_{\mathbb{R}}(\hat f - f)\|_1 + k_n\|f'\|_1\|\hat f - f\|_1\|\iota_{\mathbb{R}} f\|_1}{(1 - \max\{\hat\varrho,\varrho\})^2}|\hat\varrho - \varrho|.$$

This bound is of order $o_P(n^{-1/2})$ because $\hat\varrho - \varrho$ is of order $O_P(n^{-1/2})$ and the terms $k_n\|\hat f - f\|_1$, $\|\iota_{\mathbb{R}}(\hat f - f)\|_1$ and $\|\hat f' - f'\|_1$ are of order $o_P(1)$ in view of (5.1)–(5.3).

Next we observe the identity

$$\sum_{j=0}^{k_n} L_{\varrho,j}(\hat f - f, f) = \sum_{j=0}^{k_n} B_{\varrho^j}(\gamma_j, \hat f - f) = \sum_{j=0}^{k_n} \Gamma_j(\hat f - f)$$

where, for an integrable function h, $\Gamma_j h = B_{\varrho^j}(\gamma_j, h)$ is the function defined by

$$\Gamma_j h(x) = \int_{-\infty}^{\infty} \gamma_j(x - \varrho^j y)h(y)\,dy, \quad x \in \mathbb{R}.$$

We have $\|\Gamma_j h\|_1 \le \|h\|_1$ for all integrable h. Using this and Theorem 3 we derive

$$\sum_{j=0}^{k_n} \|\Gamma_j(\hat f_1 - \tilde f)\|_1 \le (k_n+1)\|\hat f_1 - \tilde f\|_1 = O_P(k_n/(nb^{3/2}))$$

and

$$\sum_{j=0}^{k_n} \|\Gamma_j(\hat{f}_2 - \tilde{f}) + \hat{\lambda}\Gamma_j(\iota_{\mathbb{R}} f))\|_1 \le (k_n+1)\|\hat{f}_2 - \tilde{f} + \hat{\lambda}\iota_{\mathbb{R}} f\|_1 = O_P(k_n/(nb^{3/2})).$$

Since f has mean zero and finite variance, Lemma 10 and Corollary 2 yield the inequalities

$$\|\Gamma_j(\iota_{\mathbb{R}} f)\|_1 \le \varrho^j \|\gamma_j'\|_1 \|\iota_{\mathbb{R}}^2 f\|_1 \le C_1\sigma^2\varrho^j, \quad j \ge 0.$$

This and the expansion (3.2) yield

$$\Big\|\hat{\lambda}\sum_{j=0}^{k_n}\Gamma_j(\iota_{\mathbb{R}} f) - \frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_i}{\sigma^2}\sum_{j=0}^{\infty}\Gamma_j(\iota_{\mathbb{R}} f)\Big\|_1 = o_P(n^{-1/2}).$$

Since f has mean zero, we compute

$$\Gamma_j(\iota_{\mathbb{R}} f)(x) = \int_{-\infty}^{\infty}(\gamma_j(x-\varrho^j y) - g(x))yf(y)\,dy, \quad x \in \mathbb{R},$$

and obtain the identity

$$\sum_{j=0}^{\infty}\Gamma_j(\iota_{\mathbb{R}} f)(x) = \int_{-\infty}^{\infty}\gamma(x,y)yf(y)\,dy, \quad x \in \mathbb{R}.$$

In view of $k_n^2/(nb^3) \to 0$, we obtain

$$\Big\|\sum_{j=0}^{k_n} L_{\varrho,\{j\}}(\hat{f}_1 - f) - \sum_{j=0}^{k_n}\Gamma_j(\tilde{f} - f)\Big\|_1 = o_P(n^{-1/2}) \tag{5.11}$$

and

$$\Big\|\sum_{j=0}^{k_n} L_{\varrho,\{j\}}(\hat{f}_2 - f) - \sum_{j=0}^{k_n}\Gamma_j(\tilde{f} - f) + \sum_{j=0}^{\infty}\Gamma_j(\iota_{\mathbb{R}} f)\frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_i}{\sigma^2}\Big\|_1 = o_P(n^{-1/2}). \tag{5.12}$$

For $j = 0, 1, \dots,$ we compute

$$\Gamma_j\tilde{f}(x) = \int \bar{\gamma}_j(x - \varrho^j bu)K(u)\,du, \quad x \in \mathbb{R},$$

with

$$\bar{\gamma}_j(x) = \frac{1}{n}\sum_{i=1}^{n}\gamma_j(x - \varrho^j\varepsilon_i), \quad x \in \mathbb{R}.$$

By Corollary 2, γ_j belongs to $\mathbb{H}_2$ with $\|\gamma_j''\|_1 \le C_3$. This implies that $\bar{\gamma}_j$ is $\mathbb{H}_2$-valued and $\|\bar{\gamma}_j''\|_1 \le C_3$. Lemma 11, applied with $t = b\varrho^j$, $p = \bar{\gamma}_j$ and $q = K$, yields the bound

$$\|\Gamma_j\tilde{f} - \bar{\gamma}_j\|_1 \le C_3 b^2\varrho^{2j}\|\iota_{\mathbb{R}}^2 K\|_1.$$

Since $\Gamma_j f = g$, we derive

$$\Big\|\sum_{j=0}^{k_n}(\Gamma_j(\tilde{f} - f) - (\bar{\gamma}_j - g))\Big\|_1 \le \sum_{j=0}^{k_n}\|\Gamma_j\tilde{f} - \bar{\gamma}_j\|_1 = O_P(b^2) = o_P(n^{-1/2}).$$

Finally, using Corollary 2 we obtain the rate

$$E\Big[\Big\|\sum_{j=k_n+1}^{\infty}(\bar{\gamma}_j - g)\Big\|_1\Big] \le \sum_{j=k_n+1}^{\infty}\|\gamma_j'\|_1\varrho^j E[|\varepsilon_1 - \varepsilon_2|] = O(\varrho^{k_n+1}) = o_P(n^{-1/2}).$$

¿From this we conclude

$$\Big\| \sum_{j=0}^{k_n} \Gamma_j(\tilde f - f) - \bar\Gamma_n \Big\|_1 = o_P(n^{-1/2}) \tag{5.13}$$

with

$$\bar\Gamma_n(x) = \frac{1}{n} \sum_{i=1}^n \gamma(x, \varepsilon_i), \quad x \in \mathbb{R}.$$

The first L_1-Bahadur expansion follows from (5.6), (5.7), (5.9), (5.10), (5.11) and (5.13), while the second follows from (5.6), (5.7), (5.9), (5.10), (5.12) and (5.13).

6. Proof of Theorem 2

Corollary 1 and $n^{1/2}(\varrho_n - \varrho) \to t$ imply

$$\|n^{1/2}(g_{\varrho_n,f} - g_{\varrho,f}) - t\dot g\|_1 \to 0.$$

Thus we are left to show

$$\|n^{1/2}(g_{\varrho_n,f_n} - g_{\varrho_n,f}) - Ah\|_1 \to 0. \tag{6.1}$$

Recall that hf and $\iota_{\mathbb{R}} hf$ integrate to zero, i.e.,

$$\int_{-\infty}^{\infty} h(y)f(y)\,dy = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} yh(y)f(y)\,dy = 0.$$

The second integral condition allows us to replace the function γ^* in the definition of Ah by γ, and this leads to the representation

$$Ah = \sum_{j=0}^{\infty} \Gamma_j(hf)$$

in view of the definition of γ and the first integral condition which gives

$$\int_{-\infty}^{\infty} (\gamma_j(x - \varrho^j y) - g(x))h(y)f(y)\,dy = \int_{-\infty}^{\infty} \gamma_j(x - \varrho^j y)h(y)f(y)\,dy = \Gamma_j(hf)(x).$$

It follows from Corollary 2 and Lemma 10 that $\|\Gamma_j(hf)\|_1 \le C_1 \varrho^j \|\iota_{\mathbb{R}} hf\|_1$. Using the definition of γ_0, we have the identity $\Gamma_0(hf) = B_\varrho(hf, g)$. Let k_n be a sequence of positive integers such that $k_n/(\log(n))^2 \to 1$. This implies that $n^{1/2}\varrho^{k_n} \to 0$ and therefore $\sum_{j=k_n+1}^{\infty} \|\Gamma_j(hf)\|_1 = o(n^{-1/2})$. Thus we achieve (6.1) by verifying

$$\|n^{1/2}(g_{\varrho_n,f_n} - \bar g_n) - B_\varrho(hf, g)\|_1 \to 0 \tag{6.2}$$

and

$$\Big\| n^{1/2}(\bar g_n - g_{\varrho_n,f}) - \sum_{j=1}^{k_n} \Gamma_j(hf) \Big\|_1 \to 0 \tag{6.3}$$

with

$$\bar g_n(x) = \int_{\mathbb{R}^{k_n+1}} f\Big(x - \sum_{j=1}^{k_n} \varrho_n^j y_j - \varrho^{k_n+1} z\Big) \prod_{j=1}^{k_n} f_n(y_j)\,dy_j\, g(z)\,dz, \quad x \in \mathbb{R}.$$

Let us set

$$\Delta_n = n^{1/2}(f_n - f) - hf.$$

We begin by showing that (1.5) and (1.6) imply

$$\|\Delta_n\|_1 \to 0 \quad \text{and} \quad \|\iota_{\mathbb{R}} \Delta_n\|_1 \to 0. \tag{6.4}$$

For this we set $\tau = \frac{1}{2}h\sqrt{f}$ and write

$$\begin{aligned} n^{1/2}(f_n - f) - hf &= n^{1/2}(\sqrt{f_n} - \sqrt{f})(\sqrt{f_n} + \sqrt{f}) - 2\tau\sqrt{f} \\ &= \Big(n^{1/2}(\sqrt{f_n} - \sqrt{f}) - \tau\Big)(\sqrt{f_n} + \sqrt{f}) + \tau(\sqrt{f_n} - \sqrt{f}). \end{aligned}$$

This shows that $\|\Delta_n\|_1$ is bounded by

$$\|n^{1/2}(\sqrt{f_n} - \sqrt{f}) - \tau\|_2\|\sqrt{f_n} + \sqrt{f}\|_2 + \|\tau\|_2\|\sqrt{f_n} - \sqrt{f}\|_2$$

and $\|\iota_{\mathbb{R}}\Delta_n\|_1$ is bounded by

$$\|n^{1/2}(\sqrt{f_n} - \sqrt{f}) - \tau\|_2\|\iota_{\mathbb{R}}(\sqrt{f_n} + \sqrt{f})\|_2 + \|\tau\|_2\|\iota_{\mathbb{R}}(\sqrt{f_n} - \sqrt{f})\|_2.$$

These bounds converge to 0 because (1.5) implies that $n^{1/2}(\sqrt{f_n} - \sqrt{f})$ converges in L_2 to τ and $\sqrt{f_n} - \sqrt{f}$ converges in L_2 to 0 and because (1.6) implies that $\|\iota_{\mathbb{R}}(\sqrt{f_n} - \sqrt{f}\|_2^2 \le \|\iota_{\mathbb{R}}^2|f_n - f|\|_1 \to 0$.

As a consequence of (6.4), the bilinearity of B_{ϱ_n} and $\|B_{\varrho_n}(p,q)\|_1 \le \|p\|_1\|q\|_1$ we obtain

$$\|n^{1/2}(B_{\varrho_n}(f_n, g_n) - B_{\varrho_n}(f, g_n)) - B_{\varrho_n}(hf, g_n)\|_1 \le \|n^{1/2}(f_n - f) - hf\|_1 \to 0 \tag{6.5}$$

with $g_n = g_{\varrho_n, f_n}$. For this g_n we have

$$B_{\varrho_n}(f, g_n)(x) = \int_{\mathbb{R}^{k_n+1}} f\Big(x - \sum_{i=1}^{k_n} \varrho_n^i y_i - \varrho_n^{k_n+1} z\Big) \prod_{j=1}^{k_n} f_n(y_j)\, dy_j g_n(z)\, dz$$

and obtain by an argument similar to the one used to derive (5.6),

$$\|n^{1/2}(B_{\varrho_n}(f, g_n) - \bar{g}_n)\|_1 \le \|f'\|_1 n^{1/2}\big(\varrho_n^{k_n+1}\|\iota_{\mathbb{R}} g_n\|_1 + \varrho^{k_n+1}\|\iota_{\mathbb{R}} g\|_1\big) \to 0. \tag{6.6}$$

Now we use the representations

$$B_{\varrho_n}(hf, g_n)(x) = E((hf)(x - S_n)) \quad \text{and} \quad B_{\varrho}(hf, g)(x) = E((hf)(x - S)), \quad x \in \mathbb{R},$$

with $S_n = \sum_{j=1}^\infty \varrho_n^j F_n^{-1}(U_j))$ and with $S = \sum_{j=1}^\infty \varrho^j F^{-1}(U_j))$, where $U_1, U_2, \dots$ are independent uniform random variables and F_n^{-1} and F^{-1} are the left-inverses of the distribution functions F_n and F with respective densities f_n and f. We verify

$$E[|S_n - S|] \le \sum_{j=1}^\infty |\varrho_n^j - \varrho^j|\|\iota_{\mathbb{R}} f_n\|_1 + \sum_{j=1}^\infty \varrho^j \int_0^1 |F_n^{-1}(u) - F^{-1}(u)|\, du \to 0$$

with the help of Lemma 1, properties (1.6) and the inequality

$$\int_0^1 |F_n^{-1}(u) - F^{-1}(u)|\, du = \int_{-\infty}^\infty |F_n(x) - F(x)|\, dx \le \int_{-\infty}^\infty |y||f_n(y) - f(y)|\, dy \to 0$$

where the convergence to 0 follows from (6.4). An application of Lemma 4 yields

$$\|B_{\varrho_n}(hf, g_n) - B_{\varrho}(hf, g)\|_1 \to 0. \tag{6.7}$$

The desired (6.2) follows from (6.5)–(6.7).

To verify (6.3) we begin by observing the identity $\bar{g}_n = L_{\varrho_n,\{0\}}(f, f_n)$. An argument similar to the one that produced (5.9) yields

$$n^{1/2}\Big\|\bar{g}_n - \sum_{j=1}^{k_n} L_{\varrho_n,\{j\}}(f_n - f, f)\Big\|_1 \to 0,$$

now using $k_n\|f_n - f\|_1 = O(k_n n^{-1/2})$. Next, mimicking the argument that lead to (5.10) yields

$$n^{1/2}\Big\|\sum_{j=1}^{k_n} L_{\varrho_n,\{j\}}(f_n - f, f) - \sum_{j=1}^{k_n} L_{\varrho,\{j\}}(f_n - f), f)\Big\|_1 \le \frac{C_n n^{1/2}|\varrho_n - \varrho|}{(1-\max\{\varrho_n, \varrho\})^2} \to 0$$

with $C_n = \|f'\|_1\big(\|\iota_{\mathbb{R}}(f_n - f)\|_1 + k_n\|f_n - f\|_1\|\iota_{\mathbb{R}} f\|_1\big) \to 0$. Finally, we have

$$n^{1/2}L_{\varrho,\{j\}}(f_n - f, f) - \Gamma_j(hf) = n^{1/2}B_{\varrho^j}(\gamma_j, f_n - f) - B_{\varrho^j}(\gamma_j, hf) = B_{\varrho^j}(\gamma_j, \Delta_n),$$

and Lemma 10, Corollary 2 and (6.4) imply

$$\Big\|\sum_{j=1}^{k_n} B_{\varrho^j}(\gamma_j, \Delta_n)\Big\|_1 \le \sum_{j=1}^{k_n} \|\gamma_j'\|_1 \varrho^j \|\iota_{\mathbb{R}}\Delta_n\|_1 \to 0.$$

Here we used the fact that hf integrates to zero. This completes the proof of (6.3).

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