Estimating the inter-arrival time density of semi-Markov processes under structural assumptions on the transition distribution

Priscilla (Cindy) Greenwood
School of Mathematics and Statistics
and
Mathematical and Computational Sciences Modeling Center
Arizona State University

jointly with
Anton Schick (Binghamton University)
and Wolfgang Wefelmeyer (University of Cologne)

Let $(X_0, T_0), \ldots, (X_n, T_n)$ be observations of a Markov renewal process with real state space. A nonparametric estimator for the stationary density $\varrho(v)$ at v of the inter-arrival times $T_j - T_{j-1}$ is

$$\hat{\varrho}(v) = \frac{1}{n} \sum_{j=1}^{n} k_b(v - (T_j - T_{j-1})) \quad \text{with} \quad k_b(v) = k(v/b)/b.$$

Suppose that **the inter-arrival times $T_j - T_{j-1}$ depend multiplicatively on the jump size** of the embedded Markov chain:

$$T_j - T_{j-1} = Z_j W_j \quad \text{with} \quad Z_j = |X_j - X_{j-1}|^\nu,$$

where $\nu > 0$ and the W_j's are i.i.d. and independent of the X_j's. Then we can construct estimators for $\varrho(v)$ with rate $n^{-1/2}$.
In the following we express rescalings by subscripts, $f_s(x) = f(x/s)/s$.
Let g, h denote the densities of W_j, Z_j. Then the density of $T_j - T_{j-1}$ is a scale mixture

$$\varrho(v) = \int h_w(v) g(w)\, dw = \int h(z) g_z(v)\, dz.$$

The density h of $Z_j = |X_j - X_{j-1}|^\nu$ is calculated as follows.

Let $p_1(x)$ and $q(x,y)$ denote the stationary density and the transition density of the embedded chain. The conditional density at y of $|X_j - X_{j-1}|$ given $X_{j-1} = x$ is

$$\gamma(x,y) = \Big(q(x,x+y) + q(x,x-y)\Big)\mathbf{1}(y>0).$$

Then the conditional density at y of $Z_j = |X_j - X_{j-1}|^\nu$ given $X_{j-1} = x$ is

$$\zeta(x,y) = \frac{1}{\nu} y^{\frac{1}{\nu}-1}\gamma(x, y^{\frac{1}{\nu}}).$$

Hence the stationary density at y of Z_j is

$$h(y) = \frac{1}{\nu} y^{\frac{1}{\nu}-1} \int p_1(x)\gamma(x, y^{\frac{1}{\nu}})\, dx.$$

A (“kernel”) estimator of the density $\varrho(v)$ of the inter-arrival times $T_j - T_{j-1} = Z_j W_j$ at v can be based on n^2 “observations” $Z_i W_j$; this gives the *local U-statistic*

$$\hat{\varrho}(v) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_b(v - Z_i W_j)$$

with $k_b(v) = k(v/b)/b$ a kernel k scaled by a bandwidth b.

Similar local U-statistics *for i.i.d. observations* are studied by Frees (1994) and Giné and Mason (2007). These results are not applicable here because (a) the Z_i's are not independent, and (b) an integrability condition fails.

Nevertheless, we show that our density estimator $\hat{\varrho}(v)$ has rate $n^{-1/2}$ pointwise, but that a functional central limit theorem does *not* hold, in general.

We apply the Hoeffding decomposition to our local U-statistic

$$\widehat{\varrho}(v) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_b(v - Z_i W_j).$$

The conditional mean of $k_b(v - ZW)$ given W is (change variables)

$$H(W) = \int h_W(v - bu) k(u)\, du;$$

the conditional mean given Z is

$$G(Z) = \int g_Z(v - bu) k(u)\, du.$$

Hence (by Hoeffding decomposition) $\widehat{\varrho}(v)$ has the linear approximation

$$\widehat{\varrho}(v) - E k_b(v - ZW) = \frac{1}{n} \sum_{j=1}^{n} \Big(G(Z_j) - EG(Z) + H(W_j) - EH(W) \Big) + o_P(n^{-1/2}).$$

The linear approximation is a *smoothed empirical process*.

Assume that $bn \to \infty$ and $b^4 n \to 0$. Then the smoothing can be removed, the bias is negligible, and our local U-statistic is approximated by a linear process:

$$\hat{\varrho}(v) - \varrho(v) = \frac{1}{n}\sum_{j=1}^{n}\Big(g_{Z_j}(v) - \varrho(v) + h_{W_j}(v) - \varrho(v)\Big) + o_P(n^{-1/2})$$
$$= \frac{1}{n}\sum_{j=1}^{n}\Big(\frac{1}{Z_j} g\Big(\frac{v}{Z_j}\Big) - \varrho(v) + \frac{1}{W_j} h\Big(\frac{v}{W_j}\Big) - \varrho(v)\Big) + o_P(n^{-1/2}).$$

Assume that the embedded chain is exponentially ergodic. Then our estimator $\hat{\varrho}(v)$ for the inter-arrival density has rate $n^{-1/2}$ and is asymptotically normal. (We can also show that $\hat{\varrho}(v)$ is asymptotically *efficient*).

A functional central limit theorem usually does not hold. For example, in L_2 we need finiteness of

$$\int E\Big[\frac{1}{Z^2} g^2\Big(\frac{v}{Z}\Big)\Big]\, dv = E\Big[\frac{1}{Z}\Big] \int g^2(v)\, dv,$$

but $E[1/Z]$ is typically infinite.

A nonparametric estimator for the conditional density $\kappa(x,v)$ at v of $T_j - T_{j-1}$ given $X_{j-1} = x$ is the Nadaraya–Watson estimator

$$\hat{\kappa}(x,v) = \frac{\sum_{j=1}^n k_b(x - X_{j-1}) k_b(v - (T_j - T_{j-1}))}{\sum_{j=1}^n k_b(x - X_{j-1})}.$$

Assume as above that

$$T_j - T_{j-1} = Z_j W_j \quad \text{with} \quad Z_j = |X_j - X_{j-1}|^\nu.$$

Assume, in addition, that **the embedded chain is autoregressive**:

$$X_j = \vartheta X_{j-1} + \varepsilon_j$$

with $|\vartheta| < 1$ and ε_j's i.i.d. with mean zero, finite variance, and positive density f. Then we can construct estimators for $\kappa(x,v)$ with rate $n^{-1/2}$. Write

$$Z_j = |X_j - X_{j-1}|^\nu = |\varepsilon_j - (1-\vartheta) X_{j-1}|^\nu.$$

The variables $|\varepsilon_j - (1-\vartheta)x|^\nu$ are i.i.d., follow the conditional distribution of Z_j given $X_{j-1} = x$, and are independent of the W_j's.

Note that the variables

$$\varepsilon_j - (1-\vartheta)x = X_j - x - \vartheta(X_{j-1} - x) = \varepsilon_j(x), \quad \text{say},$$

are innovations of the autoregressive process shifted by x.
Estimate ϑ by the (say, least squares) estimator $\widehat{\vartheta}$.
Estimate $\varepsilon_j(x)$ by the residual

$$\widehat{\varepsilon}_j(x) = X_j - x - \widehat{\vartheta}(X_{j-1} - x).$$

Then the conditional density of $T_j - T_{j-1}$ at v given $X_{j-1} = x$ can be estimated by the local U-statistic

$$\widehat{\kappa}(x,v) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_b(v - |\widehat{\varepsilon}_i(x)|^{\nu} W_j)$$

with $k_b(v) = k(v/b)/b$ a kernel k scaled by a bandwidth b.

The conditional density estimator $\widehat{\kappa}(x,v)$ can be shown to have rate $n^{-1/2}$. Expand about ϑ first, then proceed similarly as for $\widehat{\varrho}(v)$.

Expansion of $\widehat{\kappa}(x,v)$ about ϑ gives

$$\widehat{\kappa}(x,v) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_b(v - |\varepsilon_i(x)|^{\nu} W_j) + (\widehat{\vartheta} - \vartheta)K + o_P(n^{-1/2}) \quad (1)$$

with

$$\begin{aligned} K &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} (X_{i-1} - x)s(\varepsilon_i(x))W_j(k_b)'(v - |\varepsilon_i(x)|^{\nu} W_j) \\ &\to xv \int \frac{1}{t} g'_{|t|^{\nu}}(v) f(t + (1-\vartheta)x)\, dt \quad \text{in probability,} \end{aligned}$$

where $s(x) = \operatorname{sign}(x)\nu|x|^{\nu-1}$. For the first right-hand term of (1), Hoeffding decomposition and unsmoothing give

$$\begin{aligned} &\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_b(v - |\varepsilon_i(x)|^{\nu} W_j) \\ &= \kappa(x,v) + \frac{1}{n}\sum_{j=1}^{n} \Big(\eta_{W_j}(x,v) - \kappa(x,v) + g_{|\varepsilon_j(x)|^{\nu}}(v) - \kappa(x,v)\Big), \end{aligned}$$

where $\eta_w(x,v) = \eta(x, v/w)/w$.