Blockwise empirical likelihood and efficiency for semi-Markov processes

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Empirical likelihood and empirical estimators in the i.i.d. case.

Let $X_1, \dots, X_n$ be i.i.d. with a distribution fulfilling a linear constraint $Ph = E[h(X)] = 0$. The *empirical likelihood* of Owen (1988, 2001) uses a *weighted* empirical distribution that fulfills this constraint:

$$\mathbb{P}_w h = \frac{1}{n} \sum_{j=1}^{n} w_j h(X_j) = 0.$$

A linear functional $Pf = E[f(X)]$ is then estimated by the *weighted empirical estimator*

$$\mathbb{P}_w f = \frac{1}{n} \sum_{i=1}^{n} w_j f(X_j).$$

Take f and h one-dimensional. The weights are of the form $w_j = 1/(1 + \mu h(X_j))$, and one can show that $\mu = \mathbb{P}h/\mathbb{P}h^2 + o_P(n^{-1/2})$, where $\mathbb{P}h = \frac{1}{n}\sum_{j=1}^{n} h(X_j)$ denotes the usual empirical estimator.

The weighted empirical estimator has the stochastic expansion

$$\mathbb{P}_w f = \mathbb{P} f + \frac{Pfh}{Ph^2}\, \mathbb{P} h + o_P(n^{-1/2}). \tag{1}$$

The asymptotic variances of $\mathbb{P} f$ and $\mathbb{P}_w f$ are Var $f(X)$ and Var $f(X) - (Pfh)^2/Ph^2$. The reduction can be considerable.
From (1) we derive an alternative to $\mathbb{P}_w f$, the *additively corrected empirical estimator*

$$\mathbb{P}_{add} f = \mathbb{P} f - \frac{\mathbb{P} fh}{\mathbb{P} h^2}\, \mathbb{P} h.$$

Both estimators are asymptotically efficient.

For *dependent* data, an efficient estimator must fulfill (1) with Ph^2 and Pfh replaced by variance and covariance of $\mathbb{P} h$ and $\mathbb{P} f$. But (1) *continues* to hold for the weighted empirical estimator, which is therefore *not* efficient any more. We will now see that *blockwise* weighting gives the right result.

Empirical likelihood for Markov renewal processes.

Let $(X_0, T_0), \ldots, (X_n, T_n)$ be observations of a Markov renewal process. (The results carry over to semi-Markov processes.) Write $V_j = T_j - T_{j-1}$ for the inter-arrival times. Then $(X_1, V_1), \ldots, (X_n, V_n)$ follow a Markov chain with transition distribution not depending on the previous inter-arrival time, $S(x; dy, dv) = Q(x; dy)R(x, y; dv)$.
The *empirical estimator* for $Pf = E[f(X, Y, V)]$ is

$$\mathbb{P}f = \frac{1}{n} \sum_{j=1}^{n} f(X_{j-1}, X_j, V_j).$$

If the embedded chain is exponentially ergodic, $\mathbb{P}f$ has the *martingale approximation* $\mathbb{P}f - Pf = \mathbb{P}Af + o_P(n^{-1/2})$ with

$$\begin{aligned} Af(x, y, v) &= f(x, y, v) - Sf(x) + Sf(y) - QSf(x) \\ &\quad + \sum_{t=1}^{\infty} (Q^t Sf(y) - Q^{t+1} Sf(x)). \end{aligned}$$

Assume the linear constraint $Ph = E[h(X, Y, V)] = 0$. By MSW (2001), an efficient estimator $\hat{\vartheta}$ for Pf is characterized by

$$\hat{\vartheta} = \mathbb{P}f - \frac{PAfAh}{P(Ah)^2}\,\mathbb{P}h + o_P(n^{-1/2}).$$

Such an estimator is the *additively corrected empirical estimator*

$$\mathbb{P}_{add}f = \mathbb{P}f - \frac{\hat{\gamma}}{\hat{\sigma}^2}\,\mathbb{P}h$$

with

$$\hat{\gamma} = \mathbb{P}fh + \sum_{k=1}^{m} \frac{1}{n-k} \sum_{j=1}^{n-k} \Big(h(X_{j-1}, X_j, V_j) f(X_{j+k-1}, X_{j+k}, V_{j+k}) + h(X_{j+k-1}, X_{j+k}, V_{j+k}) f(X_{j-1}, X_j, V_j) \Big),$$

$$\hat{\sigma}^2 = \mathbb{P}h^2 + 2 \sum_{k=1}^{m} \frac{1}{n-k} \sum_{j=1}^{n-k} h(X_{j-1}, X_j, V_j) h(X_{j+k-1}, X_{j+k}, V_{j+k}).$$

An efficient estimator of $Pf = E[f(X,Y,V)]$ is also obtained with the *blockwise* empirical likelihood, introduced by Kitagawa (1997) for different purposes. Let $n = \nu m$ with $m \to \infty$ slowly. Take averages over blocks,

$$F_i = \frac{1}{m}\sum_{k=1}^{m} f(X_{(i-1)m+k-1}, X_{(i-1)m+k}, V_{(i-1)m+k}),$$
$$H_i = \frac{1}{m}\sum_{k=1}^{m} h(X_{(i-1)m+k-1}, X_{(i-1)m+k}, V_{(i-1)m+k}).$$

The empirical estimator of Pf can be written $\mathbb{P}f = \frac{1}{\nu}\sum_{i=1}^{\nu} F_i$. Define blockwise weights w_i as solutions of

$$\mathbb{P}_w h = \frac{1}{\nu}\sum_{i=1}^{\nu} w_i H_i = 0.$$

The *blockwise weighted empirical estimator* is

$$\mathbb{P}_w f = \frac{1}{\nu}\sum_{i=1}^{\nu} w_i F_i.$$

We show that this blockwise weighted empirical estimator

$$\mathbb{P}_w f = \frac{1}{\nu} \sum_{i=1}^{\nu} w_i F_i \quad \text{with weights} \quad \mathbb{P}_w h = 0$$

is asymptotically equivalent to the *blockwise additively corrected empirical estimator*

$$\mathbb{P}_{block} f = \mathbb{P} f - \frac{\sum_{i=1}^{\nu} F_i H_i}{\sum_{i=1}^{\nu} H_i^2} \, \mathbb{P} h.$$

This, in turn, is asymptotically equivalent to the above additively corrected empirical estimator

$$\mathbb{P}_{add} f = \mathbb{P} f - \frac{\widehat{\gamma}}{\widehat{\sigma}^2} \, \mathbb{P} h,$$

which we know to be efficient.

Blocks are also used to bootstrap dependent data. For empirical likelihood, we need not separate blocks by gaps. The blocks may even overlap.