Homework Set Eleven

Due Thursday, July 14.

Question 1. Suppose α is irrational quadratic (that is, $\alpha = \frac{a+\sqrt{b}}{c}$ with $b > 0$ not a square). Let $\alpha = [a_0, a_1, \ldots]$ be its continued fraction expansion. Let $\alpha_0 = \alpha$ and $\alpha_{n+1} = \frac{1}{\alpha_n - a_n}$ for $n \geq 0$, where $a_n = \lfloor \alpha_n \rfloor$ as usual. Let α'_n denote the conjugate of α_n (for example, with the notation above, $\alpha' = \frac{a-\sqrt{b}}{c}$). Finally, assume that $\alpha > 1$ and $-1 < \alpha' < 0$.

(a) Prove that $\alpha_n > 1$ and $-1 < \alpha'_n < 0$ for any $n \in \mathbb{N}$. (Hint: note that $\alpha'_{n+1} = \frac{1}{\alpha'_n - a_n}$.)

(b) Prove that $a_n = \left\lfloor -\frac{1}{\alpha'_{n+1}} \right\rfloor$ for any $n \in \mathbb{N}$.

(c) Suppose that $\alpha'_\ell = \alpha'_m$ for some $0 < m < \ell$. Prove that $\alpha'_{m-1} = \alpha'_{\ell-1}$.

(d) Prove that α has a purely periodic continued fraction expansion.

Question 2. Suppose $\alpha = [\overline{a_0, a_1, \ldots, a_{\ell-1}}]$ is purely periodic.

(a) Let $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$. Prove that $1 \leq p_0 < \cdots < p_n$ and $1 = q_0 < \cdots < q_n$, so in particular, $\alpha \geq 1$.

(b) Prove that α and α' are roots of the same quadratic polynomial $f(x) = Ax^2 + Bx + C \in \mathbb{Z}[x]$ with $A > 0$, $C < 0$. Provide your polynomial $f(x)$ (there are some minor choices one can make).

(c) Show that $f(0) < 0$ and $f(-1) > 0$. Use this to conclude that $-1 < \alpha' < 0$.

Question 3. A number $\alpha \in \mathbb{R}$ is called *algebraic* if there exists $f \in \mathbb{Z}[x]$, $f \neq 0$ such that $f(\alpha) = 0$. The integer

$$n = \min\{\deg f \mid f \neq 0, f(\alpha) = 0\}$$

is called the *degree of* α. If $\alpha \in \mathbb{R}$ is not algebraic, it is called *transcendental.*

The following is a result of Liouville.

Theorem. Let $\alpha \in \mathbb{R}$ be algebraic of degree k. Then there exists $c > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^k}$$

for all $p, q \in \mathbb{Z}$, $q > 0$.

As a corollary, Liouville also proved the following.

Corollary. Suppose that $\{\frac{p_n}{q_n} \mid n \in \mathbb{N}\}$ is a sequence of rational numbers with $q_n > 0$ for all n and λ_n is a sequence of numbers such that

$$0 < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{c}{q_n^{\lambda_n}}$$

for some $c > 0$. If $\lim_{n\to\infty} \lambda_n = \infty$ then α is transcendental.

(a) Use the corollary to prove that $\alpha = \sum_{k=0}^{\infty} \frac{1}{2^{k!}}$ is transcendental.

(b) (BONUS) Use Liouville's theorem to prove the corollary.