Homework Set One

Question 1. Prove the following statements. (These may be useful for later problems.)

1. If n is an integer and n^2 is odd, then n is odd. (Recall: n is odd if $2 \nmid n$.)
2. If n is an integer and n^2 is even, then n is even. (Recall: n is even if $2 \mid n$.)
3. If an integer d divides integers m and n, then d divides $m + n$ and $m - n$.

Question 2. Recall that $(a, b, c) \in \mathbb{N}^3$ is a PPT if $a^2 + b^2 = c^2$, and $\gcd(a, b, c) = 1$. Suppose (a, b, c) is a PPT. Prove the following statements.

1. The following two statements are equivalent:
   (i) Only one of the integers a and b is odd.
   (ii) The integer c is odd.
2. One of a and b is odd, while the other is even.

Question 3. Suppose (a, b, c) is a PPT with a odd, and note $a^2 = (c - b)(c + b)$. Let d be an integer. Suppose that $d \in \mathbb{Z}$, $d \mid c + b$ and $d \mid c - b$. Suppose that $d \in \mathbb{Z}$, $d \mid c + b$ and $d \mid c - b$. Prove the following statements.

1. The integer d is odd.
2. The integer d satisfies $d \mid c$ and $d \mid b$.
3. In fact, $d = \pm 1$.
4. Explain why $c - b = s^2$ and $c + b = t^2$ for some odd numbers $s < t$ which share no common factors. (You may use that natural numbers have a unique factorization into primes.)
5. Prove the following theorem:

   Theorem. Every primitive pythagorean triple (a, b, c) is of the form (assuming a is odd)
   $$a = st, \qquad b = \frac{t^2 - s^2}{2}, \qquad c = \frac{t^2 + s^2}{2}$$
   for integers $t > s \geq 1$ such that s and t have no common factors.

Question 4. Let L be the line with slope m that passes through the point $(-1,0)$. Consider the circle $C : x^2+y^2 = 1$. Let $(x_0, y_0) \neq (-1,0)$ be the other point of intersection between C and L.

1. Using the equation of L, show that $x_0 = \frac{1-m^2}{1+m^2}$.
2. Explain why $(x_0, y_0) \in \mathbb{Q}^2$ if and only if $m \in \mathbb{Q}$.
3. Suppose $(x_0, y_0) = \left(\frac{a}{c}, \frac{b}{c}\right) \in \mathbb{Q}^2$. Note that $(x_0, y_0) \in C$ if and only if $a^2 + b^2 = c^2$. Use this to prove:

   Theorem. Every primitive pythagorean triple (a, b, c) can be obtained by choosing integers u and v with no common factors, and setting

   $$a = u^2 - v^2, \qquad b = 2uv, \qquad c = u^2 + v^2.$$