Homework Set Seven

Due Thursday, June 16.

Question 1. Let p be an odd prime. Prove that

$$\left(\frac{p-1}{2}\right)!^2 \equiv (-1)^{\frac{p-1}{2}}(p-1)! \pmod{p}.$$

Question 2. Let k and m be non-negative integers.

(a) Prove that if $n = 8k+7$ then $n \neq x^2+y^2+z^2$ for any integers x, y, z.

(b) Prove that if $n = 4^m(8k+7)$, then $n \neq x^2+y^2+z^2$ for any integers x, y, z.

Question 3. For a non-negative integer n, set $E_n := \sum_{k=0}^{n-1} 10^k = 1 + 10 + \cdots + 10^{n-1}$.

(a) Show that

$$E_n = \frac{10^n - 1}{9}.$$

(b) Show that E_{33} is not prime. (Hint: show 67 divides E_{33}.)

Question 4. For $n \in \mathbb{N}$ set $F_n := 2^{2^n} + 1$.

(a) Prove that if F_n is prime, then $3^{\frac{F_n-1}{2}} \equiv -1 \pmod{F_n}$.

(b) Prove that if $3^{\frac{F_n-1}{2}} \equiv -1 \pmod{F_n}$, then F_n is prime.

(Note: The result just proved, that is, that F_n is prime if and only if $3^{\frac{F_n-1}{2}} \equiv -1 \pmod{F_n}$, is often referred to as Pépin's Theorem.)

Question 5. Let $p_1, \ldots, p_r$ be odd primes. Prove that

$$\frac{p_1-1}{2} + \frac{p_2-1}{2} + \ldots + \frac{p_r-1}{2} \equiv \frac{p_1p_2\cdots p_r - 1}{2} \pmod{2}.$$