Homework Set Nine

Due Thursday, June 30.

Question 1. Recall the quaternions $B = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z}\}$ which are governed by the relations

$$i^2 = j^2 = k^2 = ijk = -1, \quad \text{and} \quad ax = xa \text{ for any } x \in \{i, j, k\}, a \in \mathbb{Z}.$$

(a) Show that $\overline{\lambda\mu} = \overline{\mu}\,\overline{\lambda}$ for $\lambda, \mu \in B$. (Note: In general, $\overline{\lambda\mu} \neq \overline{\lambda}\,\overline{\mu}$ for $\lambda, \mu \in B$.)

(b) Show that $N(\lambda\mu) = N(\lambda)N(\mu)$ for $\lambda, \mu \in B$.

(c) Show that for integers a, b, c, d, A, B, C, D there exists integers $\alpha, \beta, \gamma, \delta$ such that

$$\left(a^2 + b^2 + c^2 + d^2\right)\left(A^2 + B^2 + C^2 + D^2\right) = \alpha^2 + \beta^2 + \gamma^2 + \delta^2.$$

Additionally, provide the integers $\alpha, \beta, \gamma, \delta$ in terms of a, b, c, d, A, B, C, D.

Question 2. The continued fraction of π^2 is

$$[a, b, c, d, e, f, 1, 8, 1, 1, 2, 2, 1, 1, 8, 3, 1, 10, 5, 1, 3, 1, 2, 1, 1, 3, 16, 1, 1, 2, \ldots].$$

(a) Find the values of $a, b, d, e,$ and f.

(b) For $n \in \{0, 1, 2, 3, 4\}$ compute the nth convergent to π^2.

Question 3. Let

$$\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$$

be the convergents to the continued fraction $[a_0, a_1, a_2, \ldots]$. Prove the following theorem.

Theorem. For all $n \in \mathbb{N}$,

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n.$$

Equivalently, for all $n \in \mathbb{N}$,

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_{n-1}q_n}.$$

Question 4. Assume the notation from the previous problem. Let $a = 34$ and $b = 49$.

(a) Find the continued fraction $[a_0, a_1, \ldots, a_n]$ for $\frac{b}{a}$.

(b) Find the convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \ldots, \frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n} = \frac{b}{a}$ for $\frac{b}{a}$.

(c) Use Question 3 to find an integral solution (x_0, y_0) for the equation $34x + 49y = -13$.

Question 5 (Bonus). Suppose $M, N \in \mathbb{Z}$ and $M > N$. Let a_0, r_0 and a_1, r_1 be the integers from the Euclidean Algorithm such that

$$M = a_0 N + r_0 \quad \text{and} \quad N = a_1 r_0 + r_1.$$

Let a_{j+1}, r_{j+1}, $0 \leq j \leq n$, be the remaining values of the Euclidean Algorithm so that $\gcd(M, N) = r_n$. That is, $r_{j-1} = a_{j+1} r_j + r_{j+1}$ and $r_{n+1} = 0$.

(a) Provide the continued fraction for $\frac{M}{N}$.

(b) Set $Q_0 = 1$, $P_0 = -a_0$, $Q_1 = -a_1$, and $P_1 = 1 + a_0 a_1$ so that

$$r_0 = Q_0 M + P_0 N \quad \text{and} \quad r_1 = Q_1 M + P_1 N.$$

Find a recursive formula for the expressions Q_k and P_k such that $r_k = Q_k M + P_k N$.

(c) Show that $Q_k = (-1)^k q_k$ and $P_k = (-1)^{k+1} p_k$, where q_k, p_k are as in the previous problems.