GROSS-ZAGIER FORMULA

NOTES TAKEN BY MIKE WOODBURY

These are notes taken in Tonghai Yang's course titled "The Gross-Zagier Formula" at the University of Wisconsin Fall 2009. The main goal of the course was to understand the recent paper of Yuan-Zhang-Zhang [7].

1. Wenesday, September 2, 2009

1.1. Birch's story. In "Heegner Points and Rankin L-functions," Birch has an interesting article about the history of the Gross-Zagier formula. It says something interesting about working on problems that aren't necessarily state of the art.

Heegner proved two big problems in number theory.

Theorem 1 (Heegner)**.**
- *(The Gauss class problem) For imaginary quadratic fields* $K = \mathbb{Q}(\sqrt{-D})$*, the only values of* D *for which* $h_K = 1$ *are* $D = 3, 4, 7, 8, 11, 19, 43, 67, 163$.
- *(Congruence number problem) A prime* p *is congruent if ...*

It is now accepted that his result on the congruence number problem was correct, and that the work on the class number problem was essentially correct with only a "gap" (that was later filled by Stark.) However, he was not a professional mathematician and his prose was unpolished and his notation was unorthodox. As a result very few people paid it much attention.

1.2. Some history. The idea of what Gross was trying to do is quite simple: find rational solutions to certain equations. In other words, if $f(x, y) \in \mathbb{Q}[x, y]$, find rational solutions to $f(x, y) = 0$. For example:

i Quadratics like $3x^2 + 5y^2 + 6xy = 10$.
ii Cubics: $x^3 + y^3 = p$.
iii Higher order equations

If $f(x, y)$ is smooth of degree d, the genus of the associated curve $C : f(x, y) = 0$ is $g(C) = d(d-1)/2$. So (i) has $g = 0$, in which case there are either no solutions or infinitely many. If there is one solution (x_0, y_0), then any rational line through (x_0, y_0) intersects in one more place which must also be rational.

With respect to curves with $g \geq 2$, Faltings has proved that $\#C(\mathbb{Q}) < \infty$. In the case of (iii) there have been at least two Annals papers on the topic. One by Lieman [4] and another by Tian and Diaclo (sic?).

Item (ii) is the case of $g = 1$. What is so nice about curves E with $g = 1$? In analogy to the $g = 0$ case one can consider the line L through any point P on the curve E. This line must intersect at two other points Q, R. One can then define $P + Q + R = 0$. One can check that this definition gives a valid group law on the set of points $E(F)$ for any field F.

Date: Fall 2009.

Theorem 2 (Mordell). *The set $E(\mathbb{Q})$ is a finitely generated abelian group.*

(Weil proved the analogous statement for abelian varieties, so sometimes this is called the Mordell-Weil theorem.)

As a consequence of this,

$$E(\mathbb{Q}) \simeq E(\mathbb{Q})_{\text{tor}} \oplus \mathbb{Z}^r$$

where $E(\mathbb{Q})_{\text{tor}}$ is finite. Number theorists want to know what the number r (called the rank) is. One way to do this is the method of descent which for a given elliptic curve gives a method of finding points. This is discussed in [6] chapter 9.

Birch and Swinnerton-Dyer studied, rather than $\#E(\mathbb{Q})$, the numbers $N_p(E) = \#E(\mathbb{F}_p)$. Computationally, they found that

$$\sum_{p \leq X} \frac{N_p(E)}{p} \approx C_E (\log X)^r$$

where C_E is a constant depending on E and r is the rank. One can define an L-function associated to E using the values $a_p = p + 1 - N_p$.

$$L(s, E) = \prod_p L_p(s, E) \qquad L_p(s, E) = (1 - a_p p^{-s} + p p^{2s})^{-1}.$$

Their computational results lead them to the conjecture:

Conjecture 3 (BSD). (0) *$L(s, E)$ has analytic continuation and a functional equation $F(s, E) = L(2 - s, E)$.*
(1) $\text{ord}_{s=1} L(s, E) = \text{rank}(E)$.
(2) $L^{(r)}(1, E)/r! =$ *"arithmetic infomation" on E.*

Even the fact that $L(s, E)$ has analytic continuation was not at all clear. This was known to be the case for CM elliptic curves, but up to quadratic twist there are only a finite number of such curves. Shimura, in the 1960s, proved the following theorem, which established many more examples for which (0) could be proved.

Theorem 4 (Shimura). *The curve $X_0(N) = \Gamma_0(N) \backslash \mathbb{H} \cup \{cusps\}$ is defined over $\mathbb{Q}$. One can define an L-function $L(s, H^1(X_0(N)) = \prod L_p$ such that*

$$L(s, H^1(X_0(N)) = \prod_f L(s, f)$$

where f ranges over a basis of modular forms of weight 2 and level N. Moreover, given f one obtains a map

$$J(X_0(N)) \longrightarrow A_f$$

such that $L(s, A_f) = \prod_{\sigma : \mathbb{Q}(f) \to \hookrightarrow \mathbb{C}} L(s, f^\sigma)$.

Since $\mathbb{Q}(f) = \mathbb{Q}(a_m(f))$, if f has coefficients in $\mathbb{Q}$ then A_f is an elliptic curve. (In general if f is weight two cusp form, $[\mathbb{Q}(f) : \mathbb{Q}] = \dim A_f$.) Since the L-functions of modular forms were known to have functional equations and analytic continuation, this then proved that for curves which are modular, meaning there is a map $\pi : X_0(N) \to E$, the BSD conjecture makes sense.

This leads naturally to the question: "Which curves are modular?"

2. Friday, September 4, 2009

2.1. **More stories.** The problem in the BSD conjecture is to determine first of all when $L(s,E)$ is defined at $s=1$. Shimura's work implies that $L(s,E)=L(s,f)$ if and only if there exists a so-called Weil parametrization $\pi : X_0 \to E$. In this case, E is called *modular*. At first, people doubted that many curves were modular until Weil proved his converse theorem that said if $L(s,E)$ and it's twists have a functional equation and analytic continuation then it is modular. At that point the Taniyama-Shimura conjecture that every elliptic curve is modular seemed probable.

(Remark: Two elliptic curves are isogenous if and only if their L-functions are the same.)

Birch recognized that Heegner's method to find points could be generalized and simplified by working on $X_0(N)$ instead of E. He studied the curves

$$E_D = Dy^2 = x^3 - 1728$$

and deduced from them the conjectural formula

$$\frac{d}{ds}(L'(s,E)L(s,E))|_{s=1} = 2^A 3^B \Omega \widehat{h}(Heegner)$$

where $\widehat{h}(Heegner)$ is the height of the Heegner point.

It was recognized that the LHS is related to the Rankin L-function, and from here Gross started his work.

2.2. **Start to define things.** We have the modular curves

$$\Gamma_0(N)\backslash\mathbb{H} = Y_0(N) \hookrightarrow X_0(N) = \Gamma_0(N)\backslash\mathbb{H} \cup \{\text{cusps}\}.$$

Question: How do we know that $X_0(N)$ is defined over $\mathbb{Q}$?

- Classically. Consider the modular polynomial $j(\tau)$ which is a rational function on $X_0(N)$. Since $j_N(\tau) = j(N\tau)$ is as well, and $X_0(N)$ is a curve, there must be some relation $\Phi_N(x,y) \in \mathbb{C}[x,y]$ such that $\Phi_N(j,j_N)=0$. The projective curve defined by Φ_N is isomorphic to $X_0(N)$. Since Φ_N is defined over $\mathbb{Z}$ this gives the result.
- Another way. Think of $Y_0(N)$ as the moduli space of cyclic isogenies of elliptic curves. In other words, a point x of $Y_0(N)$ is a triple (φ, E, E') where $\varphi : E \to E'$ is a degree N morphism whose kernel is cyclic. If one can show that E, E' and φ are all defined over F, then x is as well.

How does the moduli space condition relate to the upper half plane description of $Y_0(N)$? In the special case that $N=1$, a point x corresponds to an elliptic curve, so the map

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \{\text{elliptic curves}\} \qquad \tau \mapsto E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$$

gives the isomorphism. In general

$$\tau \mapsto (\varphi : E_{N\tau} \to E_\tau)$$

where φ is the obvious map.

Question: For a given $[\tau] \in Y_0(N)$, over what field is $[\tau]$ defined? The answer depends on $j(\tau)$ and $j_N(\tau)$.

Theorem 5. *If $\tau \in \mathbb{H}$ is algebraic then $[\tau] \in Y_0(N)$ is algebraic if and only if τ is (imaginary) quadratic. (Such a point is called a* CM point.

Complex multiplication: Let $K = \mathbb{Q}(\sqrt{D})$ for $D < 0$, $\mathcal{O}_K = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$. A *CM elliptic curve/L* is a pair (E, ι) where E is an elliptic curve over L together with an action $\iota : \mathcal{O}_K \to \mathrm{End}_L(E)$.

Propostion 6. *Every CM elliptic curve by $\mathcal{O}_K$ over $\mathbb{C}$ is of the form $(\mathbb{C}/\mathfrak{a}, \iota)$ where $\iota : K \to \mathbb{C}$ acts on $\mathbb{C}/\mathfrak{a}$ by multiplication.*

The idea of the proof is to write $E/\mathbb{C}$ as $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Since $\mathbb{C}$ is the universal cover of $\mathbb{C}/\Lambda$ we get

$$\begin{array}{ccc} \mathbb{C} & \dashrightarrow & \mathbb{C} \\ \downarrow & & \uparrow \\ \mathbb{C}/\Lambda & \xrightarrow{\iota(a)} & \mathbb{C}/\Lambda \end{array}$$

The induced map from $\mathbb{C}$ to $\mathbb{C}$ must be multiplication by some number α. Therefore, $\alpha\Lambda \subset \Lambda$ for all choices of $a \in \mathcal{O}_K$. Thus Λ is an $\mathcal{O}_K$-module of rank 2 as a free $\mathbb{Z}$-module. In other words, Λ is a projective $\mathcal{O}_K$-module of rank 1. One concludes that $\Lambda \simeq \mathfrak{a}$ an ideal of K.

We have the following facts.

- $(\mathbb{C}/\mathfrak{a}, \iota)$ is defined over H, the Hilbert class field of K.
- Denote the image of $[\mathfrak{a}]$ in $\mathrm{Gal}(H/K)$ by $\sigma_{\mathfrak{a}}$. Then $(\mathbb{C}/\mathfrak{a})^{\sigma_{\mathfrak{a}}} = (\mathbb{C}/\mathfrak{a}\mathfrak{b}^{-1})$.
- $H = K(j(\mathbb{C}/\mathfrak{a})) = K(j(\frac{D+\sqrt{D}}{2}))$.

These facts lead to the theorem in one direction: if τ is imaginary quadratic the $[\tau] \in Y_0(N)$ is defined over a number field.

3. Wednesday, September 9, 2009

We have $X_0(N) \supset Y_0(N) = \Gamma_0(N)\backslash\mathbb{H} = \{(\varphi : E \to E') \mid \cdots\}$.

3.1. Fricke involution.

We will define a group of actions indexed by $d \mid N$ on $Y_0(N)$. Let $x = (\phi : E \to E') \in Y_0(N)$. In particular, we have

$$w_N(x) = (\varphi^\vee : E' \to E).$$

More generally, if $d \mid N$ and $C \subset \ker\phi$ has order d then $w_d(x)$ is the composition

$$E/C \to E' \to E'/C'$$

where the first map is that induced by ϕ and $C' \subset \ker\phi^\vee$ is such that the degree of the composition is N. In other words, the size of $\phi^{-1}(C')/C$ is N.

We denote by W the group generated by w_d in either $\mathrm{End}(J_0(N))$ or in $\mathrm{End}(X_0(N))$.

3.2. Hecke operators.

For $m \geq 1$, $T_m(\phi : E \to E')$ be the formal sum $\sum_C x_C \in Z^1(X_0(N))$ where $C \subset E[m]$ is cyclic of order m and $C \cap \ker\phi = \{0\}$ and

$$x_C = (\phi : E/C \to E'/\phi(C)$$

This is called the Hecke correspondence.

As an exercise, lets calculate $T_m[\tau]$ where $[\tau] \in Y_0(N)$. Recall the notations $\Lambda_\alpha = \mathbb{Z} + \mathbb{Z}\alpha$, $E_\alpha = \mathbb{C}/\Lambda_\alpha$. Then

$$[\tau] = (\phi : E = E_\tau = \mathbb{C}/\Lambda_\tau \to \mathbb{C}/\Lambda_{\tau/N} = E')$$

Take $m = p$ a prime not dividing N. Note that

$$E_\tau[p] = (\frac{1}{p}\mathbb{Z} + \frac{\tau}{p}\mathbb{Z})/\Lambda_\tau \simeq \mathbb{Z}/p \times \mathbb{Z}/p.$$

Since $\mathbb{Z}/p \times \mathbb{Z}/p$ has $p+1$ cyclic subgroups:

$$C_\infty = \langle \frac{1}{p}, 0\rangle, \qquad C_j \simeq \langle \frac{j}{p}, \frac{1}{p}\rangle \quad (0 \le j \le p-1).$$

So the corresponding points on $Y_0(N)$ are

$$x_{C_j} = (\mathbb{C}/(\mathbb{Z} + \mathbb{Z}[\frac{j}{p} + \frac{\tau}{p}]) \to \mathbb{C}/\phi(\cdots)$$

$$x_{C_\infty} = (\mathbb{C}/(\mathbb{Z}\frac{1}{p} + \mathbb{Z}\tau) \to \mathbb{C}/\phi(\cdots)$$

and

$$T_p([\tau]) = \begin{cases} \sum_{j=0}^{n-1} [\frac{\tau+j}{p}] + [p\tau] & \text{if } p \nmid N \\ \sum_{j=0}^{n-1} [\frac{\tau+j}{p}] & \text{if } p \mid N \end{cases}$$

Remark: Hecke operators act on modular forms:

$$T_p f = \sum_{j=0} p-1 f(\frac{\tau+j}{p}) + \delta_{p,N} f(p\tau).$$

The Hecke correspondence on $X_0(N)$ bives an action on $J_0(N)$:

$$T_m((x) - (\infty)) = T_m x - \deg T_m(\infty) \in J_0(N).$$

We denote by $\mathbb{T} \subset \mathrm{End}(J_0(N)$ the subgroup generated by all of the T_m and W. This is the *Hecke algebra.*

Theorem 7 (Eichler-Shimura,Shimura (?))**.** *Given $f \in S_2^{new}(N)$, $T_m f = a_m f$ for all n. Thus there exists a unique abelian variety $A_f \subset J_0(N)$ (understood up to isogeny) which is stable by T_m action and $T_m a = a_m a$ for all $a \in A_f$.*

3.3. **Complex multiplication.** Fix $K = \mathbb{Q}(\sqrt{D})$, $\mathcal{O}_K = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$. A pair (E, ι) is a called a *CM elliptic curve by $\mathcal{O}_K$* if E is an elliptic curve and $\iota : \mathcal{O}_K \hookrightarrow \mathrm{End}(E)$.

Note that to write down abelian extension of a number field L, we know how to do this explicitly if $L = \mathbb{Q}$, in which case we adjoint all values of $e^{2\pi i z}$ where z is rational. The only other case we can do is when $L = \mathbb{Q}(j(\frac{D+\sqrt{D}}{2}), \frac{D+\sqrt{D}}{2})$ in which case we can use the theory of complex multiplication.

Theorem 8 (The Theorem of Complex Multiplication)**.** *Every CM elliptic curve over $\mathbb{C}$ is of the form $E_\mathfrak{a} = (\mathbb{C}/\mathfrak{a}, \iota)$ where $\mathfrak{a}$ is an ideal of K and $\iota : K \to \mathbb{C}$ is the standard embedding.*

(i) $(\mathbb{C}/\mathfrak{a}, \iota) \simeq (\mathbb{C}/\mathfrak{b}, \iota)$ *if and only if* $\mathfrak{a} = \alpha\mathfrak{b}$ *which is equivalent to the statement that* $[\mathfrak{a}] = [\mathfrak{b}]$ *in* $Cl(K)$*.*
(ii) $E_\mathfrak{a}$ *is defined over* $H = \mathbb{Q}(j(\frac{D+\sqrt{D}}{2}), \frac{D+\sqrt{D}}{2})$*, the Hilbert class field of* K*.*
(iii) *For all* $\sigma_\mathfrak{b} \in Gal(H/K) \simeq Cl(K)$, $[E_\mathfrak{a}]^{\sigma_\mathfrak{b}} = E_{\mathfrak{a}\mathfrak{b}^{-1}}$.
(iv) *Let* ρ *denote complex conjugation. Then* $(\mathbb{C}/\mathfrak{a}, \iota)^\rho = (\mathbb{C}/\mathfrak{a}, \bar{\iota}) = (\mathbb{C}/\bar{\mathfrak{a}}, \iota)$.

If $E_\mathfrak{a} \in Y_0(1)$, then this implies that $j(E_\mathfrak{a}) \in H$ and $j(E_\mathfrak{a})^{\sigma_\mathfrak{b}} = j(E_{\mathfrak{a}\mathfrak{b}^{-1}})$.
Consider the map

$$\pi : Y_0(N) \to Y_0(1)$$

which has degree $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$. If $(E, \iota) \in Y_0(1)$ then

$$\pi^*(E, \iota) = \sum_{x \mid \pi(x) = (E,\iota)} x$$

is defined over H even though each x need not be defined over H. When they *are* defined over H, they are called *Heegner points.*

So $x = (\phi : E \to E')$ is a Heegner point (by $\mathcal{O}_K$) if there exists an $\mathcal{O}_K$-action $\iota : \mathcal{O}_K \hookrightarrow \operatorname{End} x$ where $\operatorname{End} x$ consists of all pairs of endomorphisms (f, f') such that

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow f & & \downarrow f' \\ E & \longrightarrow & E' \end{array}$$

commutes. $\operatorname{End} x$ can be described in terms of just endomorphisms f on E such that $f(\ker \phi) \subset \ker \phi$.

Propostion 9. *Assume $(D, N) = 1$. Then there exists a Heegner point if and only if D is a square modulo $4N$. In particular, both E, E' have CM by $\mathcal{O}_K$.*

The way to prove this is to consider $x = (\phi : E \to E')$. Then, by the theorem on complex multiplication,

$$E = \mathbb{C}/\mathfrak{a} \to E' = \mathbb{C}/\mathfrak{b} \qquad \mathfrak{a} \subset \mathfrak{b},$$

$\mathfrak{b}/\mathfrak{a}$ is cyclic and $\mathfrak{b} = \mathfrak{a}\mathcal{N}$ with $N(\mathcal{N}) = N$ and $\mathcal{O}_K/\mathcal{N}$ is cyclic.

4. Friday, September 11, 2009

So Heegner points exist only if $D \equiv \square \pmod{4N}$. Assume $(D, 4N) = 1$. One question we can ask is "how many Heegner points are there?" We must have $N = \mathcal{N}\overline{\mathcal{N}}$ and $(\mathcal{N}, \overline{\mathcal{N}}) = 1$. Recall that $\mathcal{O}/\mathcal{N} \simeq \mathbb{Z}/N$.

For every $[\mathfrak{a}] \in Cl(K)$ we have a point

$$x_{[\mathfrak{a}]} = (\mathbb{C}/mfa \to \mathbb{C}/\mathfrak{a}\mathcal{N}^{-1}).$$

The class group acts on the set of Heegner points which we denote by $Heeg(D, N)$ as does W. Write $d \mid N$ and $(d, N/d) = 1$. Then

$$w_d(x_{[\mathfrak{a}]}) = (\phi^\vee : \mathbb{C}/mfa\mathcal{N}^{-1} \to \mathbb{C}/\mathfrak{a}) \simeq (\mathbb{C}/mfa\mathcal{N}^{-1} \to \mathbb{C}/\mathfrak{a}N^{-1}).$$

Lemma 10. *$Cl(K) \times W$ ats transitively on $Heeg(D, N)$.*

Question: Does T_m act on $Heeg(D, N)$? We decided that the answer is no because once you mod out by cyclic subgroups the resulting elliptic curves may only have complex multiplication by a smaller order of $\mathcal{O}_K$.

Let $\chi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times$ be an ideal class character, $f \in S_2^{new}(N)$, $x \in Heeg(D, N)$, $c = (x) - (\infty) \in J_0(N)(H)$. Define

$$c_\chi = \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi^{-1}(\sigma)\sigma(c) \in J_0(N)(H, \mu_\infty).$$

This can be projected to $c_{\chi,f} \in A_f$ which is an abelian variety over $\mathbb{Q}$, or at least over $\overline{\mathbb{Q}}$.

Example: $\chi = 1$ and $A_f = E$ an elliptic curve. Then

$$\begin{array}{rcl} c_\chi = \sum_{\sigma \in \mathrm{Gal}(H/K)} \sigma(c) & \in & J_0(N)(K) \\ & & \downarrow \pi_f \\ c_{\chi,f} = \pi(c_\chi) & \in & E(K) \end{array}$$

Question: When is $c_{\chi,f}$ non-torsion? To answer this question we use the height pairing.

4.1. **Height pairing (Neron Tate height).** Tate proved that globally there exists a canonical positive definite quadratic form[1]

$$h_{can} : J(L)/J(L)_{tor} \to \mathbb{R}^{\geq 0}$$

when J is any abelian variety. So $c_{\chi,f}$ is torsion if and only if $h_{can}(c_{\chi,f}) = 0$.

The problem with Tate's pairing is that is hard to calculate. Let H a number field and v a prime of H. Locally, Neron proved[2] the following. Let X be a curve over H, and $Z^1(X)_0$ divisors of degree zero.

(1) There exists unique biadditive symmetric continuous pairing
$$\langle \cdot, \cdot, \rangle_v : Z^1(X)_0(H_v) \times Z^1(X)_0(H_v) \to \mathbb{R}$$
which is well defined only when a, b have no common generic support. Moreover,
$$\langle a, div f \rangle_v = \log |f(a)|_v = \sum_{x \in \mathrm{supp}(a)} m_x \log |f(x)|_v.$$

(2) For all $a, b \in Z^1(X)_0(H)$, $\langle a, b \rangle_v = 0$ for almost all v. Thus $\langle a, b \rangle = \sum_v \langle a, b \rangle_v$ is a well defined global height pairing. It only depends on the image of $a, b \in J(X)$.

Theorem 11. *The two global height pairings (of Tate and Neron) are the same.*

For more on this see [1].

4.2. **Relation of height to derivative.** Birch related $h_{can}(c_{1,f})$ in the case that $A_f = E$ and $\chi = 1$ to

$$\frac{d}{ds}(L(f,s)L(f^D,s)) \mid_{s=1} = \frac{d}{ds}(L(E,s)L(E^D,s)) \mid_{s=1} .$$

The L-function appearing here is a Rankin-Selberg L-function.

4.3. **Base change.** Let $\chi : I(f) = \{\text{ideals relatively primes to } f\} \to \mathbb{C}^\times$ be a group homomorphism such that $\chi(\alpha \mathcal{O}_K) = \alpha^{k-1}$ for all $\alpha \equiv 1 \mod f$. Define

$$L(s,\chi) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{a_\chi(n)}{n^s}.$$

Theorem 12 (Hecke)**.** *There exists a unique $f_\chi \in S_k^{new}(d_K N(f))$ such that $L(s,\chi) = L(s, f_\chi)$ if $\chi \neq 1$. (If $\chi = 1$ replace f_χ with an Eisenstein series.)*

Corollary 13. *If $\chi \neq 1$ is an ideal class character of K then $f_\chi \in S_1^{new}(d_K)$. If $\chi = 1$ one gets an Eisenstein series of weight 1.*

[1] This means there is a symmetric bilinear form $(\cdot,\cdot) : J \times J \to \mathbb{R}$ such that $h_{can}(x) = \frac{1}{2}(x,x) > 0$ whener $x \in J_{tor}$.

[2] See Gross[2] for details.

5. Monday, September 14, 2009

Recall $K = \mathbb{Q}(\sqrt{D})$ and $D \equiv \square \pmod{4N}$. Up to isogeny, $J_0(N)$ is equal to

$$J_0(N)^{old} \oplus J_0(N)^{new} \qquad J_0(N)^{new} = \bigoplus_{f \in S_2^{new}(N)} A_f.$$

So, to a point $x \in J_0(N)$ we can associate to it $x_f \in A_f$.

The Heegner points are parametrized by $[\mathfrak{a}] \in Cl(K)$ and ideals $\mathcal{N}$ such that $\mathcal{N}\overline{\mathcal{N}} = N$ and $(\mathcal{N}, \overline{\mathcal{N}}) = 1$. Then

$$\tau_{[\mathfrak{a},\mathcal{N}} = (E/[\mathfrak{a}] \to E/[\mathcal{N}\mathfrak{a}]).$$

The action of $Cl(K)$ is given by $(\tau_{[\mathfrak{a},\mathcal{N}})^{\sigma_{\mathfrak{b}}} = \tau_{[\mathfrak{a}\mathfrak{b}^{-1},\mathcal{N}}$.

To describe the action of W, write $\mathcal{N} = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$. If $d \mid N$ and $(d, N) = 1$, write

$$\mathcal{N}_d = \prod_{\mathfrak{p} \mid (d,\mathcal{N})} \overline{\mathfrak{p}}^{e_{\mathfrak{p}}} \prod_{\mathfrak{p} \nmid (d,\mathcal{N})} \mathfrak{p}^{e_{\mathfrak{p}}}.$$

Then

$$w_d(\tau_{[\mathfrak{a},\mathcal{N}}) = (E/[\mathfrak{a}] \to E/[\mathfrak{a}\mathcal{N}^1].$$

Fix a Heegner point x. Write $c = (x) - (\infty) \int J_0(N)(H)$. As above $\chi : Cl(K) \to \mathbb{C}^\times$ is a character. This gives, by Hecke's Theorem, $f_\chi \in S_1^{new}(|D|, \left(\frac{D}{\cdot}\right))$ whose Fourier coefficients are $a_\chi(n) = \sum_{\mathfrak{a} | N\mathfrak{a} = n} \chi(\mathfrak{a})$.

For $c_\chi = \sum_\sigma \chi^{-1}(\sigma) c^\sigma \in J_0(N)(H, \chi)$ as before we let $c_{\chi,f}$ denote its f-isotypic component in $A_f(H, \chi, f)$.

In general, if $\chi \neq 1$ is a Hecke character of $K = \mathbb{Q}(\sqrt{D})$ of weight $k - 1$ and conductor $\mathfrak{f}$ then $f_\chi \in S_k^{new}(|D| N(\mathfrak{f}), \widetilde{\chi})$ where $\widetilde{\chi}(n) = \left(\frac{D}{n}\right) \chi(n\mathcal{O}_K)$. (If $\chi = 1$ you get an Eisenstein series.)

Another example. Take E a CM elliptic curve by $\mathcal{O}_K$. This gives a Hecke character χ_E of K associated to E such that $L(s, E/\mathbb{Q}) = L(s, \chi_E)$ if E is defined over $\mathbb{Q}$ and $L(s, E/K) = L(s, \chi_E) L(s, \overline{\chi}_E)$ if not. Again χ_E is associated to a modular form f_{χ_E} which is a weight 2 modular form. This is the reason it was known earlier that CM elliptic curves are modular.

The Rankin-Selberg L-function. Let $f = \sum_{n=1}^\infty a_n q^n \in S_k(N)$ and $g = \sum_{n=1}^\infty b_n q^n \in S_l(N)$. We can assume that $k \geq l$. Define $L(s, f, g) = \sum_{n=1}^\infty a_n b_n q^n$.

Theorem 14. *$L(s, f, g)$ has analytic continuation together with a functional equation. Actually, times some gamma factors $L(s, f, g)$ is equal to*

$$\int_{X_0(N)} f \overline{g E_{k-l}(\tau, s + a)} y^* d\mu(\tau)$$

where

$$E = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\varphi(\gamma)}{(cz + d)^{k-l} |cz + \tau|^{2s}}.$$

Moreover, $E(\tau, s) = E(\tau, -s)$ and $L(s, f) = \prod L_p(s, f)$.

In our case, $L_p(s, f) =$.

6. Monday, October 5, 2009

6.1. Theorem of Yuan, Zhang, Zhang.

6.2. **Weil representation.** Let F be a local field, (V, q) a quadratic space of dimension m. Define $\det V = \det([(e_i, e_j)])$ where $\{e_1, \ldots, e_m\}$ is a basis of V over F. Then $\det V \neq 0$ is a well-defined element of $F^\times/(F^\times)^2$. Let

$$\chi_V = ((-1)^{m(m-1)/2} \det V, \quad)_F$$

be the Hilbert symbol. Fix $\psi : F \to \mathbb{C}^\times$ an additive character.

Then we will describe a representation of $\widetilde{\mathrm{SL}_2(F)} \times O(V)(F)$ on the space $S(V)$ of Schwartz functions on V.

Some notation on groups:

$$GO(V) = \{g \in \mathrm{GL}(V) \mid (gx, gy) = \nu(g)(x, y) \text{ for all } x, y \in V\}.$$

The character $\nu : GO(V) \to \mathbb{C}^\times$ is called the *similitude character.*

$$O(V) = \{g \in GO(V) \mid \nu(g) = 1\}.$$

$$SO(V) = \{g \in O(V) \mid \det(g) = 1\}.$$

Example: Let E/F be a quadratic extension. For $a \in F^\times$ let $V = E$ and $V^a = (V, q^q)$ be the space with quadratic form $q^a(x) = ax\bar{x}$. It is easy to see that $E^\times \subset GO(V)$ with action $z \cdot x = zx$ and $\nu(z) = z\bar{z}$.

Lemma 15. *In this example*

$$GO(V^a) \simeq E^\times \ltimes < \sigma > \qquad O(V^a) \simeq E^1 \ltimes < \sigma > \qquad SO(V) \simeq E^1$$

where $< \sigma >= Gal(E/F)$ *and* $E^1 = \{z \in E \mid N_{E/F}(z) = 1\}$.

Another example: Let B be a quaternion algebra over F, $V_0 = \{x \in B \mid \mathrm{tr}\, x = x + x^\iota = 0\}$. The map $x \mapsto x^\iota$ is the main involution of B. We sometimes denote x^ι by $\bar{x}$. The quadratic form is $Q(x) = xx^\iota$. Notice that $B^\times$ acts on V_0 by $b \cdot x = bxb^{-1}$. Moreover,

$$Q(b \cdot x) = Q(bxb^{-1}) = bxb^{-1}(bxb^{-1})^\iota = bxb^{-1}(b^{-1})^\iota x^\iota b^\iota = xx^\iota = Q(x).$$

So $B^\times \subset O(V)$.

Propostion 16. *The sequence*

$$1 \longrightarrow F^\times \longrightarrow B^\times \longrightarrow SO(V) \longrightarrow 1$$

is exact.

Final example: $V = B$ and $Q(x) = xx^\iota = \det x$. In this case $B^\times \times B^\times$ acts on V by $(b_1, b_2) \cdot x = b_1 x b_2^{-1}$. Can check that $Q((b_1, b_2) \cdot x) = \det b_1 \det b_2^{-1} Q(x)$. So $B^\times \times B^\times \to GO(V)$ with kernal $F^\times$ and image of index 2.

7. Wednesday, October 7, 2009

We continue to define the Weil representation. Let F be a local field of characteristic not equal to 2, (V, Q) a quadratic space over F of dimension m. Then define

$$(x, y) = Q(x + y) - Q(x) - Q(y).$$

as above, we get groups $GO(V), O(V)$ and $SO(V)$.

In the third example from above $(V = B)$ the following sequence is exact.

$$1 \longrightarrow F^\times \longrightarrow B^\times \times B^\times \ltimes < \sigma > \longrightarrow GO(V) \longrightarrow 1$$

where the map $F^\times \to \mathbb{B}^\times \times B^\times$ is given by $a \mapsto (a, a)$.

A fourth example is $V = \mathbb{R}^m$ and $Q(x) = x_1^2 + \cdots + x_m^2$. In this case $O(V)$ is the typical orthogonal group consisting of $g \in \mathrm{GL}_n$ such that ${}^t gg = 1$.

Fix $\psi : F \to \mathbb{C}^\times$ a character. We will define $w_{V,\psi}$ a representation.

Let $W = F^2$ by the symplectic space with (standard) form $\langle x, y\rangle = x_1y_2 - x_2y_1$. From this we obtain the group

$$Sp(W) = \{g \in \mathrm{GL}(W) \mid \langle gx, gy\rangle = \langle x, y\rangle\}$$

which is isomorphic to SL_2.

Let $\mathscr{W} = W \otimes V$. This has dimension $2n$. Defining

$$\langle\langle w_1 \otimes v_1, w_2 \otimes v_2\rangle\rangle = \langle w_1, w_2\rangle(v_1, v_2)$$

gives $\mathscr{W}$ a symplectic structure (meaning that the form is alternating: $\langle\langle x, y\rangle\rangle = -\langle\langle y, x\rangle\rangle$.) We also have an injection

$$i : Sp(W) \times O(V) \longrightarrow Sp(\mathscr{W})$$

with $i(g, h)(w \otimes v) = (gw) \otimes (hv)$.

The Weil representation is a (very small, i.e. nearly irreducible) representation $w_{V,\psi}$ of $\widetilde{Sp(\mathscr{W})} = Mp(\mathscr{W})$. We have the diagram

$$\begin{array}{ccc} Mp(W) \times O(V) & \longrightarrow & Mp(\mathscr{W}) \\ \downarrow & & \downarrow \\ Sp(W) \times O(V) & \longrightarrow & Sp(\mathscr{W}) \end{array}$$

(See [5] for more discussion of the Weil representation.) If $m = \dim V$ is even, then the representation $w_{V,\psi}$ descends to a representation of $Sp(W) \times O(V)$. It is natural to ask what this restriction looks like.

(All of the above works for W any symplectic space of dimension $2n$.)

In our special case ($n = 1$ and m even) we can write down a model (the Schrodinger model) of the restriction w of $w_{V,\psi}$ to $\mathrm{SL}_2 \times O(V)$. Then the $O(V)$ action is given by

$$w(h)\varphi(x) = \varphi(h^{-1}x) \qquad h \in O(V), \varphi \in S(V).$$

Define the subgroups

$$N = \{n(b) = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right) \mid b \in F\}, \quad M = \{m(a) = \left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right) \mid a \in F^\times\},$$

$P = MN$ and K is a maximal compact subgroup of $\mathrm{SL}_2(F)$. ($K = \mathrm{SL}_2(\mathcal{O}_F)$ if F is nonarchimedean and $K = SO_2(\mathbb{R})$ if $F = \mathbb{R}$ and $K = U(2)$ if $F = \mathbb{C}$.) We have the decompositions

$$\mathrm{SL}_2 = NMK \qquad \mathrm{SL}_2 = P \cup PwP$$

where $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

We define the action of $\mathrm{SL}_2(F)$ via

$$w(n(b))\varphi(x) = \psi(bQ(x))\varphi(x),$$

$$w(m(a))\varphi(x) = \chi_V(a)\,|a|^{m/2}\,\varphi(xa),$$

$$w(w)\varphi(x) = \gamma(V)\widehat{\varphi}(x) \qquad \widehat{\varphi}(x) = \int_V \varphi(y)\psi(-(x, y))dy$$

One can check that this gives a well defined representation so long as m is even. (If m is odd you have to move to $\widetilde{\mathrm{SL}_2} = Mp(W)$.

8. Siegel-Weil formula

Now let F be a number field. We can for a Weil representation $w = \bigotimes w_v$ on $\mathrm{SL}_2(\mathbb{A}) \times O(V)(\mathbb{A})$ on $S(V_\mathbb{A}) = \bigotimes' S(V_v)$.

Weil define the *theta kernel*. For $\varphi \in S(V_\mathbb{A})$ it is defined to be

$$\theta(g, h, \varphi) = \sum_{x \in V(F)} w(g, h)\varphi(x) = \sum_{x \in V} w(g)\varphi(h^{-1}x).$$

This is a well defined function on $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}) \times O(V)(F)\backslash O(V)(\mathbb{A})$. In other words, it is an automorphic form.

The *theta integral* is

$$I(g, \varphi) = \int_{[SO(V)]} \theta(g, h, \varphi) dh.$$

For any group G, we write $[G] = G(F)\backslash G(\mathbb{A})$. The theta integral need not be convergent, but it is for example when F is totally real and V is positive then this is convergent because it implies that $O(V)(F)\backslash O(V)(\mathbb{A})$ is compact.

Exercise: Try to compute for $V = E = \mathbb{Q}(i)$, $Q(x) = x\bar{x}$ and $\varphi = \otimes\varphi_v$ with

$$\varphi_\infty(z) = e^{-2\pi z\bar{z}}, \qquad \varphi_v = 1_{\mathcal{O}_{E_v}} \quad (\text{if } n \nmid \infty).$$

Let $\psi : \mathbb{Q}\backslash\mathbb{Q}_\mathbb{A} \to \mathbb{C}$ be the character such that $\psi_\infty(x) = e^{2\pi i x}$ and $\psi_p(x) = e^{-2\pi i \lambda_p(x)}$ where

$$\lambda_p : \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}.$$

(1) Calculate $\theta(g_\tau, 1, \varphi)$ where $g_\tau = n(u)m(\sqrt{v})$. (Note that $g_\tau i = u + iv = \tau$.
(2) What is $I(g_\tau, \varphi)$?

By definition

$$\begin{aligned}
\theta(g_\tau, 1, \varphi) &= \sum_{a+ib \in \mathbb{Q}(i)} w(n(u)m(\sqrt{v}))\varphi(a + ib) \\
&= \sum_{z \in \mathbb{Z}(i)} w_\infty(n(u)m(\sqrt{v}))\varphi_\infty(z) \qquad (\varphi_f(a + ib) = 0 \text{ unless } a, b \in \mathbb{Z}) \\
&= v^{1/2} \sum_{z \in \mathbb{Z}(i)} \psi_\infty(uQ(z\sqrt{v}))\varphi_\infty(z\sqrt{v}) \qquad (\text{by the definition of the action}) \\
&= v^{1/2} \sum_{a,b \in \mathbb{Z}} e^{2\pi i u(a^2+b^2)} e^{-2\pi(a^2+b^2)v} \qquad (\text{by the definition of } \psi_\infty \text{ and } \varphi_\infty) \\
&= v^{1/2} \sum_{z \in \mathbb{Z}[i]} e^{2\pi i \tau z\bar{z}} = v^{1/2} \sum_{n=0}^{\infty} r_2(n) q^n
\end{aligned}$$

where $q = e^{2\pi i\tau}$ and $r_2(n) = \#\{z \in \mathbb{Z}[i] \mid z\bar{z} = n\}$.

Remark: The reason that the action of $O(V)$ is so nice, and that of SL_2 is messier, is because in the choice of model we have written $\mathscr{W} = W \otimes V = X \otimes V \oplus Y \otimes V$ where $W = X \oplus Y$ is a polarization. The group $O(V)$ acts on Y, and in this decomposition it remains nice, but SL_2 acts on W and it is split up.

9. Friday, October 9, 2008

We continue to discuss the Siegel Weil formula. The results of Siegel are from the 1940s, those of Weil are from around 1964, and the contribution of Kudla-Rallis is from about 1994.

Recall that $(V,Q)/F$ is a quadratic space of dimension m, $\psi : F\mathbb{F}_{\mathbb{A}} \to \mathbb{C}^\times$ is an additive character, and $w = w_{V,\psi}$ is a representation of $\widetilde{\mathrm{SL}_2} \otimes O(V)(\mathbb{A})$ on $S(V_{\mathbb{A}})$. From this we obtained the theta kernel

$$\theta(g,h,\varphi) = \sum_{x \in V} w(g,h)\varphi(x).$$

Theorem 17 (Weil)**.** *The function $\theta(g,h,\varphi)$ is an automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2}(\mathbb{A}) \times O(V)(F)\backslash O(V)(\mathbb{A})$.*

As before we let $[G] = G(F)\backslash G(\mathbb{A})$.

Thus the theta integral

$$I(g,\varphi) = \int_{[SO(V)]} \theta(g,h,\varphi)dh$$

is an automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2}(\mathbb{A})$ whenever the integral converges.

For example if $E = \mathbb{Q}(\sqrt{D})$ for $D < 0$, $V = E$ and $Q(x) = x\bar{x}$, then if we define φ and ψ as in the exercise above we get

$$\theta(g_\tau, 1, \varphi) = v^{1/2} \sum_{z \in \mathcal{O}_K} r_{\mathcal{O}_K}(n) q^n$$

where $r_{\mathcal{O}_K}(n) = \#\{z \in \mathcal{O}_K \mid z\bar{z} = n\}$. Without the factor $v^{1/2}$ this is a modular form of weight 1.

We now define the Eisenstein series. Let $\chi : F^\times\backslash F_{\mathbb{A}}^\times \to \mathbb{C}^\times$ be a character. Then we define

$$I(s,\chi) = \mathrm{Ind}_{P_{\mathbb{A}}}^{\mathrm{SL}_2(\mathbb{A})}(\chi\,|\cdot|^s) = \{\Phi : \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C} \mid \Phi(n(b)m(a)g) = \chi(a)\,|a|^{s+1}\,\Phi(g)\}$$

The space $I(s,\chi) = \bigotimes_{v \le \infty} I(s,\chi_v)$ which is generated by elements of the form $\Phi = \otimes\Phi_v$. For such an element we define

$$E(g,s,\Phi) = \sum_{\gamma \in P(F)\backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s).$$

When this converges it is clearly an automorphic form on $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})$.

Theorem 18. *The map*

$$\lambda : S(V_{\mathbb{A}}) \to I(s_0,\chi_V) \quad \varphi \mapsto \lambda(\varphi) : g \mapsto w(g)\varphi(0)$$

is $\mathrm{SL}_2(\mathbb{A})$-equivariant. (Here $s_0 = m/2 - 1$.) In other words $\lambda(w(g)\varphi) = \rho(g)\lambda(\varphi)$ where ρ is the right regular action.

Theorem 19 (Siegel-Weil)**.** *When the sum and integrals are absolutely convergent*

$$\frac{1}{\mathrm{vol}([O(V)])} I(g,\varphi) = \kappa E(g,s_0,\lambda(\varphi))$$

where $\kappa = 1$ if $m > 2$ and $\kappa = 2$ if $m \le 2$.

If r is the dimension of the maximal isotropic subspace of V, Weil proved that $I(g,\varphi)$ is absolutely convergent when

$$r = 0 \qquad \text{or} \qquad r > 0 \text{ and } m - r > 2. \tag{1}$$

On the other hand, if (1) holds and $m > 4$ then $E(g,s_0,\lambda(\varphi))$ is also absolutely convergent.

Kudla and Rallis proved that there is an analytic continuation of $E(g, s, \lambda(\varphi))$ if (1) holds in the following way. Given $\lambda(\varphi) \in I(s_0, \chi_V)$, we want to define a standard section of $\Phi(g, s) \in I(s, \chi_V)$ such that $\Phi(g, s_0) = \lambda(\varphi)$ and $\Phi(g, s) \mid_K$ is independent of K. Concretely if we write $g = n(b)m(a)k$ for some $k \in K$ then we want to define

$$\Phi(g, s) = \lambda(\varphi)(g) \left|a(g)\right|^{s-s_0}$$

where $a(g) = a$. Since the decomposition of g in this form is not unique one has to check that this gives a well defined element of $I(s, \chi_V)$, but this is the case. This gives $E(g, s, \Phi)$, for which Kudla and Rallis proved that the Siegel-Weil formula extends.

Remark: Actually, Kudla and Rallis proved even more. When E has a pole at s, they proved that there is a Siegel-Weil type formula involving the residue. We will not need this for what we are doing.

9.1. **Fourier coefficients.** Fix ψ. For any automorphic form $f : \mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}) \to \mathbb{C}$, we can write

$$f(g) = \sum_{b\in F} f_b(g) \qquad f_b(g) = \int_{F\backslash\mathbb{A}} f(n(x)g)\psi(-xb)dx.$$

The function f_b is called the *b-th Fourier coefficient.*

When f is an Eisenstein series the following proposition tells us that calculating the Fourier coefficients is pretty simple.

Remark. This simplicity is mirrored in the classical case. Here

$$E_k = 1 + * \sum \sigma_{k-1}(n)q^n \qquad \sigma_{k-1} = \sum_{d|n} d^{k-1} = \prod_{p<\infty} \sigma^{(p)}_{k-1}(n)$$

where $\sigma^{(p)}_{k-1} = 1 + p^{k-1} + (p^2)^{k-1} + \cdots + (p^r)^{k-1}$ and $r = \mathrm{ord}_p(n)$. On the other hand, finding the coefficients of cusp forms is very complicated.

Propostion 20. *For all* $a \neq 0$, $E_a(g, s, \Phi) = \prod_v E_{a,v}(g_v, s, \Phi_v)$ *where*

$$E_{a,v}(g_v, s, \Phi_v) = \int_{F_v} \Phi_v(wn(x)g_v, s)\psi(-ax)dx$$

is the local Whittaker function

Proof. We use the fact that $\mathrm{SL}_2 = P \cup PwN$ from which one can easily deduce that $P\backslash \mathrm{SL}_2 \leftrightarrow \{1\} \cup wN$. Now, using the definitions,

$$\begin{aligned}
E_a(g,s,\Phi) &= \int_{F\backslash \mathbb{A}} E(n(x)g,s,\Phi)\psi(-ax)dx \\
&= \int_{F\backslash \mathbb{A}} \sum_{\gamma \in P\backslash \mathrm{SL}_2(F)} \Phi(\gamma n(x)g,s)\psi(-ax)dx \\
&= \int_{F\backslash \mathbb{A}} \Phi(\gamma n(x)g,s)\psi(-ax)dx + \sum_{x\imath F} \Phi(wn(b)n(x)g,s)\psi(-ax)dx \\
&= \Phi(g,s)\int_{F\backslash \mathbb{A}} \psi(-ax)dx + \int_{F\backslash \mathbb{A}} \sum_{b\in F} \Phi(wn(x)g,s)\psi(-ax)dx \\
&= \Phi(g,s)\int_{F\backslash \mathbb{A}} \psi(-ax)dx + \int_{\mathbb{A}} \Phi(wn(x)g,s)\psi(-ax)dx \\
&= \Phi(g,s)\int_{F\backslash \mathbb{A}} \psi(-ax)dx + \prod_v \int_{F_v} \Phi_v(wn(x)g_v,s)\psi_v(-ax)dx
\end{aligned}$$

To finish the proof we note that $\int_{F\backslash\mathbb{A}} \chi(x)dx$ is zero unless χ is trivial (in which case it is the volume of $F\backslash\mathbb{A}$ which is finite.) $\square$

Notice that we have proved more, since we have given the formula even when $a = 0$.

Exercise. Check the Siegel Weil formula for $V = \mathbb{Q}(\sqrt{D})$, $Q(x) = x\bar{x}$ in general. If this is too hard try the cases $D = -3, -1, -7$ where the class number is 1.

10. Monday, October 12, 2009

Recall that the theta kernel

$$\theta(g,h,\varphi) = \sum_{x\in V} w(g,h)\varphi(x)$$

is an automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A}) \times O(V)(F)\backslash O(V)(\mathbb{A})$. G

Given $f \in \pi$ an irreducible cuspidal autormophic representation of $O(V)(F)\backslash O(V)(\mathbb{A})$ and $\varphi \in S(V_{\mathbb{A}})$, we define

$$\theta(h; f, \varphi) = \int_{[\mathrm{SL}_2]} \theta(g,h,\varphi) f(g) dg.$$

This is absolutely convergent because the assumption that π is cuspidal implies that f has exponential decay.

We get the space

$$\Theta(\pi) = \langle \theta(h; f, \varphi) \mid f \in \pi, \varphi \in S(V_{\mathbb{A}})\rangle.$$

This gives the so-called *theta lifting* from functions SL_2 to functions on $O(V)$. (Can go in the opposite direction but this isn't a concern for us.)

The natural questions are

(1) When is $\Theta(\pi) \neq 0$?
(2) In this case how does $\Theta(\pi)$ decompose as a representation of $O(V)$?

It is a fact that in our situation $\Theta(\pi)$ has a unique irreducible subquotient.

Example: Shimura lifting. (due to Waldspurger, Niwa, Shintani, Shimura) Set $V = B_0 = \{x \in B \mid \operatorname{tr} x = 0\}$, and let $Q(x) = ax\bar{x}$. Here we really need to replace SL_2 by $\widetilde{\mathrm{SL}}_2$ because $\dim V = 3$. For this choice $SO(V) = PB^\times = B^\times/F^\times$. In the special case that $B = M_2$, we have $SO(V) = \mathrm{PGL}_2$, and the associated lift is the Shimura correspondence.

10.1. **Extended Weil representation.** Another example (of much interest to us) is the Shimizu lifting: $V = B$, $Q(x) = x\bar{x}$. However, we'd like to enlarge the theory to treat the case $\mathrm{GL}_2 \times GO(V)$ instead of $\mathrm{SL}_2 \times O(V)$. We do this in the next section.

As normal, we assume $\dim V$ is even. The space on which $\mathrm{GL}_2 \times GO(V)$ will act is $S(V_\mathbb{A} \times F_\mathbb{A}^\times)$. The action by $GO(V)$ is simple: $w(h)\varphi(x,u) = \varphi(h^{-1}x, \nu(h)u)$.

The GL_2 action is more complicated: $w(g)\varphi(x) = w_u(g)\varphi(x)$ where

$$w_u(n(b))\varphi(x,u) = \psi(buQ(x))\varphi(x,u)$$

$$w_u(m(a))\varphi(x,u) = \chi_V(a)\,|a|^{m/2}\,\varphi(xa,u)$$

$$w_u(w)\varphi(x,u) = \gamma(V^u)\,|u|^{m/2}\int_V \varphi(y)\psi(-uxy)d_u y$$

The Shimizu lifting. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})$, and let (V,Q) be as above. We redefine the theta lift

$$\theta(h;f,\varphi) = \frac{\zeta(2)}{2L(1,\pi,\mathrm{Ad})}\int_{[\mathrm{GL}_2]} \theta(g,h,\varphi)dg.$$

Note that this is the same as before, but with a different normalization and we now are using the extended Weil representation. Since $B^\times \times B^\times \subset GO(V)$, we can think of $\Theta(\pi)$ as a representation on $B^\times \times B^\times$.

Theorem 21.
- $\Theta(\pi) \neq 0$ *if and only if* π_v *is not a principal series at all places such that* B_v *is a division algebra.*
- *When* $\Theta(\pi) \neq 0$, *we have* $\Theta(\pi) = JL(\pi) \times \widetilde{JL(\pi)}$ *as a representation of* $B_\mathbb{A}^\times \times B_\mathbb{A}^\times$.
- *If* π *is an autormophic representation of* $B^\times$, *can go in opposite direction and get an irreducible representation of* GL_2.

Now we have a $B^\times \times B^\times$ and GL_2 equivariant map

$$\theta : \pi \otimes S(V_\mathbb{A} \times F^\times) \to \pi' \otimes \widetilde{\pi}' \quad f \otimes \varphi \mapsto \theta(h;f,\varphi)$$

where $\pi' = JL(\pi)$. In other words,

$$\theta \in \mathrm{Hom}_{\mathrm{GL}_2 \times B^\times \times B^\times}(\pi \otimes S(V_\mathbb{A} \times F^\times), \pi' \otimes \widetilde{\pi}').$$

By Jacquet-Langlands this space has dimension 1, so θ is a generator.

We would like to understand what this looks like locally. Since $\theta = \prod_v \theta_v$ we want to find canonical choices of

$$\theta_v : \pi_v \otimes S(V_v \times F_v^\times) \to \pi_v \otimes \widetilde{\pi}'_v.$$

This is tricky because the Siegel-Weil formula given is completely global, and defining something similar for the local situation is a challenge.

More concretely, don't have a good way to construct the decompostion

$$S(V_v \times F_v^\times) = \bigoplus_{\pi \in Irr(\mathrm{GL}_2)} \pi \otimes \Theta(\pi).$$

11. Wednesday, October 14, 2009

Today we discuss the Shimizu lifting. Let B over F be a quaternion algebra and $\varphi \in S(V_\mathbb{A} \times F_\mathbb{A}^\times)$. Then we get

$$\theta(g,h,\varphi) - \sum_{(x,u)\in V\times F^\times} w(g)\varphi(h^{-1}x, \nu(h)u),$$

and if $f \in \pi$ an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})$ then we can define

$$\theta(h, f\otimes\varphi) = \frac{\zeta(2)}{2L(1,\pi,\mathrm{Ad})} \int_{[\mathrm{GL}_2]} \theta(g,h,\varphi) f(g) dg.$$

As discussed last time, can think of θ as an element of

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A})\times B_\mathbb{A}^\times \times B_\mathbb{A}^\times}(\pi \otimes S(V_\mathbb{A} \times F_\mathbb{A}^\times), \pi' \otimes \widetilde{\pi}')$$

which has dimension 1. Since $\theta \neq 0$ there should be θ_v such that $\theta = \prod \theta_v$. We would like to understand θ_v canonically. To do this we need a local map $\pi_v \to \pi'_v \otimes \widetilde{\pi}'_v$.

There is a canonical pairing on $\pi' \otimes \widetilde{\pi}'$. Locally, it is

$$\mathscr{F}_v : \pi'_v \otimes \widetilde{\pi}'_v \to \mathbb{C} \qquad (x, f) \mapsto f(x).$$

This is just because $\widetilde{\pi}'$ is the space of linear functionals on π'. Globally, the pairing is even more natural. Notice that π' is always cuspidal hence a subspace of $L_0^2(B^\times\backslash B_\mathbb{A}^\times)$. Since this is a Hilbert space, we have the positive definite inner form

$$\pi' \otimes \pi' \to \mathbb{C} \qquad (f_1, f_2) \mapsto \int_{FB^\times\backslash B_\mathbb{A}^\times} f_1(g)\overline{f_2(g)} dg.$$

This implies that $\widetilde{\pi}' = \{\overline{f} \mid f \in \pi'\}$, and

$$\mathscr{F} : \pi' \otimes \widetilde{\pi}' \to \mathbb{C} \qquad (f, \widetilde{f}) \mapsto \int_{FB^\times\backslash B_\mathbb{A}^\times} f(g)\widetilde{f}(g) dg.$$

11.1. The Whittaker model.

This is a local analogue of the Fourier expansion. Let π_v be an irreducible admissible representation of $\mathrm{GL}_2(F_v)$, $\psi : F_v \to \mathbb{C}$ a nontrivial character. The *ψ-Whittaker functional* on π_v is a linear map

$$\ell : \pi_v \to \mathbb{C} \quad \text{such that} \quad \ell(n(b)x) = \psi(b)\ell(x).$$

Theorem 22. *For fixed nontrivial ψ, a ψ-Whittaker functional exists and is unique up to scalar.*

Corollary 23. *For $x \in \pi_v$ is as above and ℓ is a ψ-Whittaker functional, the map $x \mapsto W^x : g \mapsto \ell(\pi_v(g)x)$ is a $\mathrm{GL}_2(F_v)$ equivariant map*

$$\pi_v \hookrightarrow \mathscr{W}(\mathrm{GL}_2, \psi) = \{W : \mathrm{GL}_2(F_v) \to \mathbb{C} \mid W(n(b)g) = \psi(b)W(g).$$

*The image of π_v is called the ψ-*Whittaker model*. It is unique.*

Notation. We will always take ψ fixed from here on out, but every other character is of the type $\psi^a(x) = \psi(ax)$, and for $f \in \pi_v$ we will denote the corresponding ψ^a-Whittaker function by $W_a^f(g)$.

We can now define θ_v. It is uniquely determined by the property that

$$\mathscr{F}_v \theta_v(f \otimes \varphi) = * \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} W_{-1}^f(g) w(g) \varphi(1,1) dg$$

and $*$ is a constant so that for unramified φ and f the value is 1.

Globally, the Whittaker model is easier. Take $f \in \pi \subset L_0^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}))$. Then

$$f \mapsto W_\psi^f(g) = \int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\psi(-b)db.$$

One can easily check that

- $f \mapsto W_\psi^f$ is injective;
- $\pi(g)f \mapsto W_\psi^{\pi(g)f} = \rho(g)W_\psi^f : h \mapsto W_\psi^f(hg)$;
- $W_\psi^f(n(b)g) = \psi(b)W_\psi^f(g)$.

Since W_ψ^f is a function on $\mathrm{GL}_2(\mathbb{A})$ this gives a local Whittaker model $g_v \mapsto W_\psi^f(g)$. By the uniqueness of the local models, one sees that the global model must also be unique.

Theorem 24. $\theta = \prod \theta_v$.

Proof. Let us write $V = B = V_0 \oplus V_1$ where $V_0 = \{b \in B \mid \operatorname{tr} b = 0\}$ and $V_1 = F$. Define $SO(V)_0 = \{(g_1, g_2) \mid \det g_1 = \det g_2\}$. The following diagram commutes.

$$\begin{array}{ccc} SO(V_0) & \longrightarrow & SO(V) \\ \simeq \uparrow & & \uparrow \\ F^\times \backslash B^\times & \xrightarrow{b \mapsto (b,b)} & SO(V)_0 \end{array}$$

So,

$$\begin{aligned} \mathscr{F}\theta(f \otimes \varphi) &= \int_{F_\mathbb{A}^\times B^\times \backslash B_\mathbb{A}^\times} \theta(g, (b,b), \varphi) dg db \\ &= \int_{[\mathrm{GL}_2]} f(g) \int_{F_\mathbb{A}^\times B^\times \backslash B_\mathbb{A}^\times} \theta(g, (b,b), \varphi) db dg \\ &= \int_{[\mathrm{GL}_2]} f(g) \int_{[SO(V_0)]} \theta(g, b, \varphi) db dg \\ &= \int_{[\mathrm{GL}_2]} f(g) \sum_{(x,u) \in V_1 \times F^\times} \sum_{x \in V_0} w(g)\varphi(b^{-1}(x_0, x_1), \nu(b)u) dg \\ &= \int_{[\mathrm{GL}_2]} f(g) \sum_{(x,u) \in V_1 \times F^\times} \sum_{x \in V_0} w(g)\varphi(b^{-1}x_0, x_1, u) dg \\ &= \int_{[\mathrm{GL}_2]} f(g) \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \sum_{(x,u) \in V_1 \times F^\times} w(g)\varphi(x, u) dg. \end{aligned}$$

In the last step we applied the Siegel-Weil formula. (We are allowed to write $P(F)\backslash \mathrm{GL}_2(F)$ because it is the same as $P^1(F)\backslash \mathrm{SL}_2(F)$.) This final integral can be unfolded. We will do this next time. $\square$

12. Friday, October 16, 2009

Recall that globally we have

$$\pi \otimes S(V_{\mathbb{A}} \times \mathbb{F}_{\mathbb{A}}^{\times}) \xrightarrow{\theta} \pi' \otimes \widetilde{\pi}' \xrightarrow{\mathscr{F}} \mathbb{C}$$

and locally,

$$\begin{array}{ccc} \pi_v \otimes S(V_v \times F_v^{\times}) & \xrightarrow{\theta_v} & \pi_v' \otimes \widetilde{\pi}_v' \\ & \searrow & \downarrow \mathscr{F}_v \\ & & \mathbb{C} \end{array}$$

where

$$\mathscr{F}_v \theta_v = \frac{\zeta_v(2)}{L_v(1, \pi_v, \mathrm{ad})} \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} W_{-1}^{f_v} w(g) \varphi_v(1,1) dg.$$

We had started to prove that $\theta = \prod \theta_v$, by first noting that it suffices to prove that $\mathscr{F}\theta = \prod \mathscr{F}_v \theta_v$. Then we decomposed $V = V_1 \oplus V_2$ and applied the Siegel-Weil formula, which says that

$$E(g, \varphi(\cdot, x_2, u)) = \sum_{\gamma \in P^1(F)\backslash \mathrm{SL}_2(F)} w(\gamma g) \varphi(\cdot, x_2, u)$$

to obtain the following.

$$\begin{aligned} \mathscr{F}\theta(f \otimes \varphi) &= \int_{B^{\times} F_{\mathbb{A}}^{\times} \backslash B_{\mathbb{A}}^{\times}} \int_{[\mathrm{GL}_2]} f(g) \theta(g, (h,h), \varphi) dg dh \\ &= \int_{[\mathrm{GL}_2]} f(g) \int_{[SO(V_1)]} \sum_{(x_1, x_2, v) \in V_1 \times V_2 \times F^{\times}} w(g) \varphi(h^{-1} x_1, x_2, u)) dh dg \\ &= \int_{[\mathrm{GL}_2]} f(g) \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \sum_{(x_2, u) \in V_2 \times F^{\times}} w(\gamma g) \varphi(x_2, u) dg \\ &+ \int_{P(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g) \sum_{(x_2, u) \in V_2 \times F^{\times}} w(\gamma g) \varphi(x_2, u) dg \end{aligned}$$

Note that we have used that $f(\gamma g) = f(g)$. We break up this final integral into two pieces. Let

$$I_1 = \sum_{u \in F^{\times}} w(g) \varphi(0, u) \qquad I_2 = \sum_{(x_2, u) \in F^{\times} \times F^{\times}} w(g) \varphi(x_2, u).$$

Let us recall/derive formulae for the action of

$$M = \{ m(a,d) = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix} \right) \mid a, d \in F^{\times} \}$$

under the Weil representation. In the case at hand $m = \dim V = 4$.

$$w(\left(\begin{smallmatrix} 1 & \\ & d \end{smallmatrix} \right)) \varphi(x, u) = |d|^{-m/4} \varphi(x, d^{-1} u),$$

$$w(\left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix} \right)) \varphi(x, u) = \chi(a) |a|^{m/2} \varphi(xa, u),$$

$$w(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix} \right)) \varphi(x, u) = \chi(a) |a|^{m/4} \varphi(xa, a^{-1} u).$$

So putting this all together we have that

$$w(\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix} \right)) \varphi(x, u) = \chi(a) |a/d|^{m/4} \varphi(xa, d^{-1} u).$$

Using this, we can write

$$I_1 = \sum_{u \in F^\times} w(\left(\begin{smallmatrix} 1 & \\ & u \end{smallmatrix}\right) g)\varphi(0,1), \qquad I_2 = \sum_{(x_2,u)\in F^\times \times F^\times} \chi^{-1}(x_2)\ w(\left(\begin{smallmatrix} x_2 & \\ & u \end{smallmatrix}\right) g)\varphi(1,1).$$

Note that $\chi\,|_F = 1$. So using the fact that $P = MN$, we have that

$$\begin{aligned} J_2 &= \int_{N(F)M(F)\backslash \mathrm{GL}_2(\mathbb{A})} \sum_{m \in M(F)} f(mg)w(mg)\varphi(1,1)dg \\ &= \int_{N(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g)w(g)\varphi(1,1)dg \\ &= \int_{N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\, \overbrace{w(n(b)g)\varphi(1,1)}^{\psi(b)w(g)\varphi(1,1)}\, dbdg \\ &= \int_{N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} w(g)\varphi(1,1)\, \overbrace{\int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\psi(b)db}^{W_{-1}^f(g)}\, dg \\ &= \prod_v \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} W_{-1}^f(g_v)w(g_v)\varphi_v(1,1)dg_v = \prod_v \mathscr{F}_v\theta_v. \end{aligned}$$

To complete the proof we must show that $J_1 = 0$. We compute in a similar fashion to above.

$$\begin{aligned} J_1 &= \int_{P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \sum_{u \in F^\times} f(\left(\begin{smallmatrix} 1 & \\ & u \end{smallmatrix}\right) g)w(\left(\begin{smallmatrix} 1 & \\ & u \end{smallmatrix}\right) g)\varphi(0,1)dg \\ &= \int_{N(F)\times\{\left(\begin{smallmatrix} * & \\ & 1 \end{smallmatrix}\right)\}\backslash \mathrm{GL}_2(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} \sum_{u\in F^\times} f(n(b)g)\, \overbrace{w(n(b)g)\varphi(0,1)}^{\psi(bQ(0))w(g)\varphi(0,1)}\, dbdg \\ &= \int w(g)\varphi(0,1) \int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)dbdg. \end{aligned}$$

Because f is cuspidal, $\int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)db = 0$. This completes the proof.

12.1. **Section 2.3 of** [7]**.** We want to represent $L(s, \pi, \chi)$ as an integral.

Recall the setup: E/F is a quadratic extension with associated character η, π is an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})$ and $\chi : E^\times \backslash E^\times_{\mathbb{A}} \to \mathbb{C}$ is a character such that $\chi\,|_{F^\times_{\mathbb{A}}}\ \omega_\pi = 1$.

From this date we get a set

$$\Sigma = \{v \mid \epsilon(\frac{1}{2}, \pi_v, \chi_v) \neq \chi_v\eta_v(-1)\}$$

and an adelic quaternion algebra $\mathbb{B}$ such that $\mathbb{B}_v$ is division algebra precisely when $v \in \Sigma$. (So $\mathbb{B}$ is global if $\#\Sigma$ is even and incoherent if odd.) Let

$$\pi' = \pi_{\mathbb{B}} = \otimes JL(\pi_v) = \otimes \pi'_v.$$

For fixed $E_{\mathbb{A}} \hookrightarrow \mathbb{B}$ we get a decomposition

$$\mathbb{V} = \mathbb{B} = E_{\mathbb{A}} \oplus E_{\mathbb{A}} j = \mathbb{V}_1 \oplus \mathbb{V}_2.$$

The element $j \in \mathbb{B}$ satisfies

$$j^2 \in \mathbb{A}_F^\times \qquad \text{and} \qquad ja = \overline{a}j \text{ for all } a \in E_{\mathbb{A}}.$$

For $\varphi \in S(\mathbb{V} \times F_{\mathbb{A}}^{\times}) = S(\mathbb{V}_1 \times \mathbb{V}_2 \times F_{\mathbb{A}}^{\times})$ we define the *Eisenstein-theta series*:

$$I(g,s,\varphi) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x_1,u)\in V_1\times F^{\times}} w(\gamma g)\varphi(x_1,u)$$

where $\delta(n(b)m(a,d)g) = \left|\frac{a}{d}\right|^{1/2}$. This is an Eisenstein series on $\mathbb{V}_2$ and a theta series on $\mathbb{V}_1$. We would like to understand its χ-component:

$$I(g,x,\chi,\varphi) = \int_{E^{\times}\backslash E_{\mathbb{A}}^{\times}} I(g,s,\overbrace{w(t)\varphi}^{\varphi(t^{-1}x,Q(t)u)})\chi(t)dt.$$

We define

$$\mathscr{P} : \pi \otimes S(\mathbb{V}\times \mathbb{F}_{\mathbb{A}}^{\times}) \to \mathbb{C} \qquad \mathscr{P}(f\otimes\varphi) = \int_{Z_{\mathbb{A}}\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} I(g,s,\chi,\varphi)f(g)dg.$$

Theorem 25 (Waldspurger)**.** *If $f = \otimes f_v$ and $\varphi = \otimes\varphi_v$ then $\mathscr{P}(f\otimes\varphi) = \otimes\mathscr{P}_v(f_v\otimes\varphi_v)$ where*

$$\mathscr{P}_v(f_v\otimes\varphi_v) = \int_{F_v^{\times}\backslash E_v^{\times}} \chi(t)\int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} \delta(g)^s W_{-1,v}(g)w(g)\varphi(t^{-1},Q(t))dgd^{\times}t.$$

Theorem 26 (Waldspurger)**.** *When everything is unramified*

$$\mathscr{P}_v(f_v\otimes\varphi_v) = \frac{L((s+1)/2,\pi_v,\chi_v)}{L(s+1,\eta_v)}.$$

13. Monday, October 19, 2009

Today we prove the second theorem of Waldspurger from the end of last time. To do this we first discuss the *local newform* of $\pi_v = \pi$ an irreducible admissible representation of $\mathrm{GL}_2(F_v)$.

Let $\varpi \in F_v$ be a uniformizer, and denote the ring of integers by $\mathcal{O}_v$. Define

$$K_1(\varpi^c) = \left\{ g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_v) \mid g \equiv \begin{pmatrix} * & * \\ & 1\end{pmatrix} \pmod{\varpi^c\mathcal{O}_v}\right\}.$$

Theorem 27 (New Whittaker function)**.** (a) *There exists a unique integer $c = c(\pi) \geq 0$ such that $\dim \pi^{K_1(\pi^c)} = 1$.*

(b) *There exists and unique $W \in \mathscr{W}_\pi$ such that $W(gk) = W(g)$ for all $k \in K_1(\varpi^c)$ and $W(1) = 1$.*

(c) $L(s,\pi) = \int_{F_v^{\times}} W(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)) |a|^{s-1/2} d^{\times}a.$

(d) *When $\pi = \pi(\mu_1,\mu_2)$ is an unramified principal series then*

$$W(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)) = \begin{cases} 0 & \text{if } a \notin \mathcal{O}_v, \\ |a|^{1/2}\,\frac{\mu_1(a\varpi)-\mu_2(a\varpi)}{\mu_1(\varpi)-\mu_2(\varpi)} & \text{if } a\in\mathcal{O}_v.\end{cases}$$

In the case that $c(\pi) = 0$ this theorem describes W completely. Indeed, $\mathrm{GL}_2(F_v) = NMK$ where $K = \mathrm{GL}_2(\mathcal{O}_v) = K_1(\pi^0)$. We know how N acts because W is a Whittaker function, the action of K is trivial by (b) and that of M is given by (d).

Since "everything is unramified," $B_v = M_2(F_v)$, φ_v is the characteristic function of $M_2(\mathcal{O}_v)\times\mathcal{O}_v^{\times}$, $c(\pi_v) = 0$ and

$$E_v = \begin{cases} \text{unramified field extension of } F_v & \text{(inert case)} \\ F_v \times F_v & \text{(split case)}\end{cases}$$

We'll do the case E_v an unramified field extension.

Lemma 28. *For this choice of φ_v, the action of K via the Weil representation is trivial. In other words $w(k)\varphi = \varphi$ for all $k \in K$.*

Note that $Z(F_v)N(F_v)\backslash \mathrm{GL}_2(F_v)/K_v \simeq \{m(a,1) \in M\}$. Let us assume that $\mathrm{vol}(K_v) = 1$. We are now ready to compute.

$$\begin{aligned}\mathscr{P}_v(f_v \otimes \varphi_v) &= \int_{Z(F_v)N(F_v)\backslash \mathrm{GL}_2(F_v)/K_v} \int_{E_v^\times} \delta(g)^s W_{-1,v}(g) w(g)\varphi(t^{-1}, Q(t))\chi(t) d^\times t dg \\ &= \int_{F_v^\times}\int_{E_v^\times} |a|^{s/2}\, W_{-1,v}(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right))\, |a|^{4/4}\, \varphi(at^{-1}, Q(t)a^{-1})\chi(at^{-1}) d^\times t \frac{da}{|a|}.\end{aligned}$$

Note that under the injection $E_v \to M_2(F_v)$, we have that the preimage of $M_2(\mathcal{O}_v)$ is precisely $\mathcal{O}_{E_v}$. So $\varphi(at^{-1}, Q(t)a^{-1}) = 0$ unless $Q(t)a^{-1} \in \mathcal{O}_v^\times$ and $at^{-1} \in \mathcal{O}_{E_v}$, in which case it is identically 1. It is straightforward to check that these conditions imply that $a \in \varpi^{2n}\mathcal{O}_v^\times$ and $t \in \varpi^n \mathcal{O}_{E_v}^\times$, and our integral becomes

$$\begin{aligned}\mathscr{P}_v(f_v \otimes \varphi_v) &= \sum_{n=0}^{\infty} \int_{\varpi^{2n}\mathcal{O}_v^\times} \int_{\varpi^n \mathcal{O}_{E_v}^\times} q^{-n(s+1)} \frac{\mu_1(a\varpi) - \mu_2(a\varpi)}{\mu_1(\varpi) - \mu_2(\varpi)} \chi(\varpi)^n d^\times t d^\times a \\ &= \frac{\mathrm{vol}(\mathcal{O}_v^\times)\,\mathrm{vol}(\mathcal{O}_{E_v}^\times)}{\mu_1(\varpi) - \mu_2(\varpi)} \left[\begin{array}{c} \mu_1(\varpi)\sum_{n=0}^{\infty}((\mu_1^2\chi)(\varpi)q^{-s-1})^n \\ -\mu_2(\varpi)\sum_{n=0}^{\infty}((\mu_2^2\chi)(\varpi)q^{-s-1})^n \end{array}\right] \\ &= \left(\frac{1}{\mu_1(\varpi) - \mu_2(\varpi)}\right)\left(\frac{\mu_1(\varpi) - (\mu_1\mu_2^2\chi)(\varpi)q^{-s-1} - (\mu_2(\varpi) - (\mu_2\mu_1^2\chi)(\varpi)q^{-s-1})}{(1-(\chi\mu_1^2)(\varpi)q^{-s-1})(1-(\chi\mu_2^2)(\varpi)q^{-s-1})}\right) \\ &= \frac{1+q^{-s-1}}{(1-\chi(\varpi)\mu_1^2(\varpi)q^{-s-1})(1-\chi(\varpi)\mu_2^2(\varpi)q^{-s-1})},\end{aligned}$$

because $(\chi\mu_1\mu_2)(\varpi) = 1$ by assumption.

Since in the case we have $\eta_v(\varpi) = 1$, $L_v(s,\eta)_v = (1+q^{-s})^{-1}$. Note that, in the split case $(E_v = F_v \times F_v)$ $\eta_v(\varpi) = -1$.

Exercise: Do the above calculation in the split case.

14. Wednesday, October 21, 2009

Recall that we have the Eisenstein-theta series:

$$I(s,g,\varphi) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x,u)\in V_1 \times F^\times} w(\gamma g)\varphi(x,u),$$

$$I(s,g,\chi,\varphi) = \int_{E^\times \backslash E_{\mathbb{A}}^\times} I(s,g,w(t,1)\varphi)\chi(t)dt.$$

We have seen that

$$\mathscr{P}(s,\chi,f,\varphi) = \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} I(s,g,\chi,\varphi)f(g)dg$$

satisfies

$$\mathscr{P}(s,\chi,f,\varphi) = \prod_v \mathscr{P}_v(s,\chi_v,f)v,\varphi_v) \tag{2}$$

where

$$\mathscr{P}_v(s,\chi_v,f)v,\varphi_v) = \int_{F_v^\times\backslash E_v^\times} \chi(t) \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} \delta(g)^s W_{-1}^{f_v}(g) \underbrace{w(g)\varphi_v(t^{-1},Q(t))}_{w(g)w(t,t)\varphi(1,1))} dgdt.$$

When everything is unramified

$$\mathscr{P}_v(s,\chi_v,f)v,\varphi_v) = \frac{L(\frac{s+1}{2},\pi_v,\chi_v)}{L(s+1,\eta_v)}. \tag{3}$$

In other words,

$$\mathscr{P}_v^0(s,\chi_v,f)v,\varphi_v) = \frac{L(s+1,\eta_v)}{L(\frac{s+1}{2},\pi_v,\chi_v)} \mathscr{P}_v(s,\chi_v,f)v,\varphi_v)$$

is 1 for all but finitely many v.

Lemma 29. *The map*

$$\mathscr{P}_v^0(s,\chi_v,\cdot,\cdot) : \pi_v \otimes S(V_1 \times F_v^\times) \to \mathbb{C}$$

given by $(f_v,\varphi_v) \mapsto \mathscr{P}_v^0(s,\chi_v,f)v,\varphi_v)$ *factors through* $\pi_v' \otimes \widetilde{\pi}_v'$.

Proof. We have seen that

$$\mathscr{F}_v\theta_v(f\otimes\varphi) = \frac{\zeta_v(2)}{2L(1,\pi_v,\mathrm{ad})} \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} W_{-1}^f(g)w(g)w(t,1)\varphi(1,1)dgdt.$$

So, by the above,

$$\mathscr{P}_v^0(s,\chi_v,f)v,\varphi_v) = \frac{2L(s+1,\eta_v)L(1,\pi_v,\mathrm{ad})}{\zeta_v(2)L(\frac{s+1}{2},\pi_v,\chi_v)} \int_{F_v^\times\backslash E_v^\times} \mathscr{F}_v\theta_v(f_v\otimes\varphi_v)\chi(t)dt. \tag{4}$$

□

If $\theta_v(f_v\otimes\varphi_v) = f'\otimes\widetilde{f}'$ where $f'\otimes\widetilde{f}' \in \pi_v'\otimes\widetilde{\pi}_v'$, we have seen that

$$\langle \pi_v'(t)f', \widetilde{f}'\rangle = \mathscr{F}_v(f'\otimes\widetilde{f}').$$

Hence we have the following important corollary.

Lemma 30 (Local Theta Lifting)**.** *Define* $\alpha_v : \pi_v'\otimes\widetilde{\pi}_v' \to \mathbb{C}$ *by*

$$\alpha_v(f'\otimes\widetilde{f}',\chi) = C\int_{F_v^\times\backslash E_v^\times} \langle \pi_v'(t)f', \widetilde{f}'\rangle\chi(t)dt$$

where C *is the constant appearing in* (4)*. Then*

$$\mathscr{P}_v^0(s,\chi_v,f_v,\varphi_v) = \alpha_v(\theta_v(f_v\otimes\varphi_v),\chi_v).$$

We are now ready to prove Waldspurger's formula. In this case we have $\#\Sigma$ even, $\mathbb{B} = B/F$ is a global quaternion algebra, $\chi\mid_{F_\mathbb{A}^\times}\omega_\pi = 1$. Define

$$\ell(f',\chi) = \int_{F_\mathbb{A}^\times E^\times\backslash E_\mathbb{A}^\times} f'(t)\chi(t)dt \qquad \ell(\widetilde{f}',\chi^{-1}) = \int_{F_\mathbb{A}^\times E^\times\backslash E_\mathbb{A}^\times} \widetilde{f}'\chi^{-1}(t)dt.$$

Theorem 31 (Waldspurger,YZZ)**.**

$$\ell(f',\chi)\ell(\widetilde{f}',\chi^{-1}) = \frac{\zeta_F(2)L(\frac{1}{2},\pi,\chi)}{2L(1,\pi,\mathrm{ad})}\prod_v \alpha_v(f'\otimes\widetilde{f}',\chi)$$

Proof. Step 1: Global theta lifting (Shimizu lifting). This is a surjective map $\theta : \pi \otimes S(V_A \times F_{\mathbb{A}}^{\times}) \to \pi' \otimes \widetilde{\pi}'$ given by

$$\theta(f, \varphi) = * \int_{\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \theta(g, h, \varphi) dg$$

where $h \in B_{\mathbb{A}}^{\times} \times B_{\mathbb{A}}^{\times}$.

Suppose that $\theta(f \otimes \varphi) = f' \otimes \widetilde{f}'$. Then

$$\ell(f', \chi)\ell(\widetilde{f}', \chi^{-1}) = \int_{(F_{\mathbb{A}}^{\times} E^{\times} \backslash E_{\mathbb{A}}^{\times})^2} \underbrace{\theta(f \otimes \varphi)(t_1, t_2)}_{\int_{[\mathrm{GL}_2]} \theta(g, t_1, t_2, \varphi) f(g) dg} \chi(t_1 t_2^{-1}) dt_1 dt_2.$$

Step 2: Seigel-Weil formula. Interchange the order of integration and make the change of variable $t_1 = tt_2$ to get

$$\int_{[\mathrm{GL}_2]} f(g) \int_{F_{\mathbb{A}}^{\times} E^{\times} \backslash E_{\mathbb{A}}^{\times}} \chi(t) \int_{F_{\mathbb{A}}^{\times} E^{\times} \backslash E_{\mathbb{A}}^{\times}} \theta(g, tt_2, t_2, \varphi) dt_2 dt dg$$

Write $V = E \oplus Ej = V_1 \oplus V_2$. Since $SO(V_2) = F^{\times} \backslash E^{\times}$, the Seigel-Weil formula allows us to replace the inner integral:

$$\int_{[\mathrm{GL}_2]} f(g) \int_{F_{\mathbb{A}}^{\times} E^{\times} \backslash E_{\mathbb{A}}^{\times}} \chi(t) I(0, g, w(t, 1)\varphi) dt dg$$

$$= \int_{[\mathrm{GL}_2]} f(g) I(0, g, \chi, \varphi) dg = \mathscr{P}(0, \chi, f, \varphi).$$

This implies that

$$ell(f', \chi)\ell(\widetilde{f}', \chi^{-1}) = C\mathscr{P}(0, \chi, f, \varphi).$$

Step 3: Unfolding and Calculation. By (2) and (3) this is equal to

$$CC' L(\frac{1}{2}, \pi, \chi) \prod_v \mathscr{P}_v^0(0, \chi_v, f_v \otimes \varphi_v)$$

Step 4: Local Theta Lifting and global to local principal. Lemma 30 now says that this is

$$CC' L(\frac{1}{2}, \pi, \chi) \prod_v \alpha_v(\theta_v(f_v \otimes \varphi_v), \chi_v).$$

The proof is complete by noting that if $f \otimes \varphi = \bigotimes f_v \otimes \varphi_v$ then □

15. Friday, October 23, 2009

From now on we will assume that $\#\Sigma$ is odd. Since the functional equation is

$$\Lambda(s, \pi, \chi) = (-1)^{\#\Sigma} \Lambda(1 - s, \pi, \chi),$$

we must have that $L(\frac{1}{2}, \pi, \chi) = 0$. So we're interested in understanding $L'(\frac{1}{2}, \pi, \chi)$.

15.1. **Kernel function.** The setup: $\mathbb{V} = \mathbb{B}$, $Q(x) = x\overline{x}$, $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$, $\mathbb{V}_1 = E_{\mathbb{A}}$ and $V_1 = E$, $\mathbb{V}_2 = E_{\mathbb{A}} j$ with $j^2 \in \mathbb{A}$ (but j is not defined over F.)

The functions:

$$I(g, s, \varphi) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^2 \sum_{(x,u)\in V_1 \times F^\times} w(\gamma g)\varphi(x, u),$$

$$I(g, s, \varphi, \chi) = \int_{E^\times \backslash E_{\mathbb{A}}^\times} I(g, s, w(t, 1)\varphi)\chi(t)dt.$$

For $f \in \pi$ we defined

$$\mathscr{P}(s, \chi, f, \varphi) = \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} I(g, s, \chi, \varphi)f(g)dg$$

which is a functional $\mathscr{P} : \pi \otimes S(\mathbb{V}_1 \times F_{\mathbb{A}}^\times) \to \mathbb{C}$.

The theorem: (Waldspurger)

$$\begin{aligned}\mathscr{P}(s, \chi, f, \varphi) =& \prod_v \mathscr{P}_v(s, \chi_v, f_v, \varphi_v)\\ =& \frac{L((s + 1/2, \pi, \chi)}{L(s, \eta)} \prod_v \mathscr{P}_v^0(s, \chi_v, f_v, \varphi_v)\end{aligned}$$

where

$$\begin{aligned}\mathscr{P}_v^0(s, \chi_v, f_v, \varphi_v) =& \frac{L_v(s, \eta_v)}{L_v((s + 1/2, \pi_v, \chi_v)}\\ &\times \int_{F_v^\times \backslash E_v^\times} \chi(t) \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} \delta(g)^s W_{-1}^{f_v}(g)w(g)w(t, 1)\varphi(1, 1)dgdt\end{aligned}$$

which is equal to 1 for all but finitely many v.

Two corollaries: We defined $\alpha_v(\chi_v, f_v, \varphi) = \mathscr{P}_v^0(0, \chi_v, f_v\varphi_v)$. Then

- If $\#\Sigma$ is even $\mathscr{P}(0, \chi, f, \varphi) = \frac{L(1/2,\pi,\chi)}{L(1,\eta)} \prod_v \alpha_v(0, \chi_v, f_v\varphi_v)$.
- If $\#\Sigma$ is odd $\mathscr{P}'(0, \chi, f, \varphi) = \frac{L'(1/2,\pi,\chi)}{2L(1,\eta)} \prod_v \alpha_v(0, \chi_v, f_v\varphi_v)$.

From here on out, we are interested in understanding $\mathscr{P}'(0, \chi, f, \varphi)$. The goal will be to relate it to a height pairing of CM points on a (global) Shimura curve X associated to $\mathbb{B}$.

15.2. **Incoherent Eisenstein series.** The definitive reference for this material is Kudla[3].

When we write $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$ we find that

$$S(\mathbb{V} \times F_{\mathbb{A}}^\times) = S(\mathbb{V}_1 \times F_{\mathbb{A}}^\times) \otimes S(\mathbb{V}_2 \times F_{\mathbb{A}}^\times).$$

The map (from right to left) is given by

$$\varphi_1 \otimes \varphi_2 \mapsto (\varphi \times \varphi_2) : (x, u) \mapsto \varphi_1(x, u)\varphi_2(x, u).$$

We denote the Weil actions on each part of the right hand side by w_1 and w_2 respectively. So

$$w_i : \mathrm{GL}_2(\mathbb{A}) \times GO(\mathbb{V}_i) \to \mathrm{Aut}(S(\mathbb{V}_i \times F_{\mathbb{A}}^\times)).$$

Note that $GO(\mathbb{V}_2) = E_{\mathbb{A}}^{\times} j$ which acts by $r \cdot x = xr^{-1}$, and $GO(\mathbb{V}_1) = E_A^{\times}$ acts by $r \cdot x = rx$. These actions come via the action of $\mathbb{B}^{\times} \times \mathbb{B}^{\times} \hookrightarrow GO(\mathbb{V})$ given by $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$. So

$$w((g, t_1 \otimes t_2))(\varphi_1 \otimes \varphi_2) = w_1(g, t_1)\varphi_1(x, u) \otimes w_2(g, t_2)\varphi_2(x, u). \tag{5}$$

Lemma 32. *The function* $I(g, s, \varphi_1 \otimes \varphi_2) = \sum_{u \in F^{\times}} \theta(g, u, \varphi_1) E(g, u, s, \varphi_2)$ *where*

$$\theta(g, u, \varphi) = \sum_{x \in V_1} w_1(g)\varphi_1(x, u), \qquad E(g, u, s, \varphi_2) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s w_2(\gamma g)\varphi_2(0, u).$$

This justifies the terminology "theta-Eisenstein series" used above, It is a theta series in φ_1 and since $\delta(\gamma g)^s w_2(\gamma g)\varphi_2(0, u) \in I(s + s_0, \eta) = \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})} \eta$ it is an Eisenstein series in φ_2.

We will give the proof of this later. (Note that actually in the proof we need the sum to be over $P^1(F)\backslash \mathrm{SL}_2(F)$ which is in bijection with $P(F)\backslash \mathrm{GL}_2(F)$ but for which the term $E(g, u, s, \varphi_2)$ is actually well-defined.)

Theorem 33. $E(g, u, o, \varphi_2) = 0$ *for all* $\varphi \in S(\mathbb{V}_2 \times F_{\mathbb{A}}^{\times})$.

The main reason for this is that $\mathbb{V}_2$ is incoherent.

Recall the Fourier expansion for $\Phi(g, u, s) \in I(s, \eta)$ ($s_0 = 0$ in the present case)

$$\begin{aligned} E(g, u, s, \Phi) = &\sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \Phi(\gamma g, u, s) \\ =& E_0(g, u, s, \Phi) + \sum_{a \in F^{\times}} E_a(g, u, s, \Phi) \end{aligned}$$

where the a-th coefficient is

$$E_a(g, u, s, \Phi) = \int_{F\backslash \mathbb{A}} E(n(b)g, u, s, \Phi)\psi(-ab)db.$$

Lemma 34. *If* $a \neq 0$,

$$E_a(g, u, s, \Phi) = \prod_v W_{a,v}(g, u, s, \Phi)$$

and

$$E_0(g, u, s, \Phi) = \Phi(g, s) + \prod_v W_{0,v}(g, u, s, \Phi)$$

where

$$W_{a,v}(g, u, s, \Phi) = \int_{F_v} \Phi_v(wn(b)g, s)\psi(-ab)db.$$

For our special case $\Phi = \delta(\gamma g)^s w_2(\gamma g)\varphi(0, u)$, if $a \neq 0$ then

$$E_a(g, u, s, \varphi) = -\prod_v \widetilde{W}_{a,v}(g, u, s, \varphi_v),$$

and

$$E_a(g, u, s, \varphi) = \delta(g)^s w_2(g)\varphi(0) - \prod_v \widetilde{W}_{0,v}(g, u, s, \varphi_v)$$

where

$$\widetilde{W}_{a,v}(g, u, s, \varphi_v) = \int_{F_v} \delta(wn(b)g)^s \int_{\mathbb{V}_{2,v}} w_2(g)\varphi(x_2, u)\psi(buQ(x_2))dx_2 dg.$$

Proof. By definition

$$\begin{aligned}W_{a,v} =& \int_{F_v} \Phi_v(wn(b)g, s)\psi(-ab)db \\ =&\psi(-ab)\delta(wn(b)g)^s w_2(wn(b)g)\varphi_2(0,u)db \\ =&\psi(-ab)\delta(wn(b)g)^s \gamma(\mathbb{V}_{2,v},\psi_v)\int_{\mathbb{V}_{2,v}} \underbrace{w_2(wn(b)g)\varphi_2(x_2,u)}_{\varphi_2(bu,Q(x_2)} dx_2 db.\end{aligned}$$

The proof follows provided that

$$\prod \gamma(\mathbb{V}_{2,v},\psi_v) = \begin{cases} 1 & \text{if } \mathbb{V}_2 \text{ is coherent,} \\ -1 & \text{if } \mathbb{V}_2 \text{ is incoherent,}\end{cases}$$

We take this as given. □

16. Monday, October 26, 2009

The goal is to show that $I(g,0,\varphi) = 0$ and to compute its derivative because

$$\begin{aligned}\mathscr{P}(s,\chi,f,\varphi) =& \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g)\int_{E^\times\backslash E^\times_{\mathbb{A}}} \chi(t) I(g,s,w(t,1)\varphi)dtdg \\ =&\frac{L(\frac{s+1}{2},\pi,\chi)}{L(s+1,\eta)}\prod \mathscr{P}^0_v(s,\chi,f_v,\varphi_v).\end{aligned}$$

which implies that

$$\mathscr{P}'(0,\chi,f,\varphi) = \frac{L'(\frac{1}{2},\pi,\chi)}{2L(1,\eta)}\prod \alpha_v(\chi,f_v,\varphi_v).$$

Reall that $I(g,0,\varphi)$ is a product of a theta series (which is independant of s) and an Eisenstein series. Today we want to see why the Eisenstein series $E(g,u,0,\varphi_2)$ is automatically zero. This is an example of an "incoherent Eiesenstein series" and "incoherent quadratic space."

The space $\mathbb{V}_2 = E_{\mathbb{A}} j$ for $j \in \mathbb{B}^\times$ and $j^2 = -\alpha \in F^\times_{\mathbb{A}} \setminus F^\times N_{E/F} E^\times_{\mathbb{A}}$ is isomorphic to $E_{\mathbb{A}}$ via $xj \mapsto x$. With this isomorphism in mind we define

$$Q(x) = Q(xj) = xj\bar{j}\bar{x} = \alpha x\bar{x}.$$

Claim: $\eta(\alpha) = \prod_v \eta(\alpha_v) = -1$. The reason for this is class field theory and the fact that

$$\eta : F^\times\backslash F^\times_{\mathbb{A}}/N_{E/F}E^\times_{\mathbb{A}} \to \{\pm 1\}.$$

Lemma 35. *If $a \in F^\times$ then*

$$E_a(g,u,s,\varphi_2) = -W_a(g,u,s,\varphi_2) := -\prod W_{a,v}(g,u,s,\varphi_2)$$

where $W_{a,v}(g,u,s,\varphi_2) = \int_{F_v} \delta(wn(b))^s \int_{\mathbb{V}_{2,v}} w_2(g)\varphi_2(x,u)\psi(bu(Q(x)-a))dadb.$

Lemma 36. *Assume $a \in F^\times$.*

(a) *If $\mathbb{V}_2$ does not represent au^{-1} then $W_{a,v}(g,u,0,\varphi_2) = 0$.*
(b) *If there exists $\xi \in \mathbb{V}_2$ with $Q(\xi) = au^{-1}$ then*

$$W_{a,v}(g,u,0,\varphi_2) = L(1,\eta_v)^{-1}\int_{E^1_v} w_2(g)\varphi_2(\xi x,u)dx.$$

For $a \in F^\times$ define

$$\begin{aligned}\mathrm{Diff}(\mathbb{V}_2, a) =& \{v \mid (\mathbb{V}_{2,v}, Q) \text{ does not represent } a\} \\ =& \{v \mid \eta_v(\alpha_v a) = -1\}\end{aligned}$$

To see why, recall $\mathbb{V}_{2,v} \simeq E_v$ and $Q(x) = x\bar{x}$. So $(\mathbb{V}_{2,v}, Q)$ represents a if and only if there exists $x \in E_v$ such that $\alpha_v x\bar{x} = a$. This happens precisely when $x\bar{x} = a\alpha_v^{-1} \in N_{E_v/F_v} E_v^\times$, which is equivalent to $\eta_v(a\alpha_v^{-1}) = -1$.

Since $\eta(\alpha) = -1$, it follows that $\#\,\mathrm{Diff}(\mathbb{V}_2, a)$ is odd. In particular, it is at least 1.

Corollary 37. *If $v \in \mathrm{Diff}(\mathbb{V}_2, a)$ then $W_{a,v}(g, u, 0, \varphi_2) = 0$. Moreover $E(g, u, 0, \varphi_2) = 0$.*

Proof. The first statement follows from the above. To prove the second statement, recall

$$E(g, u, 0, \varphi_2) = \sum_{a \in F} E_a$$

and $E_a = -\prod W_{a,v}(g, u, 0, \varphi_2)$ if $a \in F^\times$. Since $\#\,\mathrm{Diff}(\mathbb{V}_2, a)$ is odd, some $W_{a,v} = 0$.

So we only to check the constant term. There are two ways to do this. One way is to compute directly. Alternatively, write $\mathbb{V}_{2,\infty} = \otimes \mathbb{V}_{2,\sigma_i}$ where $\sigma_i : F \to \mathbb{R}$ are distinct embeddings and $\mathbb{V}_{2,\sigma_i} = E \otimes_{F,\sigma_i} \mathbb{R} = \mathbb{C}$. The signature of $\mathbb{V}_{2,\sigma_i}$ is either $(2,0)$ or $(0,2)$ which means that the weight is either 1 or -1. Since E is a modular form of nonzero weight it follows that the constant term must be zero. □

Proof of Lemma 36. To prove (a), we compute (ignoring issues of convergence)

$$\begin{aligned}W_{a,v}(g, u, 0, \varphi_2) &= \int_{F_v} \int_{\mathbb{V}_2} w_2(g)\varphi(x,y)\psi(bu(Q(x) - au^{-1}))dxdb \\ &= \lim_{m \to -\infty} \int_{\varpi^m \mathcal{O}_v} \int_{\mathbb{V}_2} w_2(g)\varphi(x,y)\psi(bu(Q(x) - au^{-1}))dxdb \\ &= \lim_{m \to -\infty} \int_{\mathbb{V}_2} w_2(g)\varphi(x,y) \int_{\varpi^m \mathcal{O}_v} \psi(bu(Q(x) - au^{-1}))dbdx\end{aligned}$$

As a consequence of the fact that $\mathcal{O}_v$ is compact,

$$\int_{\mathcal{O}_v} \psi_v(bx)db = \mathrm{vol}(\mathcal{O}_v)\,\mathrm{char}(\varpi_v^n \mathcal{O}_v)(x)$$

where $n = \min\{m \mid \psi_v \mid_{\varpi_v^m \mathcal{O}_v} = 1\}$. Using this, we have

$$\begin{aligned}\int_{\varpi^m \mathcal{O}_v} \psi(bu(Q(x) - au^{-1}))db &= \int_{\mathcal{O}_v} |\varpi_v|^m\, \psi(\varpi^m b(Q(x) - au^{-1}))db \\ &= |\varpi_v|^m\, \mathrm{vol}(\mathcal{O}_v)\,\mathrm{char}(\varpi_v^{n(\psi_v)} \mathcal{O}_v)(u\varpi_v^m(Q(x) - au^{-1}))\end{aligned}$$

This will be zero if and only if $Q(x) - au^{-1} \notin u^{-1}\varpi_v^{n(\psi_v)-m}\mathcal{O}_v$. Since $Q(x) - au^{-1}$ is never zero and we are taking the limit $m \to -\infty$ this must be the case in the limit. □

17. Wednesday, October 28, 2009

We have E/F a quadratic extension with associated character η, π an aut. representation of $\mathrm{GL}_2(F)$ and and χ a character on E. This gives Σ, $\mathbb{B} = \mathbb{V}$ and π'. We are assuming that Σ is odd.

Write

$$\mathbb{V} = E_{\mathbb{A}} + E_{\mathbb{A}} j = V_{1,\mathbb{A}} + \mathbb{V}_2.$$

Recall that

$$S(\mathbb{V} \times F_{\mathbb{A}}^{\times}) = S(V_{1,\mathbb{A}} \times F_{\mathbb{A}}^{\times}) \otimes S(\mathbb{V}_2 \times F_{\mathbb{A}}^{\times}).$$

17.1. Loose ends from before.

Lemma 38. *If* $\varphi = \varphi_1 \otimes \varphi_2 \in S(V_{1,\mathbb{A}} \times F_{\mathbb{A}}^{\times}) \otimes S(\mathbb{V}_2 \times F_{\mathbb{A}}^{\times})$*, then*

$$I(g, s, \varphi) = \sum_{u \int F^{\times}} \theta(g, u, \varphi_1) E(g, u, s, \varphi_2)$$

where

$$\theta(g, u, \varphi_1) = \sum_{x \in V_1} w(g)\varphi_1(x, u),$$

$$E(g, u, \varphi_2) = \sum_{\gamma \in P^1(F)\backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s w(\gamma g)\varphi_2(0, u).$$

Note that in [7] they have the summation of the Einsenstein series over $P(F)\backslash \mathrm{GL}_2(F)$. Although this is in bijection with $P^1(F)\backslash \mathrm{SL}_2(F)$, the summand is not invariant by $P(F)$ and so their definition is not valid.

Proof. By definition,

$$\begin{aligned} I(g, s, \varphi) &= \sum_{\gamma \in P^1(F)\backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s \sum_{(x,u) \in V_1 \times F} w_1(\gamma g)\varphi_1(x, u) w_2(\gamma g)\varphi_2(0, u) \\ &= \sum_{u \in F^{\times}} \underbrace{\sum_{\gamma \in P^1(F)\backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s w(\gamma g)\varphi_2(0, u)}_{E(g,u,s,\varphi_1)} \underbrace{\sum_{x \in V_1} w_2(\gamma g)\varphi_1(x, u)}_{T}. \end{aligned}$$

We need to show that T is not dependent on γ.

Recall that for $\varphi \in S(V_{1,\mathbb{A}})$, the theta kernel

$$\theta(g, h, \varphi) = \sum_{x \in V_1} w(g)\varphi(h^{-1}x)$$

where $g \in \mathrm{SL}_2(\mathbb{A})$ and $h \in O(V)(\mathbb{A})$ is an automorphic form on $[\mathrm{SL}_2] \times [O(V)]$. Hence $\theta(\gamma g, h, \varphi) = \theta(g, h, \varphi)$ for any $\gamma \in \mathrm{SL}_2(F)$.

In order to extend the Weil representation to $\mathrm{GL}_2 \times GO(V)$ we defined

$$w(g)\varphi(x, u) = w^u(g)\varphi_u(x)$$

for $g \in \mathrm{SL}_2(\mathbb{A})$. The function $\varphi_u(x) \in S(V_{1,\mathbb{A}})$ is equal to $\varphi(x, u)$ and w^u is the weil representation for $V_1^u = (V_1, uQ)$.

Therefore, $T = \theta(\gamma g, 1, \varphi^u) = \theta(g, 1, \varphi^u)$. ∎

17.2. **New stuff.** We have

$$\mathbb{V}_2 = E_{\mathbb{A}} j \qquad Q(xj) = -j^2 x\bar{x} = \alpha x\bar{x}$$

where $\alpha \in F_A^\times$ and $\eta(\alpha) = -1$.

We know that $E(g, u, 0, \varphi_2) = 0$ and we want to understand $E'(g, u, s, \varphi_2)\mid_{s=0}$. If $a \neq 0$ we have the Fourier coefficient

$$E_a(g, u, s, \varphi_2) = -\prod W_{a,v}(g, u, s, \varphi_2)$$

and

$$W_{a,v} = \int_{F_v} \delta(wn(b)g)^s \int_{\mathbb{V}_{2,v}} w_2(g)\varphi_2(x, u)\psi(b(uQ(x) - a))dxdb,$$

and we proved that $W_{a,v}(g, u, 0, \varphi_2) = 0$ if $v \in \mathrm{Diff}\,\{v \mid \mathbb{V}_{2,v} \text{ doesn't represent } a\}$.

Corollary 39. *If* $\#\,\mathrm{Diff}(\mathbb{V}_2, a) > 1$ *then* $E'_a(g, u, 0, \varphi) = 0$. *If* $\mathrm{Diff}(\mathbb{V}_2, a) = \{v\}$ *then*

$$E'_a(g, u, 0, \varphi) = -W'_{a,v}(g, u, 0, \varphi_2) \prod_{v' \neq v} W_{a,v'}(g, u, o, \varphi_2) = -W'_{a,v}(g, u, 0, \varphi_2) \prod_{v' \neq v} W_a^{(v)}(g, u, o, \varphi_2^{(v)}).$$

Corollary 40. *For each* v *let* $F(v) = \{a \in F^\times \mid \mathrm{Diff}(\mathbb{V}_2, au^{-1}) = \{v\}$. *Then*

$$E'(g, u, 0, \varphi_2) = E'_0(g, u, 0, \varphi_2) + \sum_{v \text{ nonsplit}} E'(g, u, 0, \varphi_2)(v)$$

where

$$E'(g, u, 0, \varphi_2)(v) = \sum_{a \in F(v)} E'_a(g, u, 0, \varphi_2).$$

We denote the *neight quaternion algebra of* $\mathbb{B}$ *at* v by $B^{(v)}$. This is the quaternion algebra over F such that $B^{(v)}_{v'} = \mathbb{B}_{v'}$ if $v' \neq v$ and $B^{(v)}_v$ satisfies $Hasse(B^{(v)}_v) = -Hasse(\mathbb{B}_v)$. We write $V^{(v)} = B^{(v)} = E + Ej(v) = V_1 + V_2^{(v)}$. Then

$$\begin{aligned} F(v) = &\left\{ a \in F^\times \;\middle|\; \begin{array}{c} \mathbb{V}_{2,v} \text{ doesn't represent } au^{-1} \\ \mathbb{V}_{2,v'} \text{ does if } v' \neq v \end{array} \right\} \\ =&\{a \mid F^\times \mid V_2^{(v)} \text{ represents } au^{-1} \text{ everwhere locally}\} \\ =&\{a \mid F^\times \mid V_2^{(v)} \text{ represents } au^{-1} \text{ globally}\} \end{aligned}$$

The final equality is the Hasse principle.

A quadratic space over F_v is determined by

$$\dim V = m, \quad \chi_V = ((-1)^{m(m-1)/2} \det V, \cdot)_V, \quad \text{and} \quad Hasse(V).$$

So if we fix χ_V and m then $Hasse(V)$ determines V. Take $V = E_v$ and $Q(x) = \alpha x\bar{x}$ for $\alpha \in F_v^\times$.

For us, $V_\alpha = E_v = F_v + F_v\sqrt{\Delta}$ and $Q(x + y\sqrt{\Delta}) = \alpha x^2 - \alpha\Delta y^2$. The Hasse invariant $Hasse(V_\alpha)$ is equal to the Hilbert symbol

$$(\alpha, -\alpha\Delta) = (\alpha, \Delta)(\alpha, -\alpha) = (\alpha, \Delta) = \eta_v(\alpha).$$

V_α represents a means there exists $x \in E_v^\times$ such that $\alpha x\bar{x} = a$. This is so if and only if $\eta_v(a\alpha) = 1$.

We have two cases. First suppose that $E_v = F_v \times F_v$, $\eta_v = 1$. Then there is only one quadratic space V_α because $\alpha \in N_{E_v/F_v} E_v^\times$. So in this case every $a \in F_v^\times$ is represented. From this we see that $v \in \mathrm{Diff}(\mathbb{V}_2, a)$ implies that v is nonsplit.

Now assume that E_v/F_v is nonsplit. Choose $v \in F_v^\times \setminus N(E_v^\times)$. Chose $\alpha \in F_v^\times \setminus NE_v^\times$. Thus $V_1 \not\simeq V_\alpha$. Then V_1 represents a if and only if $a \in NE_v^\times$ and V_α represents a if and only if $a \notin NE_v^\times$.

18. Friday, October 30, 2009

We want to understand $W'_{a,v}(g, u, 0, \varphi_2)$. We saw that

$$E'(g, u, 0, \varphi_2) = E'_0(g, u, 0, \varphi_2) + \sum_{v \text{ nonsplit}} E'(g, u, 0, \varphi_2)(v),$$

$$\begin{aligned} E'(g, u, 0, \varphi_2)(v) =& W'_{a,v}(g, u, 0, \varphi_2) W_a^{(v)}(g, u, 0, \varphi_2^{(v)}) \\ =& \sum_{y_2 \in E^1 \backslash (V_2^{(v)} \backslash \{0\})} W'_{Q(y_2)u,v}(g, u, 0, \varphi_{2,v}) W^{(v)}_{Q(y_2)u}(g, u, 0, \varphi_2^{(v)}). \end{aligned}$$

Lemma 41 (Basic Lemma $v \nmid \infty$)**.** *Assume $F/\mathbb{Q}$ is unramified at v. Let ψ_v be the unramified additive character ($\psi_v = \psi_a \circ \mathrm{tr}_{E/\mathbb{Q}}$.) Assume E/F is unramified at v. Then $\mathbb{V}_{2,v} \simeq E_v$, $Q(x) = \alpha x \bar{x}$. assume that $\alpha \in \mathcal{O}_{F_v}$. Take $\varphi_{2,v} = \mathrm{char}(\mathcal{O}_{E_v} \times \mathcal{O}_{F_v}^\times)$. Then*

$$E(1, u, s, \varphi_{2,v}) = L(1, \eta_v)^{-1} \sum_{n=0}^{\mathrm{ord}\, a} (\eta_v(\varpi_v) q_v^{-s})^n \,\mathrm{char}(\mathcal{O}_{F_v}^\times)(u) \,\mathrm{char}(\mathcal{O}_{F_v}(a).$$

In particular $E_a(1, u, 0, \varphi_{2,v}) = 0$ unless $\varphi_{2,v}(a, u) \neq 0$.

In the case that $E_a(1, u, 0, \varphi_{2,v}) \neq 0$,

$$\begin{aligned} W_{a,v}(1, u, 0, \varphi_{2,v}) =& L(1, \eta_v)^{-1} \begin{cases} \frac{1+(-1)^{\mathrm{ord}\, a}}{2} & \text{if } E_v/F_v \text{ is unramified} \\ 1 + \mathrm{ord}\, a & \text{if } E_v \simeq F_v \times F_v. \end{cases} \\ =& 0 \iff \eta(\varpi_v) = -1 \text{ and } \mathrm{ord}\, a \text{ is odd} \\ & \iff \mathbb{V}_{2,v} \text{ does not represent } a \end{aligned}$$

Moreover, when this happens,

$$W'_{a,v}(1, u, 0, \varphi_{2,v}) = \frac{1 + \mathrm{ord}\, a}{2} \log q_v.$$

Recall that

$$\begin{aligned} W_{a,v}(1, u, s, \varphi_2) &= \int_{F_v} \delta(wn(b))^s \int_{\mathcal{O}_{E_v}} \psi(buQ(x))\psi(-ba)dxdb \\ &= \int_{F_v} \delta(wn(b))^s f(b)\psi(-ba)db \end{aligned}$$

where $f(b) = \int_{\mathcal{O}_{E_v}} \psi(buQ(x))dx$.

Lemma 42. *$f(b) = 1$ if $b \in \mathcal{O}_{E_v}$ and $f(b) = \eta_v(b) |b|^{-1}$ otherwise.*

We will now drop the v in our notation.

Proof of Basic Lemma. First, suppose that $E_v = F_v \times F_v$, and write $x \in E_v$ as $x = (x_1, x_2)$ so that $x\bar{x} = x_1 x_2$. Then

$$\begin{aligned} f(b) &= \int_{\mathcal{O}_{F_v}^2} \psi(bu\alpha x_1 x_2) dx_1 dx_2 \\ &= \int_{\mathcal{O}_{F_v}} \mathrm{char}(\mathcal{O}_{F_v})(bu\alpha x_2) dx_2 \\ &= \begin{cases} \int_{\mathcal{O}_{F_v}} dx_2 = 1 & \text{if } b \in \mathcal{O}_{F_v} \\ \int_{b^{-1}\mathcal{O}_{F_v}} dx_2 = |b|^{-1} & \text{if } b \notin \mathcal{O}_{F_v} \end{cases} \end{aligned}$$

Since $\eta_v(\varpi) = 1$ in this case, we are done.

Now assume that $E_v = F_v + \sqrt{\Delta} F_v$. Note that $\eta_v(b) = \eta_v(\varpi)^{\mathrm{ord}\, b} = (-1)^{\mathrm{ord}\, b}$. The claim is easy if $b \in \mathcal{O}_{F_v}$ so we assume that $b \in \varpi^{-n}\mathcal{O}_{F_v}$ for some $n > 0$. Then

$$\begin{aligned} f(b) &= \sum_{k=0}^{\infty} \int_{\varpi^k \mathcal{O}_{E_v}^\times} \psi(bu\alpha x\bar{x}) dx \\ &= \sum_{k=0}^{\infty} q_v^{-2k} \int_{\mathcal{O}_{E_v}^\times} \psi(bu\alpha\varpi^{2k} x\bar{x}) dx. \end{aligned}$$

We calculate

$$\begin{aligned} \int_{\mathcal{O}_{E_v}^\times} \psi(bu\alpha\varpi^{2k} x\bar{x}) dx &= \int_{\mathcal{O}_{E_v}^\times} \psi(bu\alpha\varpi^{2k} x\bar{x}) d^\times x \\ &= \mathrm{vol}(E_v^1, d^\times x) \int_{\mathcal{O}_{F_v}^\times} \psi(b\varpi^{2k} t) d^\times t \\ &= \mathrm{vol}(E_v^1, d^\times x) \left(\int_{\mathcal{O}_{F_v}} \psi(b\varpi^{2k} t) dt - \int_{\varpi\mathcal{O}_{F_v}} \psi(b\varpi^{2k} t) dt \right) \\ &= \mathrm{vol}(E_v^1, d^\times x)(\mathrm{char}(\mathcal{O}_{F_V})(b\varpi^{2k}) - q_v^{-1} \mathrm{char}(\mathcal{O}_{F_V})(b\varpi^{2k+1})) \\ &= \mathrm{vol}(E_v^1, d^\times x) \begin{cases} 1 - q_v^{-1} & \text{if } b \in \varpi^{-2k}\mathcal{O}_v \\ -q_v^{-1} & \text{if } b \in \varpi^{-2k-1}\mathcal{O}_{F_v} \setminus \varpi^{-2k}\mathcal{O}_{F_v} \end{cases} \end{aligned}$$

We claim that $\mathrm{vol}(E_v^1, d^\times x) = 1 - q_v^{-1}$. To see this, note that

$$1 - q_v^{-2} = \int_{\mathcal{O}_{E_v}^\times} d^\times x = \int_{\mathcal{O}_{F_v}} \left(\int_{E_v^1} dh \right) d^\times t = \mathrm{vol}(E_v^1, d^\times x) \int_{\mathcal{O}_{F_v}^\times} d^\times t.$$

Since $\int_{\mathcal{O}_{F_v}^\times} d^\times t = 1 - q_v^{-1}$, the claim follows.

Putting all of this together, we get

$$\begin{aligned} f(b) &= \sum_{k=0}^{\infty} q_v^{-2k} \begin{cases} 1 - q_v^{-2} & \text{if } m \le k \\ -q^{-1}(1 + q_v^{-1}) & \text{if } k = m - 1 \end{cases} \\ &= -\, q^{-1}(1 + q^{-1}) q^{-2(m-1)} + \underbrace{\sum_{k=m}^{\infty} q^{-2k}(1 - q^{-1})}_{q^{-2m}} \\ &= -\, q^{-2m+1} = \eta_v(b)\, |b|^{-1} \end{aligned}$$

because $\eta_v(b) = -1$ and $|b| = q^{1-2m}$.

We have

$$\delta(wn(b)) = \begin{cases} 1 & \text{if } b \in \mathcal{O}_{F_v} \iff wn(b) \in \mathrm{GL}_2(\mathcal{O}_{F_v}) \\ |b|^{-1} & \text{if } b \notin \mathcal{O}_{F_v} \iff wn(b) = \begin{pmatrix} b^{-1} & -1 \\ & b \end{pmatrix} \begin{pmatrix} 1 & \\ b^{-1} & 1 \end{pmatrix} \end{cases}$$

It is now straightforward to derive the formulas. □

Lemma 43 (Basic Lemma $v \mid \infty$)**.** *Assume $F_v = \mathbb{R}$, $E_v = \mathbb{C}$. So $\mathbb{V}_{2,v} \simeq \mathbb{C}$ and $Q(x) = \alpha x\bar{x}$. Assume that $\alpha > 0$, and $\varphi_{2,v} = e^{-2\pi|uQ(x)|}$. Then $W_a(g_\tau, u, 0, \varphi_{2,v}) = 0$ if and only if $au < 0$, and*

$$W'_{a,v}(g_\tau, u, 0, \varphi_{2,v}) = Ei(4\pi\,|ua|\,v_0)e^{2\pi u\alpha\tau}$$

where $\tau = u_0 + iv_0$ and $Ei(g) = \int_1^\infty e^{-tr}\frac{dr}{r}$.

19. Monday, November 2, 2009

Suppose that $v \mid \infty$. Recall that F is totally real. Assume that $\mathbb{B}_v$ is a division algebra, $E_v = \mathbb{C}$. Then

$$\mathbb{B}_v = E_v + E_v j_v, \qquad j_v^2 \in \mathbb{R} \text{ and } j_v^2 < 0.$$

In other words, $\mathbb{V}_v$ has form Q which is positive definite.

In slightly more generality, we assume that (V, Q) is a positive definite quadratic space of dimension $2m$. Let

$$\varphi \in S(V \times \mathbb{R}^\times) = \{H(u)P(x)e^{-2\pi|u|Q(x)} \mid P \text{ is a polynomial on } V, H \in C_c^\infty(\mathbb{R}^\times)\},$$

$$S^0(V \times \mathbb{R}^\times) = \{[P_1(uQ(x)) + \mathrm{sgn}(u)P_2(uQ(x))]e^{-2\pi|u|Q(x)} \mid P_1, P_2 \in \mathbb{R}[x]\}.$$

Lemma 44. *Let*

$$S(V \times \mathbb{R}^\times)^{O(V)(\mathbb{R})} = \{\varphi \in S(V \times \mathbb{R}^\times) \mid w(h)\varphi = \varphi \text{ for all } h \in O(V)(\mathbb{R})\}.$$

Then the map $S(V \times \mathbb{R}^\times)^{O(V)(\mathbb{R})} \to S^0(V \times \mathbb{R}^\times)$ given by

$$\widetilde{\varphi} \mapsto \varphi(x, u) = \int_{\mathbb{R}^\times} \widetilde{\varphi}(z^{-1}x, z^2u)d^\times z$$

is surjective.

Proof. Let $\widetilde{\varphi} = H(u)P(x)e^{-2\pi|u|Q(x)}$ be $O(V)(\mathbb{R})$ invariant. Then if $h \in SO(V)(\mathbb{R})$ then $H(u)P(h^{-1}x) = H(u)P(x)$ implies that $P = \widetilde{P}(Q(x))$ for some polynomial $\widetilde{P}$. So we may assume $\widetilde{\varphi} = H(u)P(Q(x))e^{-2\pi|u|Q(x)}$. Then

$$\varphi(x, u) = \int_{\mathbb{R}^\times} \widetilde{\varphi}(z^{-1}x, z^2u)\frac{dz}{|z|} = \int_{\mathbb{R}^\times} H(z^2u)P(z^{-2}Q(x))e^{-2\pi|u|Q(x)}\frac{dz}{|z|}$$

$$= 2e^{-2\pi|u|Q(x)}\int_0^\infty H(z^2u)P(z^{-2}Q(x))\frac{dz}{|z|}.$$

If $u > 0$ let $t = z^2u$, so that

$$\varphi(x, u) = e^{-2\pi|u|Q(x)}\int_0^\infty H(t)P(t^{-1}uQ(x))\frac{dt}{t}.$$

Since P, H are polynomials, it follows that $\varphi(x, u) = e^{-2\pi|u|Q(x)}\widetilde{P}_1(uQ(x))$.

One can argue similarly if $u < 0$. □

19.1. **Standard Schwartz function (Gaussian).** We define

$$\varphi_v^0 = \frac{1}{2}(1+\operatorname{sgn} u)e^{-2\pi|u|Q(x)} = \begin{cases} e^{-2\pi Q(\sqrt{u}x)} & \text{if } u>0 \\ 0 & \text{if } u<0. \end{cases}$$

Lemma 45. *If $Q(x)\neq 0$ (which is always the case for us) then*

$$w(g)\varphi(x,u) = W_{uQ(x)}(g) \qquad g\in \mathrm{GL}_2(\mathbb{R}).$$

Writing $g = \begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} y_0 & x_0 \\ & 1\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$,

$$W_a(g) = \begin{cases} |y_0|^{m/2}\, e^{im\theta} & \text{if } a=0 \\ |y_0|^{m/2}\, e^{2\pi i t}e^{im\theta} & \text{if } ay_0>0 \\ 0 & \text{if } au_0<0. \end{cases}$$

(Recall that $\dim V = 2m$*.)*

19.2. **Degenerate Schwartz functions at finite places.** We assume v is a finite place, unramified in E. Let $\varphi_v = \mathrm{char}(\mathcal{O}_{E_v}\times\mathcal{O}_{F_v})$. Let d_v be the local different of F at v.

If v is split in E:

$$S^0(\mathbb{V}_v\times F_v^\times) = \{\varphi\in S(\mathbb{V}_v\times F_v^\times)\mid \varphi(x,u)=0 \text{ if } \mathrm{ord}_v(uQ(x))\geq -\mathrm{ord}_v\, d_v\}$$

If v is nonsplit in E:

$$S^0(\mathbb{V}_v\times F_v^\times) = \{\varphi\in S(\mathbb{V}_v\times F_v^\times)\mid \varphi(x,u)=0 \text{ if either } d_vuQ(x)\in\mathcal{O}_{F_v} \text{ or } d_vuQ(x_2)\in\mathcal{O}_{F_v}\}.$$

Functions in S^0 are called *degenerate*. Globally, $\varphi\in S(\mathbb{V}\times F_{\mathbb{A}}^\times)$ is *degenerate at* v if φ_v is degenerate.

Assumption: Fix $v_1, v_2\nmid\infty$ nonsplit in E. We assume that $\varphi\in S(\mathbb{V}\times F_{\mathbb{A}}^\times)$ satisfies

1. φ is degenerate at v_1 and v_2,
2. $\varphi_\infty\in S^0(\mathbb{V}_\infty\times F_\infty^\times)$.

Propostion 46. $I'(g,s,\varphi) = I_0'(g,0,\varphi)+\sum_{v \text{ nonsplit}} I'(g,0,\varphi)(v)$. *Under the assumption above,* $E'(g,u,0,\varphi_2)=0$ *and* $I'(g,u,0,\varphi_2)=0$.

20. Wednesday, November 4, 2009

Let $\widetilde{\varphi} = \bigotimes_v \widetilde{\varphi}_v \in \widetilde{S}(\mathbb{V}\times F_{\mathbb{A}}^\times)$,

$$E(g,s,\widetilde{\varphi}) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x,u)\in V_1\times F^\times} w(\gamma g)\widetilde{\varphi}(x,u).$$

Assume that if $v\mid\infty$ then $\widetilde{\varphi}_v = H(u)P(x)e^{-2\pi|u|Q(x)}$.

We have that if $\widetilde{\varphi}=\widetilde{\varphi}_1\otimes\widetilde{\varphi}_2$ then

$$I(g,s,\widetilde{\varphi}) = \sum_{u\in F^\times} \underbrace{\sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s w(g)\widetilde{\varphi}_2(0,u)}_{E(g,u,s,\widetilde{\varphi}_2)} \underbrace{\sum_{x\in V_1} w(g)\widetilde{\varphi}_1(x,u)}_{\theta(g,u,\widetilde{\varphi}_1)}. \tag{6}$$

Let $\varphi = \bigotimes\varphi_v \in S(\mathbb{V}\times F_{\mathbb{A}}^\times)$. This implies that $\varphi_v(x,u) = [P_1(uQ(x)) + \mathrm{sgn}(u)P_2(uQ(x))]e^{-2\pi|u|Q(x)}$, and $\varphi_f = \bigotimes_{v\nmid\infty}\varphi_v$ is K-invariant for some open compact $K\subset GO(\widehat{\mathbb{V}})$. Let $K_Z = K\cap\mathbb{A}_f^\times$, $\mu_K = K\cap F^\times$. Then set

$$\widetilde{I}(g,s,\varphi) = * \sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s \sum_{(x,u)\in\mu_K\backslash V_1\times F^\times} w(\gamma g)\varphi(x,u).$$

(To make this independent of the choice of K one should have $* = [\mathbb{A}_v^\times : F_\infty^\times K_z]$.) Set

$$\widetilde{I}(g,s,\chi,\varphi) = \int_{E^\times\backslash E_{\mathbb{A}}^\times} \chi(t)\widetilde{I}(g,s,w(t,1)\varphi)\chi(t)dt.$$

(The assumption that $\chi\mid_{F_{\mathbb{A}}^\times}\omega_\pi = 1$ implies that $\chi\mid_{F_\infty}\omega_{\pi,\infty} = 1$ but the assumptions at infinite places imply that $\omega_{\pi,\infty}$ is trivial. Hence $\chi\mid_{F_\infty} = 1$.)

Propostion 47. *$I(g,s,\chi,\widetilde{\varphi}) = \widetilde{I}(g,s,\chi,\varphi)$ where $\varphi = \int_{F_\infty^\times} w(g)\widetilde{\varphi}dz$.*

The proof is not difficult. This proposition implies that

$$\widetilde{I}(g,s,\varphi) = \sum_{u\in\mu_K^2\backslash F^\times} \underbrace{\sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s w(g)\varphi_2(0,u)}_{E} \underbrace{\sum_{x\in V_1} w(g)\varphi_1(x,u)}_{\theta}. \tag{7}$$

Propostion 48.

$$\widetilde{I}'(g,s,\chi,\varphi) = -\sum_{v\ \text{nonsplit}} \widetilde{I}'(g,0,\varphi)(v) + I_0'(g,o,\varphi)$$

where

$$\widetilde{I}'(g,o,\varphi)(v) = 2\int_{E^\times F_{\mathbb{A}}^\times\backslash E_{\mathbb{A}}^\times} \mathscr{K}_\varphi^{(v)}(g,(t,t))dt,$$

$$\mathscr{K}_\varphi^{(v)}(g,(t_1,t_2)) = \mathscr{K}_\varphi^{(v)}(g,(t_1,t_2)) = * \sum_{u\in\mu_K^2\backslash F^\times}\ \sum_{y\in V^{(v)}\backslash V_1} k_{w(t_1,t_2)\varphi_v}(g,y,u)w(g)w(t_1,t_2)\varphi^{(v)}(y,u),$$

$$k_v(g,y,u) = \frac{L(1,\eta)}{\mathrm{vol}(E_v^1)} w(g)\varphi_1(y_1,u)W'_{uQ(y_2),v}(g,u,0,\varphi_2),$$

and $y = y_1 + y_2 \in V^{(v)} = V_1 + V_2^{(v)}$.

Recall that theta function looks like

$$\theta(g,h,\varphi) = * \sum_{u\in\mu_K^2\backslash F^\times}\ \sum_{x\in V} w(g)w(h)\varphi(x,u)$$

where $\varphi\in S(V_{\mathbb{A}}^{(v)}\times F_{\mathbb{A}}^\times)$. So, in the preceding proposition we would like $k_v(g,y,u)$ to be a Schwartz function. (However, it is not, as can be seen below because $\mathrm{ord}_v(Q(y_2))$ is not a locally constant function.)

Propostion 49 (3.4.1)**.** (1) *If everything is unramified then*

$$k_{\varphi_v}(1,y,u) = \mathrm{char}(\mathcal{O}_{B_v})(y)\,\mathrm{char}(\mathcal{O}_{F_v^\times})(u)\log N_v\cdot\frac{1}{2}(\mathrm{ord}_v(Q(y_2))+1).$$

(2) *If $\varphi_v\in S^0(\mathbb{B}_v\times F_v^\times)$ is degenerate then $k_{\varphi_v}(1,y,u)$ extends to a Schwartz function on $B_v^{(v)}\times F_v^\times$.*

(3) *If v is nonsplit such that φ is degenerate at v (in addition to being degenerate at v_1,v_2) then*

$$\mathscr{K}_\varphi^v(t_1,t_2) = \theta(g,(t_1,t_2),k_{\varphi_v}\otimes\varphi^v)$$

for $g\in P(F_v)P(F_{v_1})P(F_{v_2})\mathrm{GL}_2(\mathbb{A}^{v,v_1,v_2})$.

In the third statement, $k_{\varphi_v}\otimes\varphi^v\in S(B_{\mathbb{A}}^{(v)}\times F_{\mathbb{A}}^\times)$. Also the limitation on g isn't a big assumption because of strong approximation.

21. Monday, November 9, 2009

We will discuss holomorphic projection as in section 3.6 of [7].

21.1. Classically. Let $S_k(N)$ be the space of cuspidal modular forms of weight k and level N. This space has a perfect pairing (i.e. a positive definite Hermitian form)

$$\langle \cdot, \cdot \rangle_{Pet} : S_k(N) \times S_k(N) \to \mathbb{C} \qquad \langle f, g \rangle_{Pet} = \int_{X_0(N)} f(\tau)\overline{g(\tau)} v^k d\mu(\tau).$$

In particular, this implies that $S_k(N) \simeq S_k(N)^\wedge = \mathrm{Hom}_{\mathbb{C}}(S_k(N), \overline{\mathbb{C}})$ via $f \mapsto f : g \mapsto \langle f, g \rangle_{Pet}$.

Now suppose that $f : \mathbb{H} \to \mathbb{C}$ is a continuous function such that $f(\gamma\tau) = (c\tau + d)^k f(z)$ (f need not be holomorphic) with at most polynomial growth at the cusps. Then there exists a unique $pr(f) \in S_k(N)$ such that $\langle f, g \rangle_{Pet} = \langle pr(f), g \rangle_{Pet}$. The form $pr(f)$ is called the *holomorphic projection* of f in $S_k(N)$.

How does one find $pr(f) = \sum_{n=1}^\infty a_n q^n$?

If $k > 2$ then can define the Poincare series

$$P_m(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} e(m\tau) \mid_k \gamma$$

and if $k = 2$ we define

$$P_m(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} e(m\tau) \mid_k \gamma \, \mathrm{Im}(\gamma z)^s$$

and then take limit $s \to 0$. It is a fact that $S_k(N) = \langle P_m(\tau) \rangle_{m \geq 1}$.

So one can compute (we'll assume $k > 2$ to make the calculation simple)

$$\begin{aligned}
\langle f, P_m \rangle =& \langle pr(f), P_m \rangle \\
=& \int_{X_0(N)} pr(f)(\tau) \overline{\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} e(m\tau) \mid_k \gamma} v^k d\mu(\tau) \\
=& \int_{X_0(N)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} pr(f) \overline{e(m\tau)} \, \mathrm{Im}(\gamma\tau)^k d\mu(\tau) \\
=& \int_0^\infty \int_0^1 \Big(\sum_{n \geq 1} a_n q^n \Big) \overline{q^m} v^k \frac{du dv}{v^2} \\
=& \int_0^\infty \sum_n a_n e^{-2\pi(n+m)v} \int_0^1 e^{2\pi i(n-m)u} du v^{k-2} dv \\
=& \int_0^\infty a_m e^{-4\pi m v} v^{k-2} dv = a_m \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}.
\end{aligned}$$

So we can conclude that

$$a_m = \frac{(4\pi m)^{k-1} \langle f, P_m \rangle_{Pet}}{\Gamma(k-1)}.$$

21.2. **Adelically.** We first define the space of automorphic functions we are interested in. Let $\omega : F^\times \backslash F_{\mathbb{A}}^\times \to \mathbb{C}$ be an idele class character. We define the space of automorphic functions with central character ω, denoted $\mathcal{A}(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}), \omega)$, to be the set of functions $f : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ satisfying

(1) $f(g_f g) = f(g)$ for all $g_f \in \mathrm{GL}_2(F)$;
(2) $f(gk) = f(g)$ for all $k \in K$ some compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$;
(3) $f(zg) = \omega(z) f(g)$ for all $z \in Z(\mathbb{A}) = F_{\mathbb{A}}^\times$;
(4) f is K_∞-finite, (for us, $K_\infty = SO(F_\infty)$ because F is totally real);
(5) f is smooth (i.e. locally constant on $\mathrm{GL}_2(\mathbb{A}_f)$) with compact support modulo $Z(\mathbb{A}_f)$, and continuous as a function of $\mathrm{GL}_2(F_\infty)$ with moderate growth:

$$f(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) g) = O_g(|a|_{\mathbb{A}}^\epsilon) \quad \text{for some } \epsilon > 0 \text{ as } |a|_{\mathbb{A}} \to \infty;$$

(6) f is $\mathfrak{gl}_2(F_\infty)$-finite.

It is a fact that if K is a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ then

$$\mathrm{GL}_2(\mathbb{A}) = \bigsqcup \mathrm{GL}_2(F) g_i K \mathrm{GL}_2(F_\infty) \tag{8}$$

which implies that

$$\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/K = \bigsqcup \Gamma_j \backslash \mathrm{GL}_2(F_\infty) \qquad \Gamma_j = \mathrm{GL}_2(F) \cap g_j K g_j^{-1}.$$

So under this identification, a function $f \in \mathcal{A}([\mathrm{GL}_2], \omega)$ gives a tuple $(F_{g_1}, \cdots, F_{g_n})$ of functions on $\mathrm{GL}_2(F_\infty)$ each satisfying $F_{g_i}(g_\infty) = f(g_i g_\infty)$.

Recall that we have the isomorphism

$$\mathrm{GL}_2(\mathbb{R})/SO_2(\mathbb{R}) \simeq \mathbb{H}^\pm \qquad g \mapsto gi.$$

So if f has weight $(k_1, \cdots, k_d)$, meaning $f(gk_{\vec{\theta}}) = e^{i\sum \theta_i k_i} f(g)$, then we can define

$$\widetilde{F}_i(\tau) = F_{g_i}(g_\tau)(c\tau + d)^{k_i}(\det g)^{-k/2}$$

where $g_\tau = \left(\begin{smallmatrix} v & u \\ & 1 \end{smallmatrix}\right)$. Hence we obtain from f a vector valued modular form.

Some examples of (8): Let $F = \mathbb{Q}$.

- Suppose $K = K_0(N)$. Then $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) K \mathrm{GL}_2(\mathbb{R})$.
- Suppose $K = K_1(N)$. Then $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) K \mathrm{GL}_2(\mathbb{R})$.
- Suppose $K = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{A}_f) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \mod N\}$. Then

$$\mathrm{GL}_2(\mathbb{A}) = \bigsqcup \mathrm{GL}_2(\mathbb{Q}) g_i K \mathrm{GL}_2(\mathbb{R})$$

  where we can take $g_i = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ with $a \in (\mathbb{Z}/N\mathbb{Z})^\times$.

This decomposition is related to the notion of strong approximation. For SL_2 it says that for any compact open subgroup K of $\mathrm{SL}_2(\mathbb{A}_f)$,

$$\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q}) K \mathrm{SL}_2(\mathbb{R}).$$

This easily generalizes to

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) K \mathrm{GL}_2(\mathbb{R})$$

whenever $\det : K \to \widehat{\mathbb{Z}}^\times$ is surjective.

22. Wednesday, November 11, 2009

We define two subspaces of $\mathcal{A}([\mathrm{GL}_2],\omega)$:

$$\mathcal{A}_0([\mathrm{GL}_2],\omega)=\{f\in\mathcal{A}([\mathrm{GL}_2],\omega)\mid\int_{F\backslash F_{\mathbb{A}}}f(n(b)g)db=0\},$$

$$\mathcal{A}_0^{(2)}([\mathrm{GL}_2],\omega)=\{f\in\mathcal{A}_0([\mathrm{GL}_2],\omega)\mid f(g\vec{k}_\theta=f(g)e^{2i(\theta_1+\cdots+\theta_d)}\}.$$

We define the Petersson inner product as

$$\langle f_1,f_2\rangle=\int_{Z_{\mathbb{A}}\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})}f_1(g)\overline{f_2(g)}dg.$$

This makes sense whenever at least one of the two automorphic functions is in $\mathcal{A}_0$. As in the classical case there is a map

$$pr:\mathbb{A}([\mathrm{GL}_2],\omega)\to\mathbb{A}_0^{(2)}([\mathrm{GL}_2],\omega)$$

which satisfies

$$\langle f_1,f_2\rangle=\langle pr(f_1),f_2\rangle$$

for all $f_1\in\mathbb{A}_0([\mathrm{GL}_2],\omega)$.

Propostion 50. *If* $f\in\mathbb{A}([\mathrm{GL}_2],\omega)$ *such that* $f(\left(\begin{smallmatrix}a&\\&1\end{smallmatrix}\right)g)=O_g(|a|^{1-\epsilon})$ *as* $|a|\to\infty$ *then* $pr(f)_\psi$ *can be compute explicitly via the Fourier expansion:* $pr(f)=\lim_{s\to0}f_{\psi,s}$ *where*

$$f_{\psi,s}(g)=(4\pi)^{\deg F}W_\psi^{(2)}(g)\int_{NZ\backslash\mathrm{GL}_2(\mathbb{R})}\delta(h)^sf_\psi(g_fh)\overline{W_\psi^{(2)}(h)}dh.$$

Recall that f_ψ is the ψ-Whittaker coefficient of f, and is defined by

$$f_\psi(g)=\int_{N\backslash N_{\mathbb{A}}}f(n(b)g(\psi(-b)db,$$

and $W_\psi^{(k)}$ is the standard Whittaker function of weight k. So if $g=z\left(\begin{smallmatrix}y&x\\&1\end{smallmatrix}\right)k_\theta$ then

$$W_\psi^{(k)}(g)=\begin{cases}|y|^{k/2}e^{ik\theta}e^{2\pi i(x+iy)}&\text{if }y>0\\0&\text{if }y<0.\end{cases}$$

It may seem that only knowing one of the ψ-Whittaker coefficient isn't a lot of information. However, if we start with $f_\psi(g)$ can get

$$\begin{aligned}f_{\psi_a}(g)&=\int_{[N]}f(n(b)g)\psi_a(-b)db\\&=\int_{[N]}f(n(b)g)\psi(-ab)db\\&=\int_{[N]}f(n(a^{-1}bg)\psi_a(-b)db\\&=\int_{[N]}f(\left(\begin{smallmatrix}a^{-1}&\\&1\end{smallmatrix}\right)n(b)\left(\begin{smallmatrix}a&\\&1\end{smallmatrix}\right)g)\psi_a(-b)db\\&=\int_{[N]}f(n(b)\left(\begin{smallmatrix}a&\\&1\end{smallmatrix}\right)g)\psi_a(-b)db=f_\psi(\left(\begin{smallmatrix}a&\\&1\end{smallmatrix}\right)g).\end{aligned}$$

(Note: we have used the fact that $a\in F$ more than once. Also, this also shows how GL_2 is nice—this same trick would not work for SL_2.)

Idea of proof. As we did classically, we will use Poincare series. Suppose $W : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ is a ψ-Whittaker function ($W(n(b)g) = \psi(b)W(g)$) satisfying

- $W(zg) = \omega(z)W(g)$,
- $W = W_\infty^{(k)} W_f$ where $W_\infty^{(2)}$ is the standard Whittaker function as above, and W_f has compact support modulo Z.

Define

$$\varphi_W(g) = \sum_{\gamma ZN\backslash \mathrm{GL}_2} W(\gamma g)\delta(\gamma g)^s \Big|_{s=0}.$$

(For weight $k = 2$, $\varphi_W(g)$ is the limit of the above as $s \to 0$.)

It is a fact that $|\varphi_W(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) g)| = O_g(|a|^{1-k})$ and that

$$\langle \varphi_W \mid W \text{ as above}\rangle = \mathcal{A}_0^{(k)}([\mathrm{GL}_2], \omega).$$

Now we compute

$$\begin{aligned}
\langle f, \varphi_W \rangle &= \int_{Z_\mathbb{A}\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g)\overline{\varphi_W(g)}dg \\
&= \int_{Z_\mathbb{A}\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g) \sum_{\gamma\in\cdots} \overline{W(\gamma g)\delta(g)^s}dg \\
&= \int_{Z_\mathbb{A}N(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g)\overline{W(g)\delta(g)}^s dg \\
&= \int_{Z_\mathbb{A}N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \underbrace{\int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\overline{W(n(b)g)}db}_{f_\psi(g)\overline{W(g)}}\, \overline{\delta(g)}^s dg.
\end{aligned}$$

□

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We give a definition: if $f \in \mathcal{A}([\mathrm{GL}_2], \omega)$, let $pr'(f) = \lim_{s\to 0} f_{\psi,s}(g)$ if the limit exists. The proposition from last time said that when f has the right growth conditions then $pr'(f) = pr(f)$, but this need not be the case. (We will see an example of this at a later time.)

We take

$$\varphi = \varphi_1 \otimes \varphi_2 \in S(\mathbb{V} \times F_\mathbb{A}^\times) = S(V_{1,\mathbb{A}} \times F_\mathbb{A}^\times) \otimes S(\mathbb{V}_2 \times F_\mathbb{A}^\times)$$

and assume that

$$\varphi_\infty = [P_1(uQ(x)) + \mathrm{sgn}(u)P_2(uQ(x))]e^{-2\pi|u|Q(x)},$$

and there are at least two finite places v_1, v_2 at which φ is degenerate.

Our goal is to understand the holomorphic projection of $I'(g, 0, \chi, \varphi)$. Since this is the product of a Eisenstein series and a theta series, its constant term is the product of the constant terms plus additional terms. We can see this classically: Let $E(\tau, s) = a_0(v) + \sum_{n>0} a_n q^n + \sum_{n<0} a_n(v)q^n$ and $g = \sum_{n\geq 0} b_n q^n$. The constant term of Eg is

$$b_0 a_0(v) + \sum_{n>0} a_{-n}(v)bn.$$

However, the asymptotic behavior of Eg is determined by $b_0a_0(v)$ because the remaining term decays exponentially.

With this in mind we define

$$I_{0,0}(g,s,\varphi) = \sum_{u\in\mu_K^2\backslash F^\times} I_{0,0}(g,s,u,\varphi),$$

$$I_{0,0}(g,s,u,\varphi) = \theta_0(g,u,\varphi_1)E_0(g,s,u,\varphi_2),$$

and let

$$\mathscr{J}(g,s,u,\varphi) = \sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)} I_{0,0}(\gamma g,s,u,\varphi),$$

$$\mathscr{J}(g,s,\varphi) = \sum_{u\in\mu_K^2\backslash F^\times} \mathscr{J}_{0,0}(g,s,u,\varphi).$$

We will first discuss the Fourier coefficient of $\theta(g,u,\varphi_1) = \sum_{x\in V_1} w(g)\varphi_2(x,u)$:

$$\begin{aligned}\theta_a(g,u,\varphi_1) &= \int_{F\backslash F_{\mathbb{A}}} \theta(n(b)g,u,\varphi_1)\psi(-ab)db\\ &= \int_{[F]}\sum_{x\in V_1} w(g)\varphi_1(x,u,)\psi(buQ(x))\psi(-ab)db\\ &= \sum_{x\in V_1} w(g)\varphi_1(x,u)\int_{[F]}\psi((uQ(x)-a)b)db\\ &= \sum_{\substack{x\in V_1\\ uQ(x)=a}} w(g)\varphi_1(x,u).\end{aligned}$$

Lemma 51. $\theta_0(g,u,\varphi_1) = w(g)\varphi_1(0,u)$.

Lemma 52.

$$\begin{aligned}E_0(g,s,u,\varphi_2) =&\delta(g)^s w(g)\varphi_2(0,u) - W_0(g,s,u,\varphi_2)\\ =&\delta(g)^s w(g)\varphi_2(0,u) - \frac{L(s,\eta)}{L(s+1,\eta)}W_0^0(g,s,u,\varphi_2).\end{aligned}$$

Recall that $W_0^0(g,0,u,\varphi_2) = w(g)\varphi_2(0,u)$.

Corollary 53. *$I'_{0,0}(g,0,u,\varphi) = w(g)\varphi(0,u)\log\delta(g) - c_0 w(g)\varphi(0,u) - w(g)\varphi_1(0,u)(W_0^0)'(g,0,u,\varphi_2)$ where $c_0 = \frac{d}{ds}\frac{L(s,\eta)}{L(s+1,\eta)}\Big|_{s=0}$.*

Notation:

$$J(g,s,u,\varphi) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s w(\gamma g)\varphi(0,u),$$

$$\widetilde{J}(g,s,u,\varphi) = w(\gamma g)\varphi(0,u)W_0^0(\gamma g,s,u\varphi_2).$$

These are nearly the same. Indeed we'll find that

$$\mathscr{J}'(g,0,\chi,\varphi) = J'(g,0,\chi) - \widetilde{J}(g,0,\chi) - c_0 J(g,0,\chi).$$

Propostion 54. *hh*

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