ON THE TRIPLE PRODUCT FORMULA: REAL LOCAL CALCULATIONS

MICHAEL WOODBURY

Abstract. Explicit test vectors are given and values are computed for local trilinear forms on a triple of admissible representations π_j for $j = 1, 2, 3$ of $\mathrm{GL}_2(\mathbb{R})$ of weights k_j with $k_1 \geq k_2 + k_3$ using a formula of Michel-Venkatesh. This allows one to determine the corresponding real archimedean local factors in Ichino's formula for the triple product L-function. Applications both new and old to subconvexity, quantum chaos and p-adic modular forms are discussed.

1. Introduction

Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ the ring of adeles. We consider a triple of GL_2 automorphic representations π_1, π_2, π_3 over F such that the product of the central characters is trivial. Let $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ and denote by $\Lambda(s, \Pi)$ the corresponding (completed) L-function corresponding to the natural 8-dimensional tensor product representation of the L-group $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. This L-function has a distinguished history. Indeed, if π_3, for example, corresponds to an Eisenstein series and π_1 and π_2 to modular forms with q-expansions $f = \sum_{n\geq 1} a_f(n)q^n$ and $g = \sum_{n\geq 1} a_g(n)q^n$, then, up to some additional Gamma factors, $\Lambda(s, \Pi)$ is the Rankin-Selberg convolution L-function

$$L(s, f \times g) = \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

whose importance in number theory can hardly be overstated. Thinking of this as a triple product L-function was an important point of view taken in the work of Michel and Venkatesh in [MV10] in which they established subconvexity bounds for GL_2 type L-functions simultaneously in all aspects.

The case in which all three representations are cuspidal was first taken up by Garrett in [Gar87] and by Piateski-Shapiro and Rallis in [PSR87]. Garrett, by essentially integrating a triple of cusp forms f, g, h with Fourier coefficients as above against a certain Eisenstein series for Sp_6, was able to give an integral representation for the triple product L-function

$$L(s, f \times g \times h) = \sum_{n\geq 1} \frac{a_f(n)a_g(n)a_h(n)}{n^s},$$

and he used this to prove a functional equation and meromorphic continuation. This work has since been extended by many authors. (See for example [HK91], [GK92], [Wat01] and [Ich08].) The main formula of [Ich08] (to be described below) is the culmination of these formulas. It has the advantage of being valid for any choice of test vectors, however, from the standpoint of number theoretic applications, Watson's more explicit result has been

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particularly applicable in number theory and quantum chaos due to its more explicit nature. Most notably among these applications are subconvexity results (See for example [BR05]) and to the so-called Quantum Unique Ergodicity conjecture which is now a theorem of Holowinsky and Soundarajan (see [HS10] and [Sou10] and [Wat01]).

To describe Ichino's formula, let us write $\pi_j = \otimes_v \pi_{j,v}$ as a (restricted) tensor product over the places v of F, with each $\pi_{j,v}$ an admissible representation of $\mathrm{GL}_2(F_v)$. Let $\langle \cdot, \cdot \rangle_v$ be a (Hermitian) form on π_j. Then, assuming that $\varphi_j = \otimes \varphi_{j,v} \in \pi_{j,v}$ is factorizable[1], for each v we can consider the form obtained by integrating the matrix coefficient associated to $\varphi_v = \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}$:

$$I'(\varphi_v) = \int_{\mathrm{PGL}_2(F_v)} \langle \pi_v(g_v)\varphi_{1,v}, \varphi_{1,v}\rangle_v \langle \pi_v(g_v)\varphi_{2,v}, \varphi_{2,v}\rangle_v \langle \pi_v(g_v)\varphi_{3,v}, \varphi_{3,v}\rangle_v dg_v, \tag{1.1}$$

and the normalized matrix coefficient

$$I_v(\varphi_v) = \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi_v, \mathrm{Ad})}{L_v(1/2, \Pi_v)} I'_v(\varphi_v). \tag{1.2}$$

We call I'_v and I_v trilinear forms although this is somewhat of an abuse of language since it actually defines a quadratic form on the triple product.

Ichino proved (in the case that each π_i is cuspidal) that there is a constant C (depending only on the choice of measures) such that

$$\frac{\left| \int_{[\mathrm{GL}_2]} \varphi_1(g)\varphi_2(g)\varphi_3(g)\ dg \right|^2}{\prod_{j=1}^{3} \int_{[\mathrm{GL}_2]} |\varphi_j(g)|^2\ dg} = \frac{C}{2^3} \cdot \zeta_F(2)^2 \cdot \frac{\Lambda(1/2, \Pi)}{\Lambda(1, \Pi, \mathrm{Ad})} \prod_v \frac{I_v(\varphi_v)}{\langle \varphi_v, \varphi_v \rangle_v} \tag{1.3}$$

whenever the denominators are nonzero. Note that the notation $[\mathrm{GL}_2]$ represents the quotient $\mathbb{A}^\times \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A})$. By the choice of normalizations, the product on the right hand side of (1.3) is in fact a finite product as it is identically 1 when all of the input data is unramified.

In order to derive number theoretic applications from Ichino's formula, it is necessary to compute (or at least control) the local factors at the infinite and ramified finite places. This is the topic of the author's PhD thesis [Woo11] wherein Watson's formulas and explicit generalizations are derived from (1.3) by computing local trilinear forms. (In the nonarchimedean case, for example, the trilinear forms were computed for triples of representations with—potentially distinct—squarefree level.) Using this, a certain hypothesis of Venkatesh from [Ven10] was proved thereby leading to subconvexity results analogous to [BR05], but in the level instead of eigenvalue aspect. This topic is further taken up by Hu in [Hu14a] and [Hu14b] wherein higher ramification is considered with applications similar to those of [NPS14].

In addition to the results outlined above, the triple product L-function plays an important role in the work of Darmon, Lauder and Rotger (see [DLR15]) as well as others in relation to the so-called elliptic Stark conjecture which is a generalization of Stark's conjecture and is closely related to the Birch-Swinnerton-Dyer conjecture. In this work it is critical to know that up to a computable power of π, the central critical value of the completed triple product

[1] As a restricted tensor product, we have chosen vectors $\varphi^0_{i,v} \in \pi_v$ for all but finitely many places v. We require that the local inner forms must satisfy $\langle \varphi^0_{i,v}, \varphi^0_{i,v} \rangle_v = 1$ for all such v.

L-function is rational. This work takes as a necessary starting step the evaluation of the right hand side of (1.3) in the case that the triple of representations comes from two weight one modular forms and a weight two modular form—a case which was not covered by Watson. The relevant calculation at the infinite case is one of the results of the current paper.

To be more explicit, in this paper we treat the question of determining test vectors at the real infinite places and compute the corresponding trilinear forms. This work builds in particular on the results of [Woo11] and the appendix to [SZW13]. We also remark that the particular choice of test vectors is inspired greatly by [Pop08], and that using [Lok01] one can deduce the values of the trilinear form at other test vectors besides those considered here.

Since we will be considering only the local case from this point onward, unless otherwise specified, we drop the subscript v from all local objects. Hence, for example, $I(f_1 \otimes f_2 \otimes f_3)$, $L(s, \Pi)$ etc. refer to the local normalized trilinear and L-factors of (1.2) at a real place. With this in place, the following is the main result of this paper.

Theorem 1. *Suppose that π_j for $j = 1, 2, 3$ are irreducible admissible unitary representations of $\mathrm{GL}_2(\mathbb{R})$ of weights $k_1 \geq k_2 \geq k_3$ for which the product of central characters is trivial*[2]*. If we assume*[3] *that $k_1 \geq k_2 + k_3$ then there exists a choice of test vectors $f^{(j)} \in \pi_j$ such that $I(f) \neq 0$. (See Propositions 3.2, 3.3, 3.4 and 3.5 for the choice of vectors and explicit values of $I(f)$ in each case.) In particular, if $k_1 = k_2 + k_3$, there exist a choice of (explicitly given) test vectors $f_j \in \pi_j$ such that*

$$\frac{I(f_1 \otimes f_2 \otimes f_3)}{\langle f_1, f_1 \rangle \langle f_2, f_2 \rangle \langle f_3, f_3 \rangle} = \begin{cases} 1 & \textit{if } k_1 \geq 2, \\ 2 & \textit{otherwise.} \end{cases} \tag{1.4}$$

If $k_j = 0$ for all j, then we also assume that a certain invariant $\epsilon = 0$. (See Propositions 3.2.)

Remark. In the applications in [SZW13] and [DLR15] it is essential that one has an exact formula (in the latter case at least up to rational factor) for the triple product L-function.

Remark. With the hindsight of Ichino's formula which linked the local trilinear forms to certain zeta integrals on the group Sp_6, the evaluation of the archimedean local trilinear forms in the case that two or more of the representations are (weight zero) principal series was essentially worked out by Ikeda in [Ike99] and by [Wat01] as evaluations of these zeta integrals. Moreover, [II10] gives Theorem 1 for $k_3 > 1$. As such, the principal new contribution of our work here is to give a generalized and uniform treatment. Moreover, the calculation in the case of $k_2 = 1$ and/or $k_3 = 1$ as well as the more general results in Propositions 3.2, 3.3, 3.4 and 3.5 are new. Besides giving these new results, we believe that the present proofs illustrate how the method is widely and easily applicable.

The proof of Theorem 1 is obtained on a case by case basis considering all possible combinations of representations π_1, π_2, π_3. We give an overview of the relevant representation theoretic background in Section 2 and then compute the trilinear forms Section 3. The normalizing factor in (1.2) relating I' and I can be calculated following the prescription for the local Langlands correspondence given in [Kna94]. We include an overview of this theory and record the relevant factors for each of the possible cases in an Appendix.

[2]This implies directly that $k_1 + k_2 + k_3$ is even.

[3]By [Pra90], this assumption is necessary as otherwise I_v is identically zero.

2. Background and notation

In this section we set notation and give definitions for the representation theory of $\mathrm{GL}_2(\mathbb{R})$ that will be used in the sequel. This theory is well known. See [Bum97] or [JL70] for complete details.

2.1. **Admissible representations of** $\mathrm{GL}_2(\mathbb{R})$**.** Given an automorphic representation $\otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$ as discussed in the introduction, for all real places v of F, the local factor π_v is an admissible $(\mathfrak{gl}_2, K)$-module where $\mathfrak{gl}_2$ is the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$ and

$$K = \mathrm{SO}(2) = \left\{ \kappa_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \right\}.$$

Using a slight abuse of language we refer to such a module as an admissible representation of $\mathrm{GL}_2(\mathbb{R})$.

Let $\psi_n : K \to \mathbb{C}$ be given by $\psi_n(\kappa_\theta) = e^{in\theta}$. Recall that restricting any irreducible admissible representation π to K there exists a nonnegative integer $\mathrm{wt}(\pi)$ such that

$$\pi_j\mid_K \simeq \bigoplus_{\substack{|n| \geq \mathrm{wt}(\pi) \\ n \equiv \mathrm{wt}(\pi) \pmod 2}} \mathbb{C}\psi_n.$$

An element $\phi \in \pi$ is said to have *weight* n if ϕ corresponds, via this isomorphism, to an element in $\mathbb{C}\psi_n$. The integers n appearing in the decomposition above are called the *weights* of π. Accordingly, we say that π has *even* or *odd weight* depending on whether $\mathrm{wt}(\pi)$ is even or odd respectively.

We define the following subgroups of $\mathrm{GL}_2(\mathbb{R})$:

$$\begin{aligned} A &= \left\{ a(y) = \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid y \in \mathbb{R}^\times \right\}, \\ Z &= \left\{ z(u) = \left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right) \mid u \in \mathbb{R}^\times \right\}, \\ N &= \left\{ n(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{R} \right\}. \end{aligned}$$

We can construct all such representations via the *induced representations* which are defined in terms of (quasi-)characters $\chi_j : \mathbb{R}^\times \to \mathbb{C}^\times$ of the form $\chi_j(x) = \mathrm{sgn}(x)^{\epsilon_j}|x|^{s_j}$ where $\mathrm{sgn} : \mathbb{R}^\times \to \{\pm 1\}$ is the sign character $x \mapsto x/|x|$, $\epsilon_j \in \{0,1\}$ and and $s_j \in \mathbb{C}$. Then

$$\mathcal{B}(\chi_1, \chi_2) := \left\{ f : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C} \;\middle|\; \begin{array}{c} f(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) g) = \chi_1(a)\chi_2(d)|\frac{a}{d}|^{1/2} f(g) \\ \text{for all } g \in \mathrm{GL}_2(\mathbb{R}), \\ f \text{ is smooth and } K\text{-finite.} \end{array} \right\}.$$

It is easy to see that for any $f \in \mathcal{B}(\chi_1, \chi_2)$,

$$f(z(u)a(y)g) = \mathrm{sgn}(u)^\delta |u|^\mu \,\mathrm{sgn}(y)^{\epsilon_1}|y|^s f(g)$$

where $\delta \in \{0,1\}$ is such that $\delta \equiv \epsilon_1 + \epsilon_2 \pmod 2$, $s = \frac{1}{2}(1 + s_1 - s_2)$ and $\mu = s_1 + s_2$. Given this, we define $\pi_{\delta,\epsilon}(s,\mu) := \mathcal{B}(\chi_1, \chi_2)$ where $\chi_1 = \mathrm{sgn}^\epsilon|\cdot|^{s + \frac{\mu-1}{2}}$ and $\chi_2 = \mathrm{sgn}^{\delta-\epsilon}|\cdot|^{-s+\frac{\mu+1}{2}}$. We also use the notation $\pi_{\delta,\epsilon}(s) := \pi_{\delta,\epsilon}(s, 0)$. The above makes clear that central character of $\pi_{\delta,\epsilon}(s,\mu)$ is given by $\mathrm{sgn}^\delta|\cdot|^\mu$.

Since by twisting by the determinant we have

$$|\det(\cdot)|^{\frac{\mu}{2}} \otimes \pi_{\delta,\epsilon}(s, 0) \simeq \pi_{\delta,\epsilon}(s,\mu),$$

it follows that $\pi_{\delta,\epsilon}(s)$ is the unique twist of $\pi_{\delta,\epsilon}(s,\mu)$ such that the central character is sgn^δ.

We denote by $f_{m,s}$ the weight m vector in $\pi_{\delta,\epsilon}(s)$ satisfying $f_{m,s}(\kappa_\theta) = e^{im\theta}$. Note that this is nonzero if and only if $m \equiv \delta \pmod 2$. The set of all such vectors forms a basis.

There exists an intertwining operator from $\pi = \mathcal{B}(\chi_1, \chi_2) = \pi_{\delta,\epsilon}(s,\mu)$ to $\widetilde{\pi} := \mathcal{B}(\chi_2, \chi_1) = \pi_{\delta,\delta-\epsilon}(1-s)$. If $\mathrm{Re}(s) > \frac{1}{2}$, this is given by $M(s) : \pi \to \widetilde{\pi}$ defined via

$$(2.1) \qquad \big(M(s)f\big)(g) := \int_{-\infty}^{\infty} f(wn(x)g)\; dx,$$

where $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We drop μ from the notation as the map is independent of the choice of μ within the class of twists of $\pi_{\delta,\epsilon}(s)$. By analytic continuation, $M(s)$ extends to other values of s, μ. It sends the weight m vector $f_m \in \pi$ to a multiple of $\widetilde{f_m} \in \widetilde{\pi}$. As long as $\pi_{\delta,\epsilon}(s,\mu)$ is irreducible (which is the case unless $s = \frac{k}{2}$ or $s = 1 - \frac{k}{2}$ with $k > 1$ an integer satisfying $k \equiv \delta \pmod 2$) the map $M(s)$ is an isomorphism.

Given $\pi = \mathcal{B}(\chi_1, \chi_2) = \pi_{\delta,\epsilon}(s,\mu)$ the contragradient is $\widehat{\pi} = \mathcal{B}(\chi_1^{-1}, \chi_2^{-1}) = \pi_{\delta,\epsilon}(1-s, -\mu)$ with pairing

$$(2.2) \qquad (\cdot,\cdot) : \pi \times \widehat{\pi} \to \mathbb{C}, \qquad (f,h) := \int_K f(\kappa) h(\kappa)\; d\kappa$$

(We normalize the measure $d\kappa$ on K such that $\mathrm{vol}\, K = 1$.) In the case of the unitary principal series, the characters $\overline{\chi_j} = \chi_j^{-1}$, and so we can identify the contragradient with its complex conjugate $\overline{\pi}$. Then the we have a Hermitian form $\langle f, g\rangle := (f, \overline{g})$ on π. In general, for unitary representations π one has $\widehat{\pi} = \overline{\widetilde{\pi}}$. Using the intertwining operator $M(s)$ one can define a Hermitian form $\langle f, g\rangle := (f, \overline{cM(s)g})$ for a suitable constant c.

For global applications (i.e., to be applied towards (1.3)), one only needs the results of this paper for choices of (s,μ) such that these representations are unitarizable. Note that $\pi_{\delta,\epsilon}(s,\mu)$ is unitarizable if and only if $\pi_{\delta,\epsilon}(s)$ is unitarizable and the central character is unitary, i.e., $\mu \in i\mathbb{R}$. Therefore up to twists by unitary characters, the unitary representations are differentiated as follows.

- If $s = \frac{1}{2} + \nu$ with $\nu \in i\mathbb{R}^\times$, $\pi_{\delta,\epsilon}(s)$ is called an even or odd weight *(unitary) principal series* according as $\delta = 0$ or 1 respectively. Since $\pi_{\delta,\epsilon}(\frac{1}{2}+\nu) \simeq \pi_{\delta,\delta-\epsilon}(\frac{1}{2}-\nu)$, in the case of $\delta = 1$, it suffices to consider $\pi_{1,\epsilon}(\frac{1}{2}+it)$ only in the case of $\epsilon = 0$.
- If $\delta = 0$ and $s = \frac{1}{2} + \nu$ and $s' = \frac{1}{2} + \nu'$ with $\nu, \nu' \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$, we have that $\pi_{0,\epsilon}(s) \simeq \pi_{0,\epsilon'}(s')$ if and only if $\epsilon = \epsilon'$ and $s' = 1 - s$ (meaning that $\nu' = -\nu$). These are called *complementary series.*
- If $s = \frac{k}{2}$ or $s = 1 - \frac{k}{2}$ for some $k \geq 1$ then $\pi_{\delta,\epsilon}(s) = 0$ unless $k \equiv \delta \pmod 2$. (The choice of ϵ is irrelevant.) Then, for such s with $k > 1$, $\pi_{\delta,\epsilon}(s)$ is not irreducible; however, there is an irreducible representation π_{dis}^k, called the *(holomorphic) weight k discrete series* which is isomorphic to a subrepresentation if $s = \frac{k}{2}$ and a quotient if $s = 1 - \frac{k}{2}$. We refer to π_{dis}^1 as a *limit of discrete series.* Note that $\pi_{\mathrm{dis}}^1 \simeq \pi_{1,0}(\frac{1}{2})$.

To conclude this section we record the action of the Laplace-Beltrami operator

$$(2.3) \qquad \Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y \frac{\partial^2}{\partial x \partial \theta},$$

and the raising and lowering operators

$$(2.4) \qquad R = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta}\right), \quad L = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial \theta}\right)$$

on $f \in \pi = \pi_{\delta,\epsilon}(s)$ in terms of the coordinates $n(x)a(y)\kappa_\theta$ on $\mathrm{GL}_2(\mathbb{R})$. These act via

$$\Delta f = s(1-s)f \qquad (\text{for all } f \in \pi),$$

and

$$Rf_{m,s} = \left(s + \frac{m}{2}\right) f_{m+2,s}, \qquad Lf_{m,s} = \left(s - \frac{m}{2}\right) f_{m-2,s}. \tag{2.5}$$

2.2. **Whittaker models and functions.** Given an irreducible admissible representation $\pi = \pi_{\delta,\epsilon}(s)$ or $\pi = \pi^k_{\mathrm{dis}}$ and a character $\psi : \mathbb{R} \to \mathbb{C}^\times$, there is a unique space $\mathcal{W}(\pi,\psi)$ of Schwartz functions $W : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$ such that

$$W(n(x)g) = \psi(x)W(g) \qquad \text{for all } g \in \mathrm{GL}_2(\mathbb{R}), \tag{2.6}$$

and, under the action $\rho(g)W(h) = W(hg)$, $\pi \simeq W(\pi,\psi)$.

We denote by $W_m \in \mathcal{W}(\pi,\psi)$ the unique up to constant vector of weight m, i.e., the vector which satisfies $\rho(\kappa_\theta)W_m = e^{im\theta}W_m$. There exists an explicit intertwiner $\pi \to \mathcal{W}(\pi,\psi)$ given by

$$f \mapsto W_f(g) = \int_{\mathbb{R}} f(wn(x)g)\overline{\psi(x)}\, dx.$$

Hence, if $f_{m,s} \in \pi_{\delta,\epsilon}(s)$ as defined in the previous section, the functions $W_{f_{m,s}}$ satisfy the same relations as given in (2.5) for the raising and lowering operators. Rather than work with this intertwiner directly, we simply require that $W_m \in \mathcal{W}(\pi,\psi)$ be a weight m vector such that

$$RW_m = \left(s + \frac{m}{2}\right) W_m, \qquad LW_m = \left(s - \frac{m}{2}\right) W_{m-2},$$

and

$$\rho\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)W_m = (-1)^\delta W_{-m} \tag{2.7}$$

hold for all m. This defines the collection $\{W_m\}$, therefore, up to a common constant multiple. Moreover, if $\pi = \pi_{\delta,\epsilon}(s)$ one sees via (2.6) that

$$W_m(a(-y)) = (-1)^{\epsilon+\delta}W_m(a(y)), \tag{2.8}$$

so in this case $W_m(y)$ is determined by its values on $y > 0$. In the case $\pi = \pi^k_{\mathrm{dis}}$, we will see that $W_m(a(y))$ is nonzero either for $y > 0$ or $y < 0$ (depending on ψ and m).

Following the strategy of [Pop08] (which itself is based on [JL70]), in Proposition 2.1 we describe certain functions $W \in \mathcal{W}(\pi,\psi)$. We do so in terms of the *modified Bessel function*, $K_\nu(y)$, which[4] up to a constant is the unique solution with moderate growth (as $y \to \infty$) to the differential equation

$$0 = f''(y) + \frac{1}{y}f'(y) - \left(\frac{\nu^2}{y^2} + 1\right) f(y). \tag{2.9}$$

Fixing the constant, we take for $y > 0$

$$K_\nu(y) = \tfrac{1}{2}\int_0^\infty e^{-\frac{y}{2}\left(t+\frac{1}{t}\right)} t^\nu d^\times t,$$

which is easily seen to satisfy (2.9) and have exponential decay as $y \to \infty$.

[4]Contrary to commonly used notation, in [Pop08] this function is referred to as J_ν.

In the sequel, we will make use of the identities

$$\int_0^\infty K_\nu(y/2)K_\mu(y/2)y^s d^\times y = 2^{2s-3}\frac{\Gamma(\frac{s+\mu+\nu}{2})\Gamma(\frac{s-\mu+\nu}{2})\Gamma(\frac{s+\mu-\nu}{2})\Gamma(\frac{s-\mu-\nu}{2})}{\Gamma(s)}, \tag{2.10}$$

which is valid for $\mathrm{Re}(s) > |\mathrm{Re}(\mu)| + |\mathrm{Re}(\nu)|$, and

$$\int_0^\infty e^{-y/2}K_\nu(y/2)y^s d^\times y = \pi^{1/2}\frac{\Gamma(s+\nu)\Gamma(s-\nu)}{\Gamma(s+\frac{1}{2})}, \tag{2.11}$$

which holds whenever $\mathrm{Re}(s) > |\mathrm{Re}\,\nu|$. These are equations (6.8.48) and (6.8.28) of [EMOT54] respectively. We will also need the additional fact that

$$\frac{d}{dz}K_\nu(z) = -\frac{1}{2}\big(K_{\nu-1}(z) + K_{\nu+1}(z)\big) \tag{2.12}$$

$$= \frac{\nu}{z}K_\nu(z) - K_{\nu+1}(z). \tag{2.13}$$

Proposition 2.1. *Suppose that $\psi(x) = e^{\gamma i x/2}$ with $\gamma \in \{\pm\}$. Let $W^\gamma_{\pm k} \in \mathcal{W}(\pi^k_{\mathrm{dis}}, \psi)$ be vectors of weight $\pm k$ respectively. Up to scalar, these are given by*

$$W^-_{-k}(a(y)) = W^+_k(a(y)) = \begin{cases} y^{k/2}e^{-y/2} & \text{if } y>0 \\ 0 & \text{otherwise.}\end{cases} \tag{2.14}$$

and $W^-_k(a(y)) = W^+_{-k}(a(y)) = (-1)^k W^+_k(a(-y))$.

Writing $s = \frac{1}{2}+\nu$, we may choose $W^\gamma_0, W^\gamma_{-2}, W^\gamma_2 \in \mathcal{W}(\pi_{0,\epsilon}(s), \psi)$ such that

$$W^\gamma_0(a(y)) = \mathrm{sgn}(y)^\epsilon |y|^{1/2}K_\nu(|y|/2), \tag{2.15}$$

$$(W^\gamma_{-2} - W^\gamma_2)(a(y)) = \mathrm{sgn}(y)^{\epsilon+1}|y|^{3/2}K_\nu(|y|/2), \tag{2.16}$$

and

$$\begin{aligned}(W^\gamma_{-2} + W^\gamma_2)(a(y)) = \frac{\mathrm{sgn}(y)^{\epsilon+1}}{4}\Big(&2|y|^{-1/2}K_\nu(|y|/2) \\ &- |y|^{1/2}\big(K_{\nu-1}(|y|/2) + K_{\nu+1}(|y|/2)\big)\Big).\end{aligned} \tag{2.17}$$

Finally, we may choose $W^\gamma_{\pm 1} \in \mathcal{W}(\pi_{1,0}(s), \psi)$ such that

$$(W^\gamma_{-1} + W^\gamma_1)(a(y)) = yK_{-\frac{1}{2}+\nu}(|y|/2), \tag{2.18}$$

and

$$(W^\gamma_{-1} - W^\gamma_1)(a(y)) = |y|K_{\frac{1}{2}+\nu}(|y|/2). \tag{2.19}$$

Remark. Note that our choice of character is not the same as that given in [Pop08] which results in slightly different formulas. One advantage of our choice (as will be shown) is the corresponding functions W_m will be solutions of the classical differential equation of Whittaker.

Proof. Assuming that π has central character $\pi(z(u)) = \mathrm{sgn}(u)^\delta$, we see that $W_m \in \mathcal{W}(\pi, \psi)$ satisfies

$$W_m(z(u)n(x)a(y)\kappa_\theta) = e^{i(\gamma\frac{x}{2}+k\theta)}W_m(a(y)) \tag{2.20}$$

where $\gamma = +$ if $\delta = 0$ and $\gamma = -$ if $\delta = 1$.

Suppose that $\lambda = s(1-s)$ is the eigenvalue of the action of the Laplace operator Δ on π. Then combining this with the definition of Δ from (2.3) applied to (2.20), it is easy to see that $w_m(y) = W_m(a(y))$ satisfies the differential equation

$$w'' + \left[-\frac{1}{4} + \frac{\gamma m}{2y} + \frac{\lambda}{y^2}\right] w = 0. \tag{2.21}$$

which, writing $s = \frac{1}{2} + \nu$, has solutions $W_{m,\nu}(y)$ and $W_{-m,\nu}(-y)$, the so-called Whittaker functions. Since only $W_{m,\nu}$ has moderate growth as $y \to \infty$, together with (2.8), we find—provided that[5] $W_m^\gamma \in \mathcal{W}(\pi_{\delta,\epsilon}(s), \psi)$ and $s \notin \frac{1}{2}\mathbb{Z}$—that for $m \geq 0$,

$$W_m^\gamma(a(y)) = \begin{cases} W_{m,\nu}(y) & \text{if } y > 0, \\ (-1)^{\epsilon+\delta} W_{m,\nu}(-y) & \text{if } y < 0. \end{cases}$$

Combined with (2.7), this defines $W_m(a(y))$ for all $m \equiv \delta \pmod 2$ and for all $y \neq 0$.

Applying the operators R and L given in (2.4) to W_m we find that

$$\left(\nu + \frac{1 \pm m}{2}\right) w_{m\pm 2} = \pm\left(\frac{m-y}{2}\right) w_m + y w_m'. \tag{2.22}$$

If $\pi = \pi_{\text{dis}}^k$, one has that W_k must be annihilated by L. (This fact is true for $k = 1$ as well.) Using this leads to the differential equation

$$2y w_k'(y) + \left(y - \frac{k}{2}\right) w_k = 0,$$

which can be solved using elementary methods. The restriction on the growth leads immediately to (2.14). The formula for W_{-k} follows from (2.7).

For the remainder of the proof we note that the choice of γ effects only the sign of m appearing in (2.21). This means that W_m^- is the weight m vector such that $W_m^-(a(y)) = W_{-m}^+(a(y))$. For the purpose of the rest of the proof, therefore, it suffices to take $\gamma = +$ and we drop it from the notation.

Now we consider the case of $\pi = \pi_{0,\epsilon}(s)$. If we let $w_0(y) = y^{1/2} f(y/2)$ and plug this into (2.21), after simplifying, we arrive at equation (2.9). Since $f = K_\nu$ has moderate growth as $y \to \infty$, clearly $y^{1/2} K_\nu(y/2)$ does as well. Applying (2.8), one arrives at (2.15).

Next, we apply (2.22) in the case of $m = 0$ which yields

$$2s w_2 = -y w_0 + 2y w_0' \quad \text{and} \quad 2s w_{-2} = y w_0 + 2y w_0'.$$

Hence

$$w_{-2}(y) - w_2(y) = \frac{y w_0(y)}{s}, \qquad w_{-2}(y) + w_2(y) = \frac{2y w_0'(y)}{s}.$$

Note that since π is a principal series, $s \neq 0$. In the first case, using the formula for $w_0(y)$ from above gives (2.16), and (2.17) is obtained similarly using (2.12).

[5]It is necessary, of course, that $m = \text{wt}(\pi) + 2n$ for some $n \in \mathbb{Z}_{\geq 0}$.

Finally, we now assume $\pi = \pi_{1,0}(s)$. Applying (2.22) in the case of $m = \pm 1$ leads to the system of equations

$$\nu w_1 = \left(\frac{-1-y}{2}\right) w_{-1} + y w_{-1}'(y),$$
$$\nu w_{-1} = \left(\frac{y-1}{2}\right) w_1 + y w_1'(y).$$

We now set $f = w_1 + w_{-1}$ and $g = w_1 - w_{-1}$, so that adding and subtracting these two formulas we find that

$$\begin{aligned}(2\nu+1)f &= \quad yg + 2yf' \\ (2\nu-1)g &= -\,yf - 2yg'.\end{aligned} \tag{2.23}$$

This simplifies further to

$$f'' - \frac{1}{y}f' - \left(\frac{(2\nu-1)^2-4}{4y^2} + \frac{1}{4}\right) f = 0.$$

Plugging $f(y) = yK(y/2)$ into the above, we find that K satisfies the differential equation

$$0 = K''(y) + \frac{1}{y}K'(y) - \left(\frac{(\nu-\frac{1}{2})^2}{y^2} - 1\right) K(y).$$

Comparing this with (2.9), the formula (2.18) for $f = W_1 + W_{-1}$ follows readily, using the fact that $f(y)$ is odd. (That f is odd is a direct consequence of (2.7).) On the other hand, using (2.13) and (2.23), we see that $g = W_1 - W_{-1}$ satisfies

$$\begin{aligned}yg(y) &= (2\nu+1)f(y) - 2yf'(y) \\ &= (2\nu+1)K_{\nu-\frac{1}{2}}(y/2) - 2y(-(y/2)K_{\nu+\frac{1}{2}}(y/2) + (\nu+1/2)K_{\nu-1/2}(y/2)) \\ &= y^2 K_{\nu+\frac{1}{2}}(y/2).\end{aligned}$$

This is valid for $y > 0$ and leads directly to (2.19) since g is an even function. $\square$

Proposition 2.2. *The norms of the test vectors from Proposition 2.1 are as follows. The vector $W_k \in \mathcal{W}(\pi_{\mathrm{dis}}^k, \psi)$ satisfies $\langle W_k, W_k\rangle = (k-1)!$. The vectors $W_{\pm\ell} \in \mathcal{W}\,\pi_{\delta,\epsilon}(\frac{1}{2}+\nu)$ with $\ell = 0, 1, 2$ and $\ell \equiv \delta \pmod 2$ satisfy $\langle W_\ell, W_\ell\rangle = \pi\Gamma(\frac{1+\ell}{2}+\nu)\Gamma(\frac{1+\ell}{2}-\nu)$.*

Proof. If π is a discrete series or a unitary principal series then the inner product on $\mathcal{W}(\pi,\psi)$ is given by

$$\langle W, W'\rangle = \int_K \int_{\mathbb{R}^\times} W(a(y)\kappa)\overline{W(a(y)\kappa)} d^\times y\, d\kappa.$$

Thus, using the integral representation $\Gamma(s) = \int_0^\infty y^s e^{-y} d^\times y$ for the Gamma function, we see in the case of the discrete series π_{dis}^k that

$$\langle W_k, W_k\rangle = \int_0^\infty e^{-y} y^{k-1}\, dy = \Gamma(k) = (k-1)!.$$

We write the norms of each of the functions (2.15), (2.16) and (2.19) in terms of (2.10). For $W = W_0$ this is completely straightforward.

The case of $W = W_{\pm 2}$ is somewhat more complicated, and so we go through the proof in detail. First, writing $w_m(y) = W_m(a(y))$, note that if we set $f_- = w_{-2} - w_2$ and $f_+ = w_{-2} + w_2$, then f_- is an odd function of y and f_+ is even. Thus

$$\begin{aligned}\langle W_{\pm 2}, W_{\pm 2}\rangle &= \int_{\mathbb{R}^\times} w_{\pm 2}(y)\overline{w_{\pm 2}(y)} d^\times y \\ &= \int_{\mathbb{R}^\times} \left(\frac{f_+(y) \pm f_-(y)}{2}\right)\left(\overline{\frac{f_+(y) \pm f_-(y)}{2}}\right) d^\times y \\ &= \frac{1}{4}\int_{\mathbb{R}^\times} \left(f_+(y)\overline{f_+(y)} + f_-(y)\overline{f_-(y)}\right) d^\times y \\ &= \frac{1}{2}\int_0^\infty \left(f_+(y)\overline{f_+(y)} + f_-(y)\overline{f_-(y)}\right) d^\times y.\end{aligned}$$

One calculates easily using (2.10) that

$$\int_0^\infty f_+(y)\overline{f_+(y)}\ d^\times y = \pi\Gamma(\tfrac{3}{2} + it)\Gamma(\tfrac{3}{2} - it).$$

The calculation of $\int_0^\infty f_-(y)\overline{f_-(y)}\ d^\times y$ is similar (but messier) and gives the same result. Putting this together leads to the claimed result for $W_{\pm 2}$.

The case of $W_{\pm 1}$ is similar. We leave the details to the reader. $\square$

Note that

$$\int_K W_m(a(y)\kappa)\overline{W_n(a(y)\kappa)}\ d\kappa = 0$$

unless $m = n$. In particular this implies that $\langle W_\ell \pm W_{-\ell}, W_\ell \pm W_{-\ell}\rangle = 2\pi\Gamma(\frac{1+\ell}{2} + \nu)\Gamma(\frac{1+\ell}{2} - \nu)$ if $\ell \neq 0$.

3. Computing trilinear forms

For $j = 1, 2, 3$, let π_j be irreducible admissible unitary representations of $\mathrm{GL}_2(\mathbb{R})$. We assume that the product of their central characters is trivial. Thus, without loss of generality, if ω_j is the central character of π_j we may assume that $\omega_j(z(u)) = \mathrm{sgn}(u)^{\delta_j}$ for $\delta_j \in \{0, 1\}$ satisfying

$$\delta_1 + \delta_2 + \delta_3 \equiv 0 \pmod 2. \tag{3.1}$$

As a matter of notation, we will denote an element of π_j by $f^{(j)}$, and similarly elements of $\mathcal{W}(\pi_j, \psi)$ will be denoted by $W^{(j)}$. The calculation of the trilinear form is simplified by using the Whittaker models of π_j for $j = 1, 2$ due to the following result of [MV10].

Proposition 3.1 (Michel-Venkatesh). *Let π_1, π_2, π_3 be tempered representations of $\mathrm{GL}_2(\mathbb{R})$ with π_3 a principal series. Fix isometries $\pi_1 \to \mathcal{W}(\pi_1, \psi)$ and $\pi_2 \to \mathcal{W}(\pi_2, \overline{\psi})$ for $\psi(x) = e^{ix/2}$. Via these isometries, associating to $f^{(j)} \in \pi_j$ for $j = 1, 2$ vectors $W^{(j)}$ in the Whittaker models, the form $\ell_{\mathrm{RS}} : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ given by*

$$\ell_{\mathrm{RS}}(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \sqrt{4\pi}\int_K\int_{\mathbb{R}^\times} W^{(1)}(a(y)\kappa)W^{(2)}(a(y)\kappa)f^{(3)}(a(y)\kappa)|y|^{-1}\, d^\times y\, d\kappa \tag{3.2}$$

satisfies $|\ell_{\mathrm{RS}}|^2 = I'_v(f^{(1)} \otimes f^{(2)} \otimes f^{(3)})$ where I' is the integrated matrix coefficient given in (1.1).

Remark. The constant $\sqrt{4\pi}$ is an artifact of the fact that the formula given in [MV10] (in which this constant does not appear) is valid in the particular case that $\psi(x) = e^{2\pi i x}$. Adjusting to our case of $\psi(x) = e^{ix/2}$ has the effect of multiplying by this constant.

To ease notation we will assume from henceforth that $\langle f^{(j)}, f^{(j)}\rangle = 1$. This implies that the map $\pi_j \to \mathcal{W}(\pi_j, \psi)$ given by $f^{(j)} \mapsto W^{(j)}/\langle W^{(j)}, W^{(j)}\rangle^{1/2}$ is an isometry to which we may apply Proposition 3.1.

As remarked in [MV10], the non-tempered case (including the complementary series) can also be treated with Proposition 3.1 via a polarization which we describe now. In this generality, we associate to $f \in \pi$ a vector $\widetilde{f} \in \widehat{\pi}$ such that up to constant $\widetilde{f} = \overline{M(s)f}$ and $(f, \widetilde{f}) = 1$. In the case of the weight m vector $f = f_{m,s} \in \pi_{\delta,\epsilon}(s)$, this implies that $\widetilde{f} = f_{-m,1-s}$. We denote by $\widetilde{W}^{(j)}$ the image of $\widetilde{f}^{(j)}$ in $\mathcal{W}(\widetilde{\pi}_j, \psi)$ as above.

So, under the assumption that $f^{(j)}$ and $\widetilde{f}^{(j)}$ satisfy $(f^{(j)}, \widetilde{f}^{(j)}) = 1$, we see that the polarized form of Proposition 3.1 gives

$$I'_v(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{\ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(3)})\ell_{\mathrm{RS}}(\widetilde{W}^{(1)} \otimes \widetilde{W}^{(2)} \otimes \widetilde{f}^{(3)})}{(W^{(1)}, \widetilde{W}^{(1)})(W^{(2)}, \widetilde{W}^{(2)})}. \tag{3.3}$$

Following our convention for choosing $\widetilde{f}$ from f, the calculation of norms given in Proposition 2.2 gives the correct values for $(W, \widetilde{W})$ even in the case that π is not unitarizable. In the sequel, we will use this polarized form throughout.

Remark. For $f = f^{(1)} \otimes f^{(2)} \otimes f^{(3)}$, the trilinear from $\frac{I'(f)}{\langle f,f\rangle}$ is clearly invariant under scaling $f^{(j)}$ by a nonzero constant, hence in defining the particular choice of test vectors in the sequel (or equivalently in Proposition 2.1) the exact choice of scalar is not so important. We refer to a choice such that $\langle f, f\rangle = (f, \widetilde{f}) = 1$ as *normalized*.

For the remainder of this section we adopt the notation $\mathrm{wt}(\pi_j) = k_j$, and we assume that $k_1 \geq k_2 + k_3$. The condition on the central characters implies that $k_1 + k_2 + k_3$ is even.

3.1. **The case of three principal series.** We consider first the situation in which $\pi_j = \pi_{\delta_j,\epsilon_j}(\frac{1}{2} + \nu_j)$ for all $j = 1, 2, 3$ with $\delta_1 + \delta_2 + \delta_3$ even. Attached to such a triple we define $\epsilon \in \{0, 1\}$ be such that $\epsilon \equiv \epsilon_1 + \epsilon_2 + \epsilon_3 \pmod 2$.

Proposition 3.2. *Let π_j be principal series representations with ϵ as above. Then we may arrange that $(\delta_1, \epsilon_1) = (0, \epsilon)$ and $(\delta_j, \epsilon_j) = (\delta, \epsilon')$ for $j = 2, 3$. If $\delta = 0$ there exists a choice of normalized test vectors $f^{(j)} \in \pi_j$ such that*

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \left(\frac{4\pi^4}{\lambda_j}\right)^{\epsilon} \tag{3.4}$$

where λ_j is the eigenvalue of the Laplace-Beltrami operator on π_j for either $j = 2$ or $j = 3$, i.e. $\lambda_j = \frac{1}{4} - \nu_3^2$.

When $\delta = 1$ there is a choice of normalized test vectors such that $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$.

Remark. Strictly speaking our proof is only valid for parameters ν_j such that certain integrals of the type (2.9) and (2.10) are convergent. To get the more general result one must employ analytic continuation. When the parameters correspond to unitary representations, the proof below is complete.

Proof. Suppose first that $\delta = 0$. Let $W^{(1)} = W_0$,

$$W^{(2)} = \begin{cases} W_0 & \text{if } \epsilon = 0 \\ \frac{W_{-2} - W_2}{\sqrt{2}} & \text{if } \epsilon = 1, \end{cases} \qquad \text{and} \qquad f^{(3)} = \begin{cases} f_0 & \text{if } \epsilon = 0 \\ \frac{f_{-2} + f_2}{\sqrt{2}} & \text{if } \epsilon = 1. \end{cases}$$

Note that with these choices the restriction of $W^{(1)} W^{(2)} f^{(3)}$ to A is an even function. This is because for *any* $f \in \pi_3$ the restriction to A satisfies $f(a(y)) = c \operatorname{sgn}(y)^{\epsilon} |y|^{\frac{1}{2} + \nu_3}$ for some constant c.

By Proposition 2.2 and the remark following it $\langle W^{(j)}, W^{(j)} \rangle = \pi \Gamma(\frac{1}{2} + \nu_j) \Gamma(\frac{1}{2} - \nu_j)$ for $j = 1, 2$ and any choice of ϵ, ϵ'. Also, note that $\langle f^{(3)}, f^{(3)} \rangle = 1$ in any case.

We claim that

$$\ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(3)}) = \frac{2^{\epsilon + 2\nu_3} \sqrt{\pi}}{(\frac{1}{2} + \nu_3)^{\epsilon}} \frac{\prod_{\gamma_j \in \pm} \Gamma(\frac{1+2\epsilon}{2} + \frac{\gamma_1 \nu_1 + \gamma_2 \nu_2 + \nu_3}{2})}{\Gamma(\frac{1+2\epsilon}{2} + \nu_3)}. \tag{3.5}$$

We verify this in the case of $\epsilon = 1$ by computing

$$\begin{aligned} \ell_{\mathrm{RS}}(W^{(1)} \otimes W^{(2)} \otimes f^{(3)}) &= \sqrt{\pi} \int_{\mathbb{R}^\times} (W_0 W_{-2} f_2 - W_0 W_2 f_{-2})(a(y)) |y|^{-1} d^\times y \\ &= \sqrt{\pi} \int_{\mathbb{R}^\times} W_0 (W_{-2} - W_2)(a(y)) \operatorname{sgn}(y) |y|^{-\frac{1}{2} + \nu_3} d^\times y \\ &= 2\sqrt{\pi} \int_0^\infty y^{\frac{3}{2} + \nu_3} K_{\nu_1}(y/2) K_{\nu_2}(y/2) \ d^\times y \\ &= 2^{1 + 2\nu_3} \sqrt{\pi} \frac{\prod_{\gamma_j \in \pm} \Gamma(\frac{3}{4} + \frac{\gamma_1 \nu_1 + \gamma_2 \nu_2 + \nu_3}{2})}{\Gamma(\frac{3}{2} + \nu_3)} \\ &= \frac{2^{1 + 2\nu_3} \sqrt{\pi}}{\frac{1}{2} + \nu_3} \frac{\prod_{\gamma_j \in \pm} \Gamma(\frac{3}{4} + \frac{\gamma_1 \nu_1 + \gamma_2 \nu_2 + \nu_3}{2})}{\Gamma(\frac{1}{2} + \nu_3)}. \end{aligned}$$

The other case is similar.

Computing $\ell_{\mathrm{RS}}(\widetilde{W}^{(1)} \otimes \widetilde{W}^{(2)} \otimes \widetilde{f}^{(3)})$ as above has the net effect of giving exactly the same result except with ν_3 replaced by $-\nu_3$. Thus, combining (3.5) with Proposition 2.2, we see that (3.3) now gives

$$I'_v(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \left(\frac{4}{\lambda_3} \right)^{\epsilon} \frac{\prod_{\gamma_j = \pm} \Gamma(\frac{1+2\epsilon}{4} + \frac{\gamma_1 \nu_1 + \gamma_2 \nu_2 + \gamma_3 \nu_3}{2})}{\pi \prod_{j=1}^{3} \Gamma(\frac{1}{2} + \nu_j) \Gamma(\frac{1}{2} - \nu_j)}.$$

Finally, we divide by the normalizing factor in Table A.2 for Π^1 and thus obtain the desired result.

Now suppose that $\delta = 1$. We choose $f^{(1)} = f_{0, \frac{1}{2} + \nu_1}$,

$$f^{(2)} = \frac{f_{1, \frac{1}{2} + \nu_2} - (-1)^{\epsilon} f_{-1, \frac{1}{2} + \nu_2}}{\sqrt{2}}, \quad \text{and} \quad f^{(3)} = \frac{f_{1, \frac{1}{2} + \nu_3} - f_{-1, \frac{1}{2} + \nu_3}}{\sqrt{2}}.$$

Thus $W^{(1)} = W_0$ and $W^{(2)} = \frac{W_1 - (-1)^{\epsilon} W_{-1}}{\sqrt{2}}$. Note again that having made these choices the product $W^{(1)} W^{(2)} f^{(3)}$ has the property that its restriction to A is an even function.

By a computation very similar to that above, we find

$$\begin{aligned}\ell_{\mathrm{RS}}(W^{(1)}\otimes W^{(2)}\otimes f^{(3)}) &= \sqrt{\pi}\int_{\mathbb{R}^\times}\big(W_0W_1f_{-1}+(-1)^\epsilon W_{-1}f_1\big)(a(y))|y|^{-1}\ d^\times y\\ &= \sqrt{\pi}\int_{\mathbb{R}^\times}\big(W_0(W_1+(-1)^\epsilon W_{-1})\big)(a(y))\,\mathrm{sgn}(y)|y|^{\nu_3-\frac12}\ d^\times y\\ &= 2\sqrt{\pi}\int_0^\infty y^{1+\nu_3}K_{\nu_1}(y/2)K_{\nu_2-\frac12+\epsilon}(y/2)\ d^\times y\\ &= \sqrt{\pi}2^{2\nu_3}\frac{\prod_{\gamma\in\pm}\Gamma(\frac{1+2\epsilon}{2}+\frac{\gamma\nu_1+\nu_2+\nu_3}{2})\Gamma(\frac{3-2\epsilon}{2}+\frac{\gamma\nu_1-\nu_2+\nu_3}{2})}{\Gamma(1+\nu_3)}.\end{aligned}$$

Multiplying by the appropriate polarizing factor, and dividing by the appropriate norms as before, we find that

$$I'(f^{(1)}\otimes f^{(2)}\otimes f^{(3)}) = \frac{\prod_{\gamma_j\in\pm}\Gamma(\frac{1+2\epsilon}{2}+\frac{\gamma_1\nu_1+\gamma_2(\nu_2+\nu_3)}{2})\Gamma(\frac{3-2\epsilon}{2}+\frac{\gamma_1\nu_1+\gamma_2(\nu_2-\nu_3)}{2})}{\pi\prod_{\gamma\in\pm}\Gamma(\frac12+\gamma\nu_1)\Gamma(1+\gamma\nu_2)\Gamma(1+\gamma\nu_3)}.$$

Since this agrees with the corresponding factor in Table A.2 it follows that the normalized trilinear form satisfies $I(f^{(1)}\otimes f^{(2)}\otimes f^{(3)})=1$ as claimed. □

Remark. As discussed in the introduction, the case of three weight zero principal series representations was treated by Watson in [Wat01] but only in the case $\delta=0$. We remark that the way in which he uses ϵ agrees with our notation. He did not give a test vector in the case that $\epsilon=1$, but showed that if one takes $f^{(j)}$ to be the weight zero vector for each of $j=1,2,3$, the resulting trilinear from will be zero. This is immediately evident from our method above, as the resulting function $\big(W^{(1)}W^{(2)}f^{(3)}\big)(a(y))$ will be an odd function of y.

3.2. **The case of two principal series and a discrete series.** Note that this case was worked out in [SZW13] when k is even and both principal series are weight zero. We extend the result here to arbitrary k and allow that the principal series be odd.

Proposition 3.3. *Suppose that $\pi_1=\pi_{\mathrm{dis}}^k$ and $\pi_j=\pi_{\delta_j,\epsilon_j}(\frac12+\nu_j)$ such that $k+\delta_2+\delta_3$ is even. Then if $\delta=\delta_1+\delta_2\le 1$ there exists a choice of normalized test vectors $f^{(j)}\in\pi_j$ such that*

$$I(f^{(1)}\otimes f^{(2)}\otimes f^{(3)}) = \frac{(2\pi)^{k-1}}{\pi^{\delta-1}}\frac{\Gamma(\frac{1+\delta_m}{2}+\nu_m)\Gamma(\frac{1+\delta_m}{2}-\nu_m)}{\Gamma(\frac{k+1}{2}+\nu_m)\Gamma(\frac{k+1}{2}-\nu_m)}, \tag{3.6}$$

where $\{\ell,m\}=\{2,3\}$ satisfies $\delta_\ell=0$.

Otherwise, (if $\delta_2=\delta_3=1$),

$$I(f^{(1)}\otimes f^{(2)}\otimes f^{(3)}) = 2\lambda_{k,j}(2\pi)^{k-2}\frac{\Gamma(1+\nu_j)\Gamma(1-\nu_j)}{\Gamma(\frac{k}{2}+1+\nu_j)\Gamma(\frac{k}{2}+1-\nu_j)}, \tag{3.7}$$

for $j=2$ or $j=3$ and $\lambda_{k,j}=(\frac{k}{2})^2-\nu_j^2$.

Proof. We arrange the representations and take test vectors such that

$$\begin{aligned}W^{(1)} &= W_k\in\mathcal{W}(\pi_{\mathrm{dis}}^k,\psi),\\ W^{(2)} &= W_{-\delta_2}^-\in\mathcal{W}(\pi_{\delta_2,\epsilon_2}(\tfrac12+\nu_2),\overline{\psi}),\\ f^{(3)} &= f_{-k+\delta_2,\frac12+\nu_3}\in\pi_{\delta_3,\epsilon_3}(\tfrac12+\nu_3).\end{aligned}$$

Since $W_k(a(y))$ is supported on $y>0$, we see that

$$\ell_{\mathrm{RS}}(W^{(1)}\otimes W^{(2)}\otimes f^{(1)}) = 2\sqrt{\pi}\int_0^\infty y^{\frac{k-1}{2}+\nu_3}e^{-y/2}W^-_{-\delta_2}(a(y))\ d^\times y$$

In the case that $\delta_2=0$, one follows the same procedure as in the proof of Proposition 3.2 to arrive at

$$I'(f^{(1)}\otimes f^{(2)}\otimes f^{(3)}) = \frac{4\pi}{(k-1)!}\frac{\prod_{\gamma_j\in\pm}\Gamma(\frac{k}{2}+\gamma_2\nu_2+\gamma_3\nu_3)}{\Gamma(\frac{k}{2}+\nu_3)\Gamma(\frac{k}{2}-\nu_3)\Gamma(\frac{1}{2}+\nu_2)\Gamma(\frac{1}{2}-\nu_2)},$$

from which (3.7) follows as before.

We now consider the case $\delta_2=1$, for which

$$W^-_{-1}(a(y)) = \frac{|y|}{2}\Big(K_{\nu_2-\frac{1}{2}}(|y|/2)+\mathrm{sgn}(y)K_{\nu_2+\frac{1}{2}}(|y|/2)\Big).$$

Therefore,

$$\begin{aligned}\ell_{\mathrm{RS}}(W^{(1)}\otimes W^{(2)}\otimes f^{(1)}) &= \sqrt{\pi}\int_0^\infty y^{\frac{k+1}{2}+\nu_3}e^{-y/2}\Big(K_{\nu_2-\frac{1}{2}}(|y|/2)+K_{\nu_2+\frac{1}{2}}(|y|/2)\Big)\ d^\times y\\ &= 2\pi(\frac{k}{2}+\nu_3)\frac{\prod_{\gamma_j\in\pm}\Gamma(\frac{k}{2}+\gamma_2\nu_2+\gamma_3\nu_3)}{\Gamma(\frac{k}{2}+1+\nu_3)},\end{aligned}$$

where in the final step we have used (2.11) and the functional equation $s\Gamma(s)=\Gamma(s+1)$ in order to simplify. Again polarizing this and dividing by the appropriate norms, this implies that

$$I'(f^{(1)}\otimes f^{(2)}\otimes f^{(3)}) = \frac{4\pi\lambda}{(k-1)!}\frac{\prod_{\gamma_j\in\pm}\Gamma(\frac{k}{2}+\gamma_2\nu_2+\gamma_3\nu_3)}{\prod_{\gamma\in\pm}\Gamma(\frac{k}{2}+1+\gamma\nu_3)\Gamma(\frac{1}{2}+\gamma\nu_2)},$$

where $\lambda=(\frac{k}{2})^2-\nu_3^2$. Dividing this by the appropriate normalizing factor from Table A.2 and simplifying gives (3.7) in the case of $j=3$. Switching the roles of π_2 and π_3 gives the other case. $\square$

3.3. **The case of one principal series and two discrete series.** We now assume that $\pi_j=\pi^{k_j}_{\mathrm{dis}}$ for $j=1,2$ and $\pi_3=\pi_{\delta,\epsilon}(\frac{1}{2}+\nu)$.

Proposition 3.4. *Let $\pi_j=\pi^{k_j}_{\mathrm{dis}}$ for $j=1,2$ with $k_1\geq k_2$. Let π_3 be a principal series representation of weight zero if k_1+k_2 is even and of weight one otherwise. Then there exists a choice of normalized test vectors $f^{(j)}\in\pi_j$ such that*

$$I(f^{(1)}\otimes f^{(2)}\otimes f^{(3)}) = \frac{(2\pi)^{k_1-k_2}}{\pi^\delta}\frac{\Gamma(\frac{1+\delta}{2}+\nu)\Gamma(\frac{1+\delta}{2}-\nu)}{\Gamma(\frac{k_1-k_2+1}{2}+\nu)\Gamma(\frac{k_1-k_2+1}{2}-\nu)}. \tag{3.8}$$

where $\lambda=\frac{1}{2}-\nu^2$ is the eigenvalue of Δ on π_3.

Proof. Let $\delta=\mathrm{wt}(\pi_3)$. We take $f^{(1)}$ to be the weight k_1 vector, $f^{(2)}$ the weight $-k_2$ vector and $f^{(3)}$ the weight k_2-k_1 vector. Then $W^{(1)}=W^+_{k_j}$ and $W^{(2)}=W^-_{-k_2}$. Since

$$\ell_{\mathrm{RS}}(W^{(1)}\otimes W^{(2)}\otimes f^{(3)}) = \sqrt{4\pi}\int_0^\infty y^{\frac{k_1+k_2}{2}-\frac{1}{2}+\nu}e^{-y}\ d^\times y = \sqrt{4\pi}\Gamma\left(\tfrac{k_1+k_2-1}{2}+\nu\right),$$

using (3.3) we find that

$$I'(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{4\pi\Gamma(\frac{k_1+k_2-1}{2}+\nu)\Gamma(\frac{k_1+k_2-1}{2}-\nu)}{(k_1-1)!(k_2-1)!}.$$

We again divide by the normalizing factor for Π^3 from Table A.2 to obtain the desired result. $\square$

3.4. **The case of three discrete series.** Let us assume that $\pi_j = \pi_{\mathrm{dis}}^{k_j}$ for $j = 1, 2$ and $\pi_3 = \pi_{\delta,0}(\frac{k_3}{2})$ where $\delta \in \{0, 1\}$ has the same parity as k_3, so that $\pi_{\mathrm{dis}}^{k_3} \subset \pi_3$. Note then that $\widetilde{\pi}_3 = \pi_{\delta,0}(1 - \frac{k_3}{2})$ which has $\pi_{\mathrm{dis}}^{k_3}$ as a quotient.

In this situation, the form ℓ_{RS} descends in fact to a trilinear form on $\pi_{\mathrm{dis}}^{k_1} \otimes \pi_{\mathrm{dis}}^{k_2} \otimes \pi_{\mathrm{dis}}^{k_3}$, and we will take as a hypothesis that the polarization of the form $|\ell_{\mathrm{RS}}|^2$ in fact gives the correct trilinear form on $\pi_{\mathrm{dis}}^{k_1} \otimes \pi_{\mathrm{dis}}^{k_2} \otimes \pi_{\mathrm{dis}}^{k_3}$. Note that this is unconditionally true if $k_3 = 1$, and in the special case that $k_1 = k_2 + k_3$ the answer that we obtain by this method is correct.

Proposition 3.5. *Let $\pi_j = \pi_{\mathrm{dis}}^{k_j}$ for $j = 1, 2, 3$ and $k_1 - (k_2 + k_3) = 2m \geq 0$. There exists a choice of normalized test vectors $f^{(j)} \in \pi_j$ such that*

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{2(2\pi)^{2m}}{\binom{k_3+m-1}{k_3-1}}. \tag{3.9}$$

Proof. As test vectors, we choose $f^{(1)}$ to be the weight k_j vector, $f^{(2)}$ to be the weight $-k_2$ vector, and $f^{(3)} = f_{k_2-k_1}$. Then the computation of ℓ_{RS} proceeds exactly as in the previous section but with $\nu = \frac{k_3-1}{2}$. This immediately implies that

$$I'(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = \frac{4\pi\Gamma(\frac{k_1+k_2+k_3}{2}-1)\Gamma(\frac{k_1+k_2-k_3}{2})}{(k_1-1)!(k_2-1)!}$$

in the case at hand. Note, however, that the normalizing factor of the previous section does not agree with that here unless $k_3 = 1$. Indeed, the triple product local L-factor $L(\frac{1}{2}, \Pi)$ for $\Pi = \pi_{\mathrm{dis}}^{k_1} \otimes \pi_{\mathrm{dis}}^{k_2} \otimes \pi_{\delta,0}(\frac{s}{2})$ does not specialize to that for $\Pi = \pi_{\mathrm{dis}}^{k_1} \otimes \pi_{\mathrm{dis}}^{k_2} \otimes \pi_{\mathrm{dis}}^{k_3}$ as $s \to k_3$. Moreover, the adjoint L-factor $L(s, \pi_{\delta,0}(\frac{k_3}{2}), \mathrm{Ad}\,) = L(s, \pi_{\delta,0}(1 - \frac{k_3}{2}), \mathrm{Ad}\,)$ has a pole at $s = \frac{1}{2}$ if $k_3 \geq 2$.

Dividing by the correct normalizing factor gives the result. $\square$

Remark. One may ask why the above proof doesn't also apply in the case that $k_1 < k_2 + k_3$. By [Pra90], the form must be zero in this case, although at first glance it may not appear to be so. However, it is easy to see from Proposition 2.1 that for weight m_j vectors $f^{(j)}$ if the form $\ell_{\mathrm{RS}}(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) \neq 0$ then the weights m_1 and m_2 must have opposite parity. Thus, in the special case that $m_1 = k_1$ and $m_2 = -k_2$, the vector $f^{(3)}$ has weight $-k_3 + m$ where $m = k_3 + k_2 - k_1$ which corresponds to a vector in $\pi_{\mathrm{dis}}^{k_3}$ if and only if $k_1 \geq k_2 + k_3$.

3.5. **Proof of Theorem 1.** The calculations of the previous sections cover all possible cases π_1, π_2, π_3 satisfying the hypotheses of Theorem 1, and in each case the corresponding test vectors $f^{(j)}$ are shown to satisfy $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) \neq 0$.

In particular, if all three representations are principal series, then $k_1 = k_2 = \delta$ and $k_3 = 0$. In the case that $\delta = 0$ if we assume moreover that $\epsilon = 0$, from Proposition 3.2 one sees by (3.4) that $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$. In the case $\delta = 1$ Proposition 3.2 says immediately that $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$.

If exactly one of the representations is a discrete series then either $k_1 = 1$ and the other representations are a weight 1 and a weight 0 principal series, or $k_1 = 2$ and both of the other representations are weight 1 principal series. In the first case, the result follows either from specializing (3.6) to the case $k = \delta = 1$ or from Proposition 3.3 in the case that $k = 1$, $\delta_2 = 0$ and $\delta_3 = 1$. In the latter case one applies (3.7) with $k = 2$, from which we obtain

$$I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 2\lambda_{2,j} \frac{\Gamma(1+\nu_j)\Gamma(1-\nu_j)}{\Gamma(2+\nu_j)\Gamma(2-\nu_j)} = 2,$$

since $\Gamma(2+\nu_j)\Gamma(2-\nu_j) = (1-\nu_j^2)\Gamma(1+\nu_j)\Gamma(1-\nu_j) = \lambda_{2,j}\Gamma(1+\nu_j)\Gamma(1-\nu_j)$.

If two of the representations are discrete series, then we may assume that π_3 is a principal series of weight $k_3 = \delta \in \{0,1\}$ and $k_1 = k_2 + \delta > 1$. This case corresponds to Proposition 3.4. Specializing (3.8) gives $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 1$ if $\delta = 0$, and $I(f^{(1)} \otimes f^{(2)} \otimes f^{(3)}) = 2$ if $\delta = 1$, as claimed.

Finally, the case that all three representations are discrete series is treated in Proposition 3.5. The assumption that $k_1 = k_2 + k_3$ means that $m = 0$ in which case the right hand side of equation 3.9 is obviously 2. □

Appendix A. Normalizing L-factors

The goal of this appendix is to record the normalizing L-factors for the triple product L-function appearing in (1.2). These factors are determined by applying the local Langlands correspondence relating finite dimensional semisimple representations of the Weil group $W_{\mathbb{R}}$ to admissible representations of $\mathrm{GL}_2(\mathbb{R})$ as detailed in [Kna94]. The local factors will be described in terms of

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \text{and} \qquad \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s).$$

We recall the following elementary facts:

$$\overline{\Gamma(s)} = \Gamma(\bar{s}), \quad \Gamma_{\mathbb{R}}(1) = 1, \quad \Gamma_{\mathbb{R}}(2) = \frac{1}{\pi}, \quad \Gamma_{\mathbb{C}}(m) = \frac{(m-1)!}{2^{m-1}\pi^m}.$$

A.1. **Local Langlands parameters for** $\mathrm{GL}_2(\mathbb{R})$**.** We recall briefly the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{R})$. (See [Kna94] for complete details.) Let $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$ be the Weil group. For $\delta \in \{0,1\}$ and $t \in \mathbb{C}$, we have the 1-dimensional representation of $W_{\mathbb{R}}$ given by

$$\rho_1(\delta,t): \begin{array}{c} z \mapsto |z|^t \\ j \mapsto (-1)^\delta. \end{array}$$

Moreover, if $m \in \mathbb{Z}$ and $t \in \mathbb{C}$ we have the 2-dimensional representation

$$\rho_2(m,t): \begin{array}{c} re^{i\theta} \mapsto \begin{pmatrix} r^{2t}e^{im\theta} & 0 \\ 0 & r^{2t}e^{-im\theta} \end{pmatrix} \\ j \mapsto \begin{pmatrix} 0 & (-1)^m \\ 1 & 0 \end{pmatrix} \end{array},$$

which is easily checked to be irreducible except when $m = 0$. The following is a simple exercise.

Lemma A.1. *Every semisimple finite-dimensional representation of $W_{\mathbb{R}}$ is a direct sum of irreducibles of the type ρ_1 and ρ_2 as defined above. Under the operations of direct sum and tensor product, the following is a complete set of relations.*

$$\begin{aligned}
\rho_2(m,t) &\simeq \rho_2(-m,t)\\
\rho_2(0,t) &\simeq \rho_1(0,t)\oplus\rho_1(1,t)\\
\rho_1(\delta_1,t_1)\otimes\rho_1(\delta_2,t_2) &\simeq \rho_1(\delta,t_1+t_2) \qquad (\delta\equiv\delta_1+\delta_2 \pmod 2)\\
\rho_1(\delta,t_1)\otimes\rho_2(m,t_2) &\simeq \rho_2(m,t_1+t_2)\\
\rho_2(m_1,t_1)\otimes\rho_2(m_2,t_2) &\simeq \rho_2(m_1+m_2,t_1+t_2)\oplus\rho_2(m_1-m_2,t_1+t_2).
\end{aligned}$$

Moreover, if $\widetilde{\rho}$ denotes the contragradient of ρ then

$$\widetilde{\rho_1(\delta,t)} \simeq \rho_1(\delta,-t), \qquad \textit{and} \qquad \widetilde{\rho_2(m,t)} \simeq \rho_2(m,-t).$$

Given an irreducible admissible representation π of $\mathrm{GL}_2(\mathbb{R})$ we associate to it a representation $\rho(\pi)$ of $W_{\mathbb{R}}$. For example, if $\mathcal{B}(\chi_1,\chi_2)=\mathcal{B}(\mathrm{sgn}^{\epsilon_1}|\cdot|^{s_1},\mathrm{sgn}^{\epsilon_2}|\cdot|^{s_2})$ is irreducible the corresponding representation of $W_{\mathbb{R}}$ is $\rho_1(\epsilon_1,s_1)\oplus\rho_1(\epsilon_2,s_2)$. We record how this correspondence works in Table A.1 for representations with central character sgn^{δ}. (We let $\overline{\delta+\epsilon}\in\{0,1\}$ be the reduction of $\epsilon+\delta$ modulo 2.) Note that the third column of the table is calculated using Lemma A.1 and the identity

$$\mathrm{Ad}(\rho)\simeq\rho\otimes\widetilde{\rho}\ominus\rho_1(0,0).$$

π	$\rho(\pi)$	$\mathrm{Ad}(\rho(\pi))$
$\pi_{\delta,\epsilon}(\frac{1}{2}+\nu)$	$\rho_1(\epsilon,\nu)\oplus\rho_1(\overline{\delta+\epsilon},-\nu)$	$\rho_1(0,0)\oplus\rho_1(\delta,2\nu)\oplus\rho_1(\delta,-2\nu)$
π_{dis}^{k}	$\rho_2(k-1,0)$	$\rho_1(1,0)\oplus\rho_2(2k-2,0)$

TABLE A.1. Representations of $W_{\mathbb{R}}$ attached to admissible unitary representations of $\mathrm{GL}_2(\mathbb{R})$

A.2. **Triple product and adjoint L-factors.** We associate to each of $\rho_1(\delta,t)$ and $\rho_2(m,t)$ the L-functions

$$L(s,\rho_1(\delta,t))=\Gamma_{\mathbb{R}}(s+\delta+t), \qquad L(s,\rho_2(m,t))=\Gamma_{\mathbb{C}}(s+\tfrac{m}{2}+t). \tag{A.1}$$

More generally, given $\rho\simeq\rho_1\oplus\rho_2\oplus\cdots\oplus\rho_r$ a (semisimple) representation of $W_{\mathbb{R}}$ with ρ_j irreducible we define

$$L(s,\rho)=\prod_{j=1}^{r}L(s,\rho_j).$$

Using this definition it follows, setting $L(s,\pi,\mathrm{Ad})=L(s,\mathrm{Ad}(\rho(\pi)))$ and combining (A.1) with Table A.1, that

$$L(1,\pi,\mathrm{Ad}\,))=\begin{cases}\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(k) & \text{if } \pi=\pi_{\mathrm{dis}}^{k}\\ \dfrac{\Gamma(\frac{1+\delta}{2}+\nu)\Gamma(\frac{1+\delta}{2}-\nu)}{\pi^{1+\delta}}, & \text{if } \pi=\pi_{\delta,\epsilon}(\frac{1}{2}+\nu).\end{cases} \tag{A.2}$$

Recall that we are considering admissible representations π_1,π_2,π_3 of $\mathrm{GL}_2(\mathbb{R})$ such that $\Pi=\pi_1\otimes\pi_2\otimes\pi_3$ has trivial central character. This means that we may assume without loss of generality the central character of each π_j is of the form sgn^{δ_j} with $\delta_1+\delta_2+\delta_3 \equiv 0 \pmod 2$.

Proposition A.2. *Consider $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ a triple product of admissible $\mathrm{GL}_2(\mathbb{R})$ representations. Defining*

$$L(s,\Pi) = L(s, \rho(\pi_1) \otimes \rho(\pi_2) \otimes \rho(\pi_3)),$$

and

$$L(s,\Pi,\mathrm{Ad}\,) = L(s, \mathrm{Ad}(\rho(\pi_1)) \otimes \mathrm{Ad}(\rho(\pi_2) \otimes \mathrm{Ad}(\rho(\pi_3)),$$

the normalizing factors relating I to I' in (1.2) *for each possible choice of π_1, π_2, π_3 are given by Table A.2 where*

$$\begin{aligned}
&\Pi^1 = \pi_{0,\epsilon}(\tfrac{1}{2}+\nu_1) \otimes \pi_{\delta,\epsilon'}(\tfrac{1}{2}+\nu_2) \otimes \pi_{\delta,\epsilon'}(\tfrac{1}{2}+\nu_3), && \\
&\Pi^2 = \pi^{k}_{\mathrm{dis}} \otimes \pi_{\delta_2,\epsilon_2}(\tfrac{1}{2}+\nu_2) \otimes \pi_{\delta_3,\epsilon_3}(\tfrac{1}{2}+\nu_3), && \\
&\Pi^3 = \pi^{k_1}_{\mathrm{dis}} \otimes \pi^{k_2}_{\mathrm{dis}} \otimes \pi_{\delta,\epsilon}(\tfrac{1}{2}+\nu) && \textit{(with } k_1 \geq k_2+\delta\textit{)}, \\
&\Pi^4 = \pi^{k_1}_{\mathrm{dis}} \otimes \pi^{k_2}_{\mathrm{dis}} \otimes \pi^{k_3}_{\mathrm{dis}} && \textit{(with } k_1 \geq k_2+k_3\textit{)}.
\end{aligned}$$

Π	$\dfrac{\Gamma_{\mathbb{R}}(2)^2 L(\frac{1}{2},\Pi)}{L(1,\Pi,\mathrm{Ad}\,)}$
Π^1	$\dfrac{\prod_{\gamma_j\in\pm}\Gamma(\frac{1+2\epsilon}{4}+\frac{\gamma_1\nu_1+\gamma_2(\nu_2+\nu_3)}{2})\Gamma(\frac{1+2\epsilon+2\delta(1-2\epsilon)}{4}+\frac{\gamma_1\nu_1+\gamma_2(\nu_2+\nu_3)}{2})}{\pi^{1+4\epsilon(1-\delta)}\prod_{j=1}^{3}\Gamma(\frac{1+\delta_j}{2}+\nu_j)\Gamma(\frac{1+\delta_j}{2}-\nu_j)}$
Π^2	$\dfrac{(2\pi)^{3-k}\pi^{\delta_2+\delta_3-2}}{(k-1)!}\dfrac{\prod_{\gamma_j=\pm1}\Gamma(\frac{k}{2}+\gamma_2\nu_2+\gamma_3\nu_3)}{\prod_{j=1}^{2}\Gamma(\frac{1+\delta_j}{2}+\nu_j)\Gamma(\frac{1+\delta_j}{2}-\nu_j)}$
Π^3	$\dfrac{2\pi^{\delta}}{(2\pi)^{k_1-k_2-1}}\dfrac{\prod_{\gamma\in\pm}\Gamma(\frac{k_1+k_2-1}{2}+\gamma\nu)\Gamma(\frac{k_1-k_2+1}{2}+\gamma\nu)}{(k_1-1)!(k_2-1)!\Gamma(\frac{1+\delta}{2}+\nu)\Gamma(\frac{1+\delta}{2}-\nu)}$
Π^4	$\dfrac{\Gamma(\frac{k_1+k_2+k_3}{2}-1)\Gamma(\frac{k_1+k_2-k_3}{2})\Gamma(\frac{k_1-k_2+k_3}{2})\Gamma(\frac{k_1-k_2-k_3}{2}+1)}{(2\pi)^{k_1-k_2-k_3-1}(k_1-1)!(k_2-1)!(k_3-1)!}$

TABLE A.2. Normalizing factors for triple product L-function at a real place

Proof. A simple exercise in applying Lemma A.1 gives the following.

$$\rho(\Pi^1) = \left(\bigoplus_{\gamma_j \in \pm} \rho_1(\epsilon, \gamma_1\nu_1 + \gamma_2(\nu_2 + \nu_3))\right) \oplus \left(\bigoplus_{\gamma_j \in \pm} \rho_1(\overline{\epsilon + \delta}, \gamma_1\nu_1 + \gamma_2(\nu_2 - \nu_3))\right)$$

$$\rho(\Pi^2) = \bigoplus_{\gamma_j \in \pm} \rho_2(k-1, \gamma_2\nu_2 + \gamma_3\nu_3)$$

$$\rho(\Pi^3) = \rho_2(k_1 + k_2 - 2, \nu) \oplus \rho_2(k_1 + k_2 - 2, \nu) \oplus \rho_2(k_1 - k_2, -\nu) \oplus \rho_2(k_1 - k_2, -\nu)$$

$$\begin{aligned}\rho(\Pi^4) = {} & \rho_2(k_1 + k_2 + k_3 - 3, 0) \oplus \rho_2(k_1 + k_2 - k_3 - 1, 0) \\ & \oplus \rho_2(k_1 - k_2 + k_3 - 1, 0) \oplus \rho_2(k_1 - k_2 - k_3 + 1, 0)\end{aligned}$$

Combining each of these with the appropriate factors for $L(1, \pi, \mathrm{Ad}\,)$ from (A.2) together with $\Gamma_{\mathbb{R}}(2)^2$ gives the result. □

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