NOTES ON EISENSTEIN SERIES: ADELIC VS. CLASSICAL FORMULATION

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These notes discuss the adelic interpretation of Eisenstein series and relate it to the classical definitions.

Let F be a number field, $\mathbb{A}$ its ring of ideles, $G = PK$ an algebraic group with K maximal compact. (e.g. $G = \mathrm{GL}_n$, P the set upper triangular matrices and $K = O(n)$.) Let Δ be the modulus character of P. If χ is a representation of $P(\mathbb{A})$ we form

$$\begin{aligned} I(\chi) &= \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\Delta^{1/2}\chi) \\ &= \left\{ f : G(\mathbb{A}) \to \mathbb{C} \mid f(pg) = \Delta(p)^{1/2}\chi(p)f(g) \text{ for all } p \in P, g \in G \right\} \end{aligned}$$

(We usually work with f satisfying some other conditions. e.g. continuous, K-finite, etc.) This is the *unitary induction of* χ.

Suppose that $f \in I(\chi)$. We can form from this the Eisenstein series

$$E(f, \chi, g) = \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma g).$$

Supposing that the sum is convergent, this is obviously *automorphic*. This means that $E(f, \gamma g) = E(f, g)$ for all $\gamma \in G(F)$. In other words, it is a function on $G(F)\backslash G(\mathbb{A})$.

Example

Let $F = \mathbb{Q}$, $G = \mathrm{GL}_2$, and P be the standard Borel. Define

$$\chi : P(\mathbb{A}) \to \mathbb{C} \qquad \chi(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right)) = |a_1|^{s_1} |a_2|^{s_2}.$$

Since $\Delta(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right)) = \left|\frac{a_1}{a_2}\right|$, we have that $I(\chi) = I(s_1, s_2)$ consists of functions transforming according to

$$f(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) g) = |a_1|^{s_1+1/2} |a_2|^{s_2-1/2} f(g). \tag{1}$$

Note that for $a = (a_v) \in \mathbb{A}$, we are defining $|a| = \prod_v |a_v|_v$. Therefore, by the product formula, if $\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) \in P(\mathbb{Q})$ then $(\Delta^{1/2}\chi)(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right)) = 1$.

Lemma 1. *The space* $P(\mathbb{Q})\backslash G(\mathbb{Q})$ *is in bijection with* $\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$ *where* $\Gamma_\infty = P(\mathbb{Q}) \cap \mathrm{SL}_2(\mathbb{Z})$.

Proof. We claim that both sets are in bijection with $\mathbb{P}^1(\mathbb{Q})$. Indeed, we may identify $\mathbb{P}^1(\mathbb{Q})$ with row vectors $(a\ b)$ modulo the usual equivalence relation. Then the action of $G(\mathbb{Q})$ on $e_1 = (0\ 1)$ by multiplication on the right is transitive with stabilizer $P(\mathbb{Q})$. Similarly, the action of $\mathrm{SL}_2(\mathbb{Z})$ is transitive with stabilizer Γ_∞. □

Let $z = x + iy \in \mathbb{H}$, and $g_z = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}$ so that $g_z \cdot i = z$. We consider g_z as an element of $\mathrm{GL}_2(\mathbb{A})$. This means that at the infinite place it is $\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}$ and it is the identity at all other places.

We define $f = \otimes f_v$ as follows. Note that the Iwasawa decomposition implies that $G(\mathbb{Q}_p) = P(\mathbb{Q}_p)\mathrm{GL}_2(\mathbb{Z}_p)$. This implies that $f_p \in \mathrm{Ind}_{P(\mathbb{Q}_p}^{G(\mathbb{Q}_p)}(\Delta_p^{1/2}\chi_p)$ is determined by its restriction to $\mathrm{GL}_2(\mathbb{Z}_p)$. With this in mind, let f_p be the unique such function such that its restriction to $\mathrm{GL}_2(\mathbb{Z}_p)$ is the characteristic function of

$$K_0(p^{\mathrm{ord}_p(N)}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(\mathbb{Z}_p) \mid c \equiv 0 \pmod{p^{\mathrm{ord}_p(N)}} \right\}.$$

In the real case $v = \infty$, the Iwasawa decomposition is $G(\mathbb{R}) = P(\mathbb{R})\mathrm{SO}(2)$. Analogous to the above, we let $f_\infty \in \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\Delta_\infty^{1/2}\chi_\infty)$ such that

$$f(gk_\theta) = e^{ik\theta}f(g), \qquad \text{and} \qquad f(e) = 1$$

for $\kappa_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right) \in \mathrm{SO}(2)$ and k a non-negative even integer.

As a result of our definition for f, we have that if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in G(\mathbb{Q})$ then $f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g_z) = 0$ unless $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N)$.

Finally set $s = s_1 = -s_2$. We'd like to calculate $E(f, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g_z)$, and show that it is a classical Eisenstein series for a particular choice of s.

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{R})$, we write

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g_z = \begin{pmatrix} y_1^{1/2} & x_1 y_1^{-1/2} \\ & y_1^{-1/2} \end{pmatrix} k_\theta.$$

Such a decomposition is possible since $\mathrm{SL}_2(\mathbb{R}) = P^1(\mathbb{R})SO_2(\mathbb{R})$ where $P^1(\mathbb{R}) = P(\mathbb{R}) \cap \mathrm{SL}_2(\mathbb{R})$. Hence

$$f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g_z) = f(\begin{pmatrix} y_1^{1/2} & x_1 y_1^{-1/2} \\ & y_1^{-1/2} \end{pmatrix} k_\theta) = y_1^{s+1/2}e^{ik\theta} \tag{2}$$

To calculate y_1 and $e^{ik\theta}$, we consider the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g_z$ on $\mathbb{H}$. In particular,

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g_z \cdot i = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot z = \frac{az+b}{cz+d} = \frac{iy}{|cz+d|^2} + \mathrm{Re}(\frac{az+b}{cz+d}).$$

On the other hand $\begin{pmatrix} y_1^{1/2} & x_1 y_1^{-1/2} \\ & y_1^{-1/2} \end{pmatrix} k_\theta \cdot i = x_1 + iy_1$. Hence $y_1 = \frac{y}{|cz+d|^2}$.

Also k_θ must be equal to

$$\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right) = \begin{pmatrix} y_1^{-1/2} & -x_1 y_1^{1/2} \\ & y_1^{1/2} \end{pmatrix} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g_z = \begin{pmatrix} * & * \\ c(yy_1)^{1/2} & (cx+d)y^{-1/2}y_1^{1/2} \end{pmatrix},$$

which implies that

$$e^{i\theta} = \cos\theta + i\sin\theta = \frac{cx+d}{|cz+d|} - \frac{icy}{|cz+d|} = \frac{c\bar{z}+d}{|cz+d|} = \frac{|cz+d|}{cz+d}.$$

Putting together these formulas,

$$E(f,s,g) = \sum_{\Gamma_\infty \backslash \Gamma_0(N)} \frac{y^{s+1/2}}{(cz+d)^k \, |cz+d|^{2s+1-k}}$$

which, upon setting $s = (k-1)/2$, is exactly equal to $y^{k/2}E_k(z)$ where $E_k(z)$ is the classical Eisenstein series of weight k.

Note: it is straightforward to see that $E(f,s,g)$ is absolutely convergent whenever $\operatorname{Re} s > 1$ (for arbitrary f,g) which is consistent with the fact that $E_k(z)$ is convergent only for $k>2$.

1. Whittaker coefficients

Generally speaking, if is a generic representation then one can define a Whittaker model. We work with $G = \mathrm{GL}_2(F)$ for F a number field for which every irreducible automorphic representation is generic. Let π be such a representation and $f \in \pi$. Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a nontrivial character. We define

$$W_f^\psi(g) = \int_{F\backslash\mathbb{A}} f(n(b)g)\psi(-b)db$$

where $n(b) = \left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right)$. This gives a map

$$\pi \to \mathscr{W}(\pi,\psi) = \{W : G \to \mathbb{C} \mid W(n(b)g) = \psi(b)W(g)\}$$

which is a G-isomorphism onto its image.

The function $f(g)$ may be reconstructed from W_f^ψ:

$$f(g) = \sum_{\alpha\in F} W_f^\psi(\left(\begin{smallmatrix} \alpha & \\ & 1\end{smallmatrix}\right)g).$$

The *Whittaker coefficients* are the functions $W_f^\psi(\left(\begin{smallmatrix} \alpha & \\ & 1\end{smallmatrix}\right)g)$. Notice that the Whittaker coefficient

$$\begin{aligned} W_f^\psi(\left(\begin{smallmatrix} \alpha & \\ & 1\end{smallmatrix}\right)g) &= \int_{\mathbb{Q}\backslash\mathbb{A}} f(\left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & \\ & 1\end{smallmatrix}\right)g)\psi(-b)db \\ &= \sum_{\alpha\in F}\int_{F\backslash\mathbb{A}} f(\left(\begin{smallmatrix} \alpha & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \alpha^{-1}b \\ & 1\end{smallmatrix}\right)g)\psi(-b)db \\ &= \int_{F\backslash\mathbb{A}} f(\left(\begin{smallmatrix} \alpha & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right)g)\psi_\alpha(-b)db = W_f^{\psi_\alpha}(g). \end{aligned}$$

because $\alpha \in F$. (This implies that $f(\left(\begin{smallmatrix} \alpha & \\ & 1\end{smallmatrix}\right)g) = f(g)$ and that $d(\alpha b) = |\alpha|\,db = db$.)

1.1. Connection with Fourier coefficients.

We now describe how these are related to classical Fourier coefficients. Let $F=\mathbb{Q}$ and let $\widetilde{f}$ be a classical modular form of level N and weight k. We obtain an automorphic form f via the rule

$$f(gkg_\infty) = \widetilde{f}\,|_i[g_\infty]$$

where $g_f \in \mathrm{GL}_2(\mathbb{Q})$, $k \in K_0(N)$ and $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})$ and

$$\widetilde{f}\,\big|_k\big[\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\big]\;(z) = (cz+d)^k\widetilde{f}(\frac{az+b}{cz+d}).$$

Let $\psi = \otimes\psi_v$ be defined by $\psi_\infty(x) = e^{2\pi i x}$ and $\psi_p(x) = e^{-2\pi i\lambda(x)}$ where λ is the composition of the maps

$$\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

Since $\mathbb{A} = \mathbb{Q}\cdot\widehat{\mathbb{Z}}\cdot\mathbb{R}$ and $\mathbb{Q}$ has class number 1 we have

$$\mathbb{Q}\backslash\mathbb{A} = \widehat{\mathbb{Z}}\times\mathbb{Z}\backslash\mathbb{R}.$$

Finally let $g_\tau = \left(\begin{smallmatrix} v & u \\ & 1\end{smallmatrix}\right)$ where $\tau = u+iv \in \mathbb{H}$.

Therefore,

$$\begin{aligned} W_f^{\psi_\alpha}(g_\tau) &= \int_{\mathbb{Q}\backslash\mathbb{A}} f(n(b)g_\tau)\psi_\alpha(-b)db \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} \int_{\widehat{\mathbb{Z}}} f(n(b_\infty)n(\hat{b})g_\tau)\psi_\alpha(-b)d\hat{b}db_\infty \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} f(n(b_\infty)n(\hat{b})g_\tau)\psi_\alpha(-b_\infty)db_\infty. \end{aligned}$$

since f is right invariant by $K_0(N) \ni n(\hat{b})$ and $\psi(-\alpha\hat{b}) = 1$. Now, writing this in terms of $\widetilde{f}$, yields

$$\begin{aligned} W_f^{\psi_\alpha}(g_\tau) &= \int wtf\,|_k[g_\tau n(b_\infty)]\,(i)\psi(-\alpha b\infty)db_\infty \\ &= \int_0^1 \widetilde{f}(\tau+y)e^{2\pi i\alpha\tau}e^{2\pi i\alpha y}dy. \end{aligned}$$

Since

$$W_f^{\psi_\alpha}(g_\tau) = \int_{\mathbb{Z}\backslash\mathbb{R}} \widetilde{f}(\tau+y+1)e^{2\pi i\alpha}e^{2\pi i\alpha(\tau+y)}dy = e^{2\pi i\alpha}W_f^{\psi_\alpha}(g_\tau),$$

we see that this is nonzero only if $\alpha \in \mathbb{Z}$, and

$$W_f^{\psi_n}(g_\tau) = a_n(\widetilde{f})q^n$$

where $a_n(\widetilde{f})$ is the nth Fourier coefficient.

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