AUTOMORPHIC FORMS ON GL_2 – JACQUET-LANGLANDS

Contents

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These notes were derived primarily from a course taught by Yannan Qiu at the University of Wisconsin in Spring of 2008. Briefly, the course was an introduction to the book of Jacquet and Langlands on Automorphic forms on GL_2. I am greatly in debt to Yannan for teaching the class, and to Rob Rhoades who typed most of the original lecture notes. Additional details beyond Rob's notes (including solutions to many of the exercises) are included, and some additions and restructuring of the material has been made.

The main reference for the course (and these notes) is Jacquet and Langlands book [5]. Some other references are Bump's book [2] and Jacquet's book [3]. Bump's book is easier to read but the real material is in [5]. Jacquet's book develops theory of GL_n automorphic forms.

1. Introduction and motivation

This course will be about L-functions and automorphic forms. There are two sorts of L-functions.

(1) Artin L-functions: Suppose E/F is a finite extension of number fields and $\mathrm{Gal}(E/F)$ acts on a finite dimensional vector space over $\mathbb{C}$ by ρ. Then we get $L(s,\rho)$.
(2) Automorphic L-function: Let G be a reductive group over F and π an automorphic representation. Then we get $L(s,\pi)$.

The *Langlands Philosophy* says that every Artin L-function is automorphic. More precisely, if $L(s,\rho)$ is the Artin L-function corresponding to a non-trivial irreducible finite dimensional Galois representation, then there exists π, a cuspidal automorphic representation of GL_n, such that $L(s,\rho) = L(s,\pi)$.

Remark. The full Langlands conjectures says more than just what is above, but they predict that irreducible ρ correspond to cuspidal π.

The Langlands philosopy is useful because there are many analytic techniques that can be used to study automorphic L-functions. In particular, they are known to have holomorphic continuation and other such properties. However, many conjectures (including Artin's conjecture) of number theory would be true if the same properties could be shown to hold for Artin L-functions.

Remark. Artin's conjecture is that Artin L-functions are analytic (i.e. holomorphic on the entire complex plane.) It is known that they admit a meromorphic continuation, but it appears that the best way to prove holomorphicity is via the Langlands philosophy.

The truth of the Langlands philosophy has been shown in the "GL_1 case." The 1-dimensional Artin L-functions arise from characters $\rho : \mathrm{Gal}(E/F) \to \mathbb{C}^\times$. By class field theory, we have

$$\mathrm{Gal}(E/F) \to \mathrm{Gal}(E/F)^{ab} = \mathrm{Gal}(E'/F) \simeq \mathbb{A}_F^\times / F^\times Nm(\mathbb{A}_{E'}^\times),$$

for some field E'. We construct the automorphic representation of GL_1 over F to be

$$\chi : \mathbb{A}_F^\times / F^\times Nm(\mathbb{A}_{E'}^\times) \to \mathbb{C}^\times.$$

(Note that $\mathrm{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times$.) It is true that $L(s,\rho) = L(s,\chi)$. The study of $L(s,\chi)$ can be found in Tate's thesis[7].

As in the example above, given a Galois representation, the first goal to proving the truth of Langlands philosopy would be to associate to it an automorphic representation. Then, one would want to show that the corresponding L-functions are the same.

The primary goal of this course will be to explain what automorphic L-functions and automorphic representations are.

2. Representation Theoretic Notions

2.1. **Preliminaries on adeles, groups and representations. General Set Up:** Let G be a group over a global field F (which will typically be $\mathbb{Q}$.) Possible examples include GL_n, Sp_{2m}, $O(n)$. In the case of Sp_{2m} we mean that the skew symmetric form is defined over F. To such a group, the notion of an *automorphic representation* can be defined.

We take $G = \mathrm{GL}_2$. Then the adelic points of G can be viewed as the restricted product

$$G(\mathbb{A}_F) = \prod_{v \text{ places of } F}^{*} G(F_v)$$

with respect to the subgroups $K_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$. That is, $(g_v) \in G(\mathbb{A}_F)$ if and only if $g_v \in K_v$ for almost all v[1]. K_ν is a maximal compact subgroup of $G(F_\nu)$.

This is a generalization of the adeles

$$\mathbb{A}_F^\times = \prod_v F_v^\times.$$

Here, the restricted product is taken with respect to the valuation rings $\mathcal{O}_{F_\nu} \subset F_v$.

Example 2.1.1.

$$\mathbb{A}_{\mathbb{Q}}^\times = \mathbb{R}^\times \times \prod_{p \text{ prime}}^{*} \mathbb{Q}_p^\times.$$

Note that $\mathrm{GL}_1(\mathbb{Z}_p) = \mathbb{Z}_p^\times$.

Exercise 2.1.2. Show that $\mathrm{GL}_2(\mathbb{Z}_p)$ is a maximal compact subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$.

It is a fact that if π is an irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ then it is the restricted tensor product $\pi = \otimes \pi_v$, where π_v are irreducible representations of $\mathrm{GL}_2(\mathbb{F}_v)$. Attached to each π_v is an L-factor $L(s,\pi_v)$, and $L(s,\pi) = \prod L(s,\pi_v)$ converges absolutely when $s \gg 0$, and it has mermomorphic continuation.

We do not describe what the restricted tensor product means at this time, but instead begin by classifying all *local representations* π_v. We often treat the archimedean and nonarchimedean cases separately.

2.2. **Some categories of G_F-modules and $\mathcal{H}_F$-modules.** In this section, we consider representations of $G_F = \mathrm{GL}_2(F)$, where F is a non-archimedean local field. F is a topological field so G_F inherits the subspace topology from $M_2(F) = F^4$, and it is totally disconnected. A *representation* (π, V) of G_F is a complex vector space V together with an action of G_F denoted $\pi(g)$. In other words, V is a G_F-module. We may also refer to the representation as π (when V is clear from context), or we may just say V is a *G_F-module* (when the action π is clear from context.)

[1] i.e for all but finitely many.

2.2.1. *Finite dimensional representations.* Usually V will be infinite dimensional, but, in this section, suppose V is finite dimensional.

Proposition 2.2.1. *If* (π, V) *is a continuous representation of* G_F*, then* $\ker(\pi)$ *must contain an open subgroup.*

For example, $\mathrm{GL}_2(\mathcal{O}_F)$ is an open subgroup.

Proof. We first prove the GL_1 case. Say $\pi = \chi : \mathbb{Q}_p^\times \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, this is the case that V is one dimensional. So the preimage of 1 equals $\ker(\chi)$. Let R be a small open set around 1 in $\mathbb{C}$, then $\chi^{-1}(R) \supseteq N$, N an open normal subgroup and $\chi(N) \subset R$. Since N is a group, $\chi(N)$ must be a subgroup of $\mathbb{C}^\times$. For R sufficiently small, the only such group is $\{1\}$. Hence, $\chi(N) = \{1\}$, and $N \subseteq \ker(\chi)$.

This proof readily generalizes to GL_n for any n. It is a fact from complex Lie theory, that any open neighborhood of the identity generates the connected component of the identity in the group. $\square$

The topologies of $\mathbb{C}$ and $\mathbb{Q}_p$ are not compatible, and that is what makes this possible. $\mathbb{C}$ has an archimedean topology and $\mathbb{Q}_p^\times$ is totally disconnected.

Remark. An open problem is to understand finite dimensional p-adic vector space representations which are much more complicated.

2.2.2. *Smoothness and admissibility.* Proposition 2.2.1 is not true in general when V is infinite dimensional. In the spirit of the proposition, and to obtain more manageable representations, we will restrict ourselves to the class of representations satisfying one or more of the following conditions.

(A) (π, V) is such that for any vector $v \in V$, $\{g \in G_F \mid \pi(g)v = v\}$ contains an open subgroup.
(B) For all $N \subset G_F$ open, the set $\{v \in V : Nv = v\}$ is finite dimensional

If (π, V) satisfies (A), we call π *smooth*, and if it satisfies by (A) and (B), we say it is *admissible*.

There is a good theory of admissible representations because it often reduces to that of finite groups where one can use orthogonality conditions and Schur's lemma.

2.2.3. *The Hecke algebra.* It is a fact from Lie theory that every topological group has a unique up to scale *Haar measure*, which is a left invariant measure on the group. Symbolically, this means that if U is any measureable set, then

$$\mu(g'U) = \int_{g'U} d(g) = \int_U d(g'g) = \int_U = \mu(U).$$

Example 2.2.2. The Lebesgue measure on $\mathbb{R}^n$, a topological group under vector addition, is translation invariant, hence a Haar measure.

Example 2.2.3. We can define the Haar measure dx on $\mathbb{Q}_p$ by defining $\mu(p^n\mathbb{Z}_p) = p^{-n}$. Just as in the case above, the additive Haar measure on $\mathbb{Q}_p^n$ is $dx_1 \cdots dx_n$. We leave it as an exercise to show that the measure $(1 - \frac{1}{p})^{-1} \frac{dx}{|x|_p}$ is a Haar measure for $\mathbb{Q}_p^\times$, and that the measure of $\mathbb{Z}_p^\times$ is one.

Example 2.2.4. The Haar measure on $\mathrm{GL}_n(\mathbb{Q}_p)$ is

$$dg = \frac{dg_{11} dg_{12} \cdots dg_{nn}}{|\det g|_p^n}.$$

where $g = (g_{ij})$ and dg_{ij} is additive measure on $\mathbb{Q}_p$ from the previous example. Notice that when $n = 1$ this agrees with the previous example (up to a scalar multiple.)

Example 2.2.5. Let $P \subset \mathrm{GL}_n(F)$ be the subgroup of upper triangular matrices. Any $p \in P$ can be uniquely written in the form

$$p = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

Let dx denote the translation invariant measure on F. Then one can directly verify that the left and right invariant measures on P are given by

$$d_L p := \frac{da_1 da_2 dx}{|a_1 a_2|}, \quad \text{and} \quad d_R p := \frac{da_1 da_2 dx}{|a_2|^2}.$$

So $d_L p = \left|\frac{a_1}{a_2}\right| d_R p$. Thus P is not unimodular, meaning that the left and right Haar measures do not agree.

Let $\mathcal{H}_F$ be the set of all locally constant compactly supported functions on G_F valued in $\mathbb{C}$. This is an algebra under *convolution* which is defined for $f_1, f_2 \in \mathcal{H}_F$ by

$$f_1 * f_2(h) := \int_{G_F} f_1(g) f_2(g^{-1}h) dg \tag{2.2.1}$$

where dg is the Haar measure such that the measure of $\mathrm{GL}_2(\mathcal{O}_F)$ is 1.

If (π, V) is a smooth G_F-module then V is an $\mathcal{H}_F$-module as well. The action is given by

$$\pi(f)v = \int_{G_F} f(g)\pi(g)v \; dg. \tag{2.2.2}$$

Since f is locally constant and compactly supported, this is actually a finite sum, so it makes sense. Indeed, we can find $N \subseteq G_F$ open so that $Nv = v$ and f is constant on each gN (since π is smooth.) There exists finitely many g_i such that $\mathrm{supp}(f) = \cup_i g_i N$. This implies that

$$\begin{aligned} \int f(g)\pi(g)v dg &= \sum_i \int_{g_i N} f(g)\pi(g) v dg \\ &= \sum_i f(g_i)\pi(g_i)v \cdot \mu(N). \end{aligned}$$

2.2.4. *Idempotents in $\mathcal{H}_F$.* Let π_i, $1 \le i \le n$ be non-equivalent finite dimensional irreducible representations of $\mathrm{GL}_2(\mathcal{O}_F)$. Note that $\ker(\pi_i)$ are normal, and that $\mathrm{GL}_2(\mathcal{O}_F)/\ker(\pi_i)$ is finite. Define

$$\zeta_i(g) := \dim(\pi_i)\,\mathrm{tr}(\pi_i(g^{-1})). \quad \text{and} \quad \zeta = \sum_i \zeta_i(g). \tag{2.2.3}$$

Now extend the definition of ζ_i to a function on all of G_F by setting $\zeta_i(g) = 0$ if $g \notin \mathrm{GL}_2(\mathcal{O}_F)$.

Then $\zeta_i * \zeta_j = \delta_{ij}\zeta_i$ and thus ζ_i and ζ are idempotents.

Exercise 2.2.6. Prove that $\zeta * \zeta = \zeta$. Hint: use Schur's orthogonality relations for matrix coefficients. Such ζ are called *elementary idempotents.*

Consider the following conditions on $\mathcal{H}_F$-modules.

(A') For all $v \in V$ there exists an $f \in \mathcal{H}_F$ such that $\pi(f)v = v$.
(B') For all elementary idempotents ζ, $\pi(\zeta)V$ is finite dimensional.

Definition 2.2.7. *If condition (A') holds, we say that (π, V) is* smooth*, and we call (π, V)* admissible *if both (A') and (B') hold.*

Remark. There are notions of smoothness and admissibility for a representation of G_F when F is a local archimedean field as well.

Our goal is to show that the category of smooth (respectively, admissible) G_F-modules is equivalent to the category of smooth (resp., admissible) $\mathcal{H}_F$-modules. The following is straightforward.

Proposition 2.2.8. *If (π, V) is a smooth (admissible) G_F-module, then it is a smooth (admissible) $\mathcal{H}_F$-module.*

Proof. We prove the statement for smoothness and leave that of admissibilty as an exercise. Recall that
$$\pi(f)v := \int_{G_F} f(g)\pi(g)vdg.$$
Since π is smooth, this is a finite sum, and the $\mathcal{H}_F$ action is therefore well defined. Moreover, by Proposition 2.2.1 there exists an open subgroup N, such that for all $g \in N$ $\pi(g)v = v$. For $f = \frac{1}{\mu(N)}1_N$, it follows that $\pi(f)v = v$. □

We would like to have the other direction as well, but we first need to define an action of G_F on an $\mathcal{H}_F$-module (π, V). Let us begin with some heuristic calculations. If $\mathcal{H}_F$ acts smoothly on V, then for $v \in V$, $g \in G_F$, there exists $f \in \mathcal{H}_F$ such that $\pi(v) = v$. We expect the G_F action to satisfy
$$\begin{aligned}\pi(g)v = \pi(g)\pi(f)v =&\pi(g)\int f(h)\pi(h)vdh\\ =&\int f(h)\pi(gh)vdh\\ =&\int f(g^{-1}h)\pi(h)vd(g^{-1}h)\\ =&\int f(g^{-1}h)\pi(h)vdh\\ =&\int \lambda_g f(h)\pi(h)vdh\\ =&\pi(\lambda_g f)v,\end{aligned}$$
where λ is the left regular representation. That is, $(\lambda_g f)(h) = \lambda(g)f(h) := f(g^{-1}h)$.

This calculation suggests that if (π, V) is a smooth $\mathcal{H}_F$-module, we should define an action of G_F on V via
$$\pi(g)v = \pi(g)\left(\pi(f)v\right) = \pi(\lambda_g f)v$$
where we choose f so that $\pi(f)v = v$. Such an f exists because π is smooth, but we must still show that the action does not depend on the choice of f. This result is a consequence of the following.

Lemma 2.2.9. *Suppose (π, V) is a smooth $\mathcal{H}_F$-module, $g \in G_F$, and $v \in V$. If $\sum_{i=1}^r \pi(f_i)v = 0$, then $\sum_{i=1}^r \pi(\lambda_g f_i)v = 0$.*

Proof. Write $w = \sum \pi(\lambda_g f_i)v \in V$. By smoothness there exists an $f \in \mathcal{H}_F$ so that $\pi(f)w = w$. But

$$w = \pi(f)w = \pi(f)\sum \pi(\lambda_g f_i)v = \sum_i \pi(f * \lambda_g f_i)v.$$

On the other hand,

$$\begin{aligned} f * \lambda_g f_i(h) =& \int f(r)\lambda_g f_i(r^{-1}h)dr \\ =& \int f(r) f_i(g^{-1}r^{-1}h)dr \\ =& \int f(rg^{-1}) f_i(r^{-1}h)d(rg^{-1}) \\ =& \int \rho_{g^{-1}} f(r) f_i(r^{-1}h)dr \\ =& (\rho_{g^{-1}} f) * f_i(h), \end{aligned}$$

where $\rho_{g^{-1}}$ is right translation. Furthermore, we used the fact that GL$_2$ is reductive and thus the left and right Haar measures are the same. Thus,

$$w = \sum_i \pi(\rho_{g^{-1}} f * f)v = \pi(\rho_{g^{-1}} f)\sum_i \pi(f_i)v,$$

and it is clear that if $\sum \pi(f_i)v = 0$ then $\sum \pi(\lambda_g f_i)v = 0$ as desired. □

The final goal of this section is to prove the following

Proposition 2.2.10. *If (π, V) is a smooth (resp. admissible) $\mathcal{H}_F$-module, then action of G_F on V defined above makes it a smooth (resp. admissible) G_f-module.*

Proof. Let $v \in V$ then there exists an $f \in \mathcal{H}_F$ with $\pi(f)v = v$. Note that f is left invariant by an open subgroup N. Then for any $n \in N$ $\pi(n)v = \pi(n)\pi(f)v = \pi(\lambda_n f)v = \pi(f)v = v$. Now, $f \in \mathcal{H}_F$ is compactly supported and is locally constant, which implies that it is invariant by an open subgroup. This is a topology argument. This should be done for homework. For $f \in \mathcal{H}_F$ then, there exists $N \subseteq \mathrm{GL}_2(\mathcal{O}_F)$ open and normal such that $f(n_1 g n_2) = f(g)$ for every $n_1, n_2 \in N$. This is true because it is a local non-archimedean field. We leave equivalent statement for admissible representations as an exercise. □

Remark. We have defined $\mathcal{H}_F$ to be the space of locally constant functions with compact support. It is easy to see that this is the set of functions $f : G_F \to \mathbb{C}$ such that $f = \sum c_i 1_{g_i N}$. Indeed, for $f \in \mathcal{H}_F$, let $U = \mathrm{supp}(f)$. For each $g \in U$ let N_g be an open group[2] such that $gN_g \subset U$ and f is constant on gN_g. We have that

$$U = \bigcup_{g \in U} gN_g$$

Since U is compact, it is the union of finitely many of these, which we denote by $g_i N_i$. Let N be the intersection of these finitely many N_i. Then

$$U = \bigcup_{g \in U} gN = \bigcup_{\text{finite}} g_i N,$$

[2]Since F is nonarchimedean, G_F has a basis of open neighborhoods of the identity consisting of (normal) compact open subgroups.

and $f = \sum_i c_i 1_{g_i N}$.

Exercise 2.2.11. Show that

$$\mathcal{H}_F = span\,\{1_{NgN} \mid N \text{ is open in } G_F, g \in G_F\}\,.$$

In other words $\mathcal{H}_F = \{f : G_F \to \mathbb{C} \mid f(n_1 g n_2) = f(g)\}$ for some N open, and all $n_1, n_2 \in N$. We call such a function N-*biinvariant*.

Solution to Exercise 2.2.11. If N is an open (compact) subgroup of G_F and $g \in G_F$, then

$$NgN = \bigcup_{n \in N} ngN = \bigcup_{\text{finite}} g_i N$$

because NgN is also open and compact. This implies that if f is N-biinvariant, then $f \in \mathcal{H}_F$.

We now show that if $f \in \mathcal{H}_F$ then f is biinvariant. Without loss of generality, we may assume that $f = 1_{gN}$, and thus we need to show that 1_{gN} is N'-biinvariant for some N'. As remarked in the footnote, G_F has a basis of neighborhoods of the identity consisting of normal open subgroups. Using the same type of argument as above, it follows that if N is an open subgroup then $N = \cup_i g_i N'$ where the N_i are a finite number of normal subgroups. Hence

$$gN = \bigcup g g_i N' = \bigcup N' g_i' N'$$

where $g_i' = g g_i$. Note that we use the fact that if N is normal, $gN = Ng = NgN$. ☐

2.3. **Contragredient representation.** Let (π, V) be a smooth representation of G_F, V^* the set of linear functionals over $\mathbb{C}$. We define an action of G_F on V^* by $(v, \pi^*(g)v^*) = (\pi(g^{-1})v, v^*)$ for $v \in V$ and $v^* \in V^*$, where the pairing $(w, v^*) = v^*(w)$. In other words, $\pi^*(g)v^*(v) = v^*(\pi^(g^{-1})v)$. This action may not be smooth!

To get a smooth action we go through $\mathcal{H}_F$. Define the action of $\mathcal{H}_F$ on V^* by

$$\pi^*(f)v^*(v) = v^*(\pi(\check{f})v),$$

where $\check{f}(g) := f(g^{-1})$. In general this action is still not smooth. V^* might be too big.

Set $\widetilde{V} := \pi^*(\mathcal{H}_F)V^* = \{\sum \pi^*(f_i)v_i^*\}$ and $\widetilde{\pi} = \pi^* \mid_{\widetilde{V}}$. In the case that (π, V) is smooth, we call $\widetilde{V}$ the set of *smooth vectors* in V^*. Obviously, if $v^* \in V^*$ is fixed by some $f \in \mathcal{H}_F$, $v^* \in \widetilde{V}$. This fact, together with the following lemma, implies that $\widetilde{V}$ is equal to

$$\{v^* \in V^* : v^* \text{ is fixed by some } L \subseteq K\text{open}\},$$

and explains the terminology "smooth vectors."

Lemma 2.3.1. *(π, V) is smooth implies $(\widetilde{\pi}, \widetilde{V})$ is smooth. Moreover, (π, V) is admissible implies $(\widetilde{\pi}, \widetilde{V})$ is admissible.*

Remark. This is an example where infinite dimensional spaces are more complicated than finite dimensional ones. Passing to $\widetilde{V}$ makes the infinite dimensional representation behave more like a finite dimensional representation.

Proof of Lemma 2.3.1. Let $\widetilde{v} \in \widetilde{V}$. We need to show there is an $f \in \mathcal{H}_F$ such that $\widetilde{\pi}(f)\widetilde{v} = \widetilde{v}$. By definition of $\widetilde{V}$, every element is of the form $\widetilde{v} = \sum_{i=1}^r \pi^*(f_i)v_i^*$ for some $f_i \in \mathcal{H}_F$.

We observe that there exists elementary idempotents so that $\zeta * f_i = f_i$ for $1 \le i \le r$. The proof will be given later. From this we deduce that

$$\widetilde{\pi}(\zeta)\widetilde{v} = \pi^*(\zeta)\sum_i \pi^*(f_i)v_i^* = \sum_i \pi^*(\zeta * f_i)v_i^* = \sum \pi^*(f_i)v_i^* = \widetilde{v}.$$

This proves the smoothness but we need to show the existence of a ζ.

That (π, V) admissible implies $(\widetilde{\pi}, \widetilde{V})$ is also admissibile follows from the following exercise. □

Exercise 2.3.2. Show that $\widetilde{V}(\zeta) = V(\check{\zeta})^*$ with $\check{\zeta}(g) = \zeta(g^{-1})$.

Remark. In the proof we made use of the fact that every $\widetilde{v} \in \widetilde{V}$ can be written as a finite sum $\sum_{i=1}^r \pi^*(f_i)v_i^*$. In fact it is true that every $\widetilde{v}$ is equal to $\pi^*(f)v^*$ for some f and v^*.

The following lemma contains some basic facts about the Hecke algebra and helps us deduce the existence of the ζ used in the previous lemma.

Lemma 2.3.3. *Let $N \subseteq \mathrm{GL}_2(\mathcal{O}_F)$ be open. Then the following hold.*

(1) $1_N * 1_{Ng} = 1_{Ng}$
(2) $1_{gN} * 1_N = 1_{gN}$
(3) *Assume that N is normal. Let $\pi_1, \cdots, \pi_r$ be irreducible representations of $\mathrm{GL}_2(\mathcal{O}_F)/N$ and $\zeta = \sum_i \zeta_i$ then $\zeta_i * 1_N = 1_N * \zeta = 1_N$.*
(4) *If $f \in \mathcal{H}_F$ and f is bivariant by N, open and normal, then $\zeta * f = f * \zeta = f$.*

Proof. We do not prove (1) or (2). (The proofs are straight forward.) For the proof of (3), let $\{\pi_1, \cdots, \pi_r\}$ be all of the distinct representations of the finite group $\mathrm{GL}_2(\mathcal{O}_F)/N$. Then $\{\zeta_1, \cdots, \zeta_r\}$ are a basis for the class functions on this finite group. So $1_N = \sum_i c_i\zeta_i$ for some c_i. Then

$$1_N * \zeta = (\sum c_i\zeta_i) * (\sum \zeta_i) = \sum_{i,j} c_i\zeta_i * \zeta_j = 1_N,$$

by the orthogonality of the idempotents. A similar calculation shows that $\zeta * 1_N = 1_N$.

To prove (4) we note that $\zeta * f = f * (1_N * f) = (\zeta * 1_N) * f = 1_N * f = f$. The opposite relation is similar. □

3. Classification of local representations: F nonarchimedean

Our goal is to classify irreducible admissible representations of $G_F = \mathrm{GL}_2(F)$ with F a local nonarchimedean field. Since $K = \mathrm{GL}_2(\mathcal{O}_F)$ is a compact (sub)group, its representations are much easier to classify. Indeed, the representation theory of compact groups mimics that of finite groups. Since any G_F-module V is also a K-module, we can consider the decomposition of V as such. In the following section, we discuss this decomposition. After introducing the notion of irreducibility, we then proceed to classify irreducible G_F-modules.

3.1. (π, V) **considered as a K-module.** In this section, we see what the consequences of assuming that (π, V) is smooth or admissible G_F-module imply with respect to the decomposition of V as a K-module.

The Hecke algebra[3] $\mathcal{H}_F = C_c^\infty(G_F)$, and, as we have seen in exercise 2.2.11 and the accompanying discussion, $\mathcal{H}_F$ is equal to the span of the set

$$\{1_{LgL} : L \subseteq \mathrm{GL}_2(\mathcal{O}_F) \text{ open }, g \in G_F\}.$$

Notice that this span description makes $\mathcal{H}_F$ look a lot more like the classical Hecke algebra, i.e. that given by the double coset description.

3.1.1. *Elementary idempotents.* For $L \subseteq \mathrm{GL}_2(\mathcal{O}_F)$ open normal, we may define an elementary unipotent associated to L by

$$\zeta_L(g) := \sum_{\pi \in \mathcal{E}(K/L)} \dim(\pi_i) \operatorname{tr}(\pi_i(g^{-1})) \tag{3.1.1}$$

for $g \in K$ and $\mathcal{E}(K/L)$ the (finite since K/L is finite) set of non-equivalent representations of K/L. Lemma 2.3.3 implies that

$$\zeta_L * 1_{LgL} = 1_{LgL} * \zeta_L = 1_{LgL}$$

for any $g \in G_F$.

3.2. **A new formulation of smoothness and admissibility.** Let (π, V) G_F-module. Then we define

$$V^L := \{v \in V : Lv = v\} \tag{3.2.1}$$

for $L \subseteq K$ open. Obviously, V is smooth if and only if $V = \bigcup V^L$. Moreover, V is admissible if and only if, additionally, each V^L is finite dimensional.

Since (π, V) is also an $\mathcal{H}_F$-module, we can define

$$V(\zeta_L) := \pi(\zeta_L)V. \tag{3.2.2}$$

We call $V(\zeta_L)$ *the L-part* of V. Similarly to the above statement, V is smooth if $V = \bigcup V(\zeta_L)$, and V is admissible if additionally $V(\zeta_L)$ is finite dimensional for each L. The fact that this characterization for $\mathcal{H}_F$-modules is true follows from the observations about G_F-modules above and the following.

Exercise 3.2.1. If (π, V) a smooth $\mathcal{H}_F$-module then $V^L = V(\zeta_L)$.

Remark. This formulation demonstrates one of the many uses of the elementary idempotents.

3.2.1. *Decomposition of K/L-modules.* Let (π, V) be a smooth G_F-module, $L \subseteq K$ open and normal. Then V^L is a K/L-module and

$$V^L = \bigoplus_{\delta \in \mathcal{E}(K/L)} V_{(\delta)}$$

where $V_{(\delta)}$ is the *isotipic component* of $\delta \in \mathcal{E}(K/L)$ defined by

$$V_{(\delta)} = \sum_{\varphi \in \mathrm{Hom}_K(W_\delta, V^L)} \varphi(W_\delta) \tag{3.2.3}$$

[3]in the case of a nonarchimedean local field, a function is smooth precisely when it is locally constant.

with W_δ is the vector space for δ. That is, (δ, W_δ) is a finite dimensional irreducible representation of K/L.

If $v \in V^L$ it can be expressed as $v = \sum_\delta v_\delta$, where $v_\delta \in V^L_{(\delta)}$. In fact, $v_\delta = \zeta_\delta v$ where $\zeta_\delta(g) = \mathrm{tr}(\delta(g^{-1}))\dim(\delta)$.

Remark. We intend to include an appendix on the representation theory of finite (and compact?) groups. The decomposition in equation (3.2.3) and the other facts above, are corollaries to Schur's lemma in this setting.

3.2.2. *Decomposition of K-modules.* This description of K/L-modules extends to K-modules.

Remark. The reason for this may follow from the fact that K is the inverse limit of the groups K/L where L is open and normal in K. Maybe we can explore this question in the appendix(?)

Let $\delta \in \mathcal{E}(K)$ the equivalence classes of finite dimensional irreducible continuous representations of K. (Note that since δ is continuous it factors through K/L for some L open and normal in K.) Let (π, V) be smooth, and write $V = \oplus_{\delta\in\mathcal{E}(K)} V_{(\delta)}$ where

$$V_{(\delta)} = \sum_{\varphi\in\mathrm{Hom}_K(W_\delta, V)} \varphi(W_\delta).$$

Then V is admissible if and only if each $V_{(\delta)}$ is finite dimensional. That is, every δ occurs with finite multiplicity.

Remark. One of the facts of finite dimensional representation theory that is being mirrored here is that if π is a finite dimensional complex representation of a finite group G, then

$$\pi \simeq \bigoplus_i n_i \pi_i$$

where $\{\pi_1, \ldots, \pi_r\}$ are irreducible representations of G. The number n_i equals the cardinality of $\mathrm{Hom}_G(\pi_i, \pi)$. If W_i is the space corresponding to π_i then

$$\pi = \bigoplus_{\varphi\in\mathrm{Hom}(\pi_i,\pi)} \varphi(W_i).$$

Again, this is a consequence of Schur orthogonality.

3.2.3. *The contragradient representation.* Again, assume that (π, V) is smooth, and let $\widetilde{V} := \pi^*(\mathcal{H}_F)V^*$ with $\widetilde{\pi} = \pi^* \mid_{\widetilde{V}}$ be the contragradient representation. Recall that

$$\widetilde{V} = \{v^* \in V^* : v^* \text{ is fixed by some } L \subseteq K\text{open}\},$$

the collection of smooth vectors in V^*. One sees there is a map $V \mapsto \widetilde{V}^*$ that induces a monomorphism $V \hookrightarrow \widetilde{\widetilde{V}}$. When V is admissible this is an isomorphism, hence $\widetilde{\widetilde{V}} \simeq V$. The key to proving this is an understanding of Exercise 2.3.2.

3.3. **Irreducibility.** A G_F-module (π, V) is called *irreducible* if the only G_F-invariant subspaces of V are $\{0\}$ and V itself.

Proposition 3.3.1 (Schur's Lemma)**.** *Let (π, V) be an irreducible G_F-module. If $A : V \to V$ satisfies $A\pi(g)v = \pi(g)Av$ for all $g \in G_F$ then $A = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof. First, we show that $\dim_{\mathbb{C}} V$ is countable. Choose $v \in V$, $v \neq 0$. Since V is irreducible, $V = \pi(\mathcal{H}_F)v$. So, it suffices to show that $\mathcal{H}_F$, which is equal to the span of $\{1_{LgL} : g \in K, L\}$ has a countable basis. This follows from the fact that K has a countable (topological) basis of the identity consisting of open subgroups, and that for each such subgroups the number of distinct double cosets LgL is also countable. (We leave the proof of these facts as an exercise.)

Next, assume that A is not a scalar. If V is finite dimensional, then A must have an eigenvector of eigenvalue λ. This would imply that there is a non trivial eigenspace, but, by irreducibility, we would have our desired result.

Even if V is infinite dimensional, we will show that A has an eigenvector, and, just as in the finite dimensional case, irreducibility of V would imply the proposition. By means of contradiction, assume that A has no eigenvectors. Then $p(A) \neq 0$ for all nonzero polynomials $p(t) \in \mathbb{C}[t]$. Indeed, if $p(A) = 0$ then, since

$$p(t) = \prod f_i(t)^{e_i}$$

where each f_i is linear, the kernel of $p(A)$ must be nontrivial, and A must have an eigenvalue. Thus $p(A)$ is injective, and (again by irreducibility) a bijection for all nonzero $p(t) \in \mathbb{C}[t]$. So, we have an injective map (of vector spaces) $\mathbb{C}(t) \to V$ given by

$$\frac{p(t)}{q(t)} \mapsto p(A)q(A)^{-1}v$$

for a fixed nonzero vector $v \in V$. However, $\dim_{\mathbb{C}}(\mathbb{C}(t))$ is uncountable, which can be seen by observing that $\{\frac{1}{t-\lambda} : \lambda \in \mathbb{C}\}$ are independent and uncountable. □

Let (π, V) be a representation of G_F. Even if V is infinite dimensional, the use of the so called *Hecke algebra of level* L, defined to be the set

$$\mathcal{H}_L := \{f \in \mathcal{H}_F : f \text{ is bi-invariant under } L\}, \tag{3.3.1}$$

allows us to determine the irreducibility of V at a finite level.

Lemma 3.3.2. *Let (π, V) a smooth representation of G_F. Then V is smooth and irreducible if and only if for all $L \subseteq K$ open V^L is an irreducible $\mathcal{H}_L$-module.*

Proof. Suppose that for all $L \subseteq K$ open V^L is an irreducible $\mathcal{H}_L$-module. If V is reducible then there exists a W so that $V \supsetneq W \supsetneq \{0\}$ that is G_F-invariant. Let $w \in W$ and $v \in V \setminus W$ be nonzero vectors. By smoothness, there is an L fixing w and v. Thus $V^L \supsetneq W^L \supsetneq \{0\}$, since $v \in V^L$ and $w \in W^L$. This implies that V^L is not an irreducible $\mathcal{H}_L$ module.

The other direction is left as an exercise. □

Remark. The fact that V is not irreducible does not, in general, imply that $V = W \oplus U$ for nontrivial submodules W and U. In the finite dimensional case, it does.

Proposition 3.3.3. *If (π, V) is smooth and irreducible then it is admissible.*

The proof of this is very hard and is left as homework.

Lemma 3.3.4. *Suppose that (π, V) is admissible. Let $(\widetilde{\pi}, \widetilde{V})$ be the contragradient representation. Then (π, V) is irreducible if and only if $(\widetilde{\pi}, \widetilde{V})$ is irreducible.*

This is an easy lemma which follows by chasing definitions.

Proposition 3.3.5. *If (π, V) is a finite dimensional continuous irreducible representation then V is 1-dimensional, and there exists a quasicharacter*[4] $\chi : F^\times \to \mathbb{C}^\times$ *such that $\pi(g) = \chi(\det(g))$.*

Proof. We first show that $\mathrm{SL}_2(F) \subset \ker(\pi)$. As we saw in Proposition 2.2.1, $\ker(\pi)$ contains an open subgroup. Consider

$$N := \{n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F\}. \tag{3.3.2}$$

Intersecting the kernel of π with N, we find an $\epsilon > 0$ so such that if $|x| < \epsilon$ then $n(x) \in \ker(\pi)$. Note that for any $\alpha \in F^\times$ we can define an element of G_F

$$a(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

and $a(\alpha)n(x)a(\alpha^{-1}) = n(\alpha^2 x)$. If $y \in F$ choose x with $|x| < \epsilon$ and α such that $y = \alpha^2 x$. Then

$$\pi(n(y)) = \pi(a(\alpha))\pi(n(x))\pi(a(\alpha^{-1})) = \pi(a(\alpha))\pi(a(\alpha^{-1})) = \pi(a(\alpha)a(\alpha)^{-1}) = 1.$$

So $N \subset \ker(\pi)$. Similarly, one may deduce that $N' \subseteq \ker(\pi)$ with

$$N' := \{n'(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in F\}.$$

Since N and N' generate $\mathrm{SL}_2(F)$, we conclude that π factors through $\mathrm{GL}_2/\mathrm{SL}_2$. Hence one gets an irreducible action of the abelian group $F^\times$ on V. Therefore V is 1-dimensional, and the action is via a character χ. □

With this we now turn to irreducible admissible infinite dimensional representations.

3.4. **Induced representations.** One method of obtaining representations of $\mathrm{GL}_n(F)$ is to start with a representation of a subgroup and "lift" it to a representation of $\mathrm{GL}_n(F)$. Some natural subgroups are the parabolic groups (those fixing a subspace of F^n) and the tori (abelian subgroups.) For our group $G_F = \mathrm{GL}_2(F)$, we have the subgroups

$$P := \left\{ \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \mid a_i \in F^\times \right\} \quad \text{and} \quad A := \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in P \right\}.$$

Since $A \simeq F^\times \times F^\times = \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$, a representations of A is the product of representations of GL_1, i.e. characters. Essentially, an induced representation of G_F is one which agrees with this one when we restrict it to A.

Definition 3.4.1. *Let χ_1, χ_2 be two quasicharacters of $F^\times$. The* induced representation *of χ_1, χ_2, denoted by $B(\chi_1, \chi_2)$, is the set of functions $f : G_F \to \mathbb{C}$ satisfying*

(i) $f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \chi_1(a_1)\chi_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} f(g).$

(ii) *Under the right regular action ρ of G_F, f is smooth.*

Remark. We could define $B(\chi_1, \chi_2)$ without the factor $\left|\frac{\cdot}{\cdot}\right|^{1/2}$ if we like. However, this normalization will become easier to work with later. It arises because the group group P is not unimodular.

[4]A *quasicharacter* of a group G is a homomorphism from G to $\mathbb{C}^\times$.

Note that condition (ii) is equivalent to saying that f is fixed by some open compact subgroup L. Thus $B(\chi_1, \chi_2)$ is smooth as a G_F-module. Furthermore, it is admisssible. Indeed, by the Iwasawa decomposition,

$$G = PK, \tag{3.4.1}$$

So, condition (i) implies that f is determined by its values on K. Furthermore, f is invariant by $L \subseteq K$ open, so f is determined by its values on K/L which is finite. Hence, if we fix L, $B(\chi_1, \chi_2)^L$ is finite dimensional which means that it admissible.

3.5. **Supercuspidal representations and the Jacquet module.** We will see in this section that an irreducible representation of G_F that does not occur as the subquotient or submodule of an induced representation is *supercuspidal*.

Jacquet came up with a simple but powerful idea for determining if a G_F-module is supercuspidal. First, restrict the representation to P. By condition (i) of Definition 3.4.1, an induced representation is trivial on the unipotent subgroup N defined in equation (3.3.2).

So, if we start with a G_F-module V and restrict to the P action, in order for the representation to come from a pair of characters (as in the induced representations) N must act trivially. Note that $A = P/N$. We define the *Jacquet module* to be $V_N := V/V(N)$ where

$$V(N) := \operatorname{span}\{\pi(n)v - v \mid n \in N, v \in V\}. \tag{3.5.1}$$

Lemma 3.5.1. *If the (π, V) is a smooth G_F-module, then V_N is a smooth A-module.*

Proof. In order for V_N to be an A-module, we must check that P preserves $V(N)$. Since $P = AN$, it suffices to show that both A and N preserve $V(N)$. Since N is commutative ($N \simeq F$), if $n, n' \in N$ then

$$\pi(n')(\pi(n)v - v) = \pi(n'n)v - \pi(n')v = \pi(n)v' - v'$$

for $v' = \pi(n')v$. To see that A preserves $V(N)$, first note that

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a_1 x a_2^{-1} \\ 0 & 1 \end{pmatrix}.$$

In other words, A normalizes N. Hence

$$\pi(a)(\pi(n)v - v) = \pi(an)v - \pi(a)v = \pi(ana^{-1})(\pi(a)v) - \pi(a)v \in V(N).$$

The action is clearly smooth being the quotient of a smooth module. $\square$

Notice that we have gone in the opposite direction as we did in the previous section. $B(\chi_1, \chi_2)$ is a G_F-module induced from an A-module, and V_N is an A-module that is derived from a G_F-module.

Remark. If V is an admissible G_F-module then V_N is an admissible A-module. This is hard to prove. We don't prove it, as we won't need it. However, it may simplify things. We'll see.

Proposition 3.5.2. *$V \longrightarrow V_N$ is an exact functor from smooth G-modules to smooth A-modules. That is, if*

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

is exact then so is

$$0 \longrightarrow V'_N \longrightarrow V_N \longrightarrow V''_N \longrightarrow 0.$$

Lemma 3.5.3. *Suppose that (π, V) is smooth. Then $v \in V(N)$ if and only if there exists $U \subset N$ compact and open such that $\int_U \pi(u)vdu = 0$.*

Example 3.5.4. If $F = \mathbb{Q}_p$ then U can be taken to be $p^n\mathbb{Z}_p$ for some n.

Proof of Lemma. If $v \in V(N)$ then $v = \sum_i(\pi(n_i)v_i - v_i)$. Choose U such that U contains all n_i. Then

$$\begin{aligned}\int_U \pi(u)vdu =& \sum \int \pi(u)(\pi(n_i)v_i - v_i)du \\ =& \int \left(\sum_i \pi(un_i)v_i - \sum \pi(u)v_i\right) du \\ =& \sum_i \int_U \pi(u)v_i d(un_i^{-1}) - \sum_i \int_U \pi(u)v_i du \\ =&0\end{aligned}$$

since $d(un) = du$ for any n.

Conversely, say $\int_U \pi(u)vdu = 0$ for some U. We show that $v \in V(N)$. Recall v is smooth so there exists $U' \subset U$ compact open such that $U'v = v$. Write $U = \bigsqcup u_iU'$. Since U is compact, this is a finite union. Then

$$0 = \int_U \pi(u)vdu = \sum_{i=1}^{l} \int_{u_iU'} \pi(u)vdu = \mu(U') \sum_{i=1}^{l} \pi(u_i)v.$$

Thus $\sum \pi(u_i)v = 0$, and

$$v = -\frac{1}{l}\sum_{i=1}^{l}(\pi(u_i)v - v) \in V(N).$$

□

Proof of Proposition 3.5.2. Denote the maps from V' to V and from V to V'' by α β respectively, and let α_N and β_N be defined similarly. Checking exactness V''_N amounts to showing that β_N is surjective, but this is true because β is, and we are dealing with quotient groups. Exactness at V_N is also easy.

So, to complete the proof, we must check the exactness at V'_N. In other words, show that α injective implies α_N is injective. If $[v'] \in V'_N$ with $v' \in V'$ such that $\alpha_N[v'] = 0$, we to show that $v' \in V'(N)$. Actually, $0 = \alpha_N[v'] = [\alpha v']$. So $\alpha v' \in V(N)$. By Lemma 3.5.3 gives U compact and open subgroup so that

$$0 = \int_U \pi(u)\alpha v'du = \int_U \alpha(\pi'(u)v')du = \alpha\left(\int_U \pi'(u)v'du\right).$$

The second equality follows since α is a homomorphism of G_F-modules, i.e. an intertwining map. Since α is injective, we know that $\int_U \pi'(u)v'du = 0$. One more application of Lemma 3.5.3 implies that $v' \in V'(N)$. □

Definition 3.5.5. *An admissible G_F-module (π, V) is called* supercuspidal *(or* absolutely cuspidal*) if $V_N = 0$.*

Exercise 3.5.6. If (π, V) is an irreducible G_F-module. Then Z, the center of G_F, acts by a character. In other words, if $z = \left(\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}\right)$, then $\pi(z)v = \omega(a)v$ for some ω a quasicharacter of $F^\times$ and all $v \in V$.

Solution to Exercise 3.5.6. Fix $z \in Z$. Then the map

$$A : V \longrightarrow V \quad v \mapsto \pi(z)V$$

is an intertwining operator. That is, because $zg = gz$ for all $z \in Z$ and $g \in G_F$, it is a linear map such that

$$\pi(g)Av = \pi(g)\pi(z)v = \pi(z)\pi(g)v = A\pi(g)v.$$

Let v be an eigenvector of A with eigenvalue λ. Then, the above equation becomes

$$\pi(z)\pi(g)v = \pi(g)(\lambda v) = \lambda\pi(g)v.$$

By irreducibility, every vector $w \in V$ is of the form $\pi(g)v$ for some $g \in G_F$. Hence $\pi(z)$ acts by a scalar, call it $\omega(z)$, for any $z \in Z$. That ω is a quasicharacter is immediate from the fact that π is a representation. $\square$

Examing the motivation for the definition of the Jacquet module, we expect that for an irreducible smooth module V, $V_N \neq 0$ if and only if V is not obtained by inducing a representation of G_F from a representation of A. The following proposition formalizes and verifies this intuition.

Theorem 3.5.7. *Suppose that (π, V) is a smooth irreducible G_F-module. Then π is not supercuspidal if and only if there exists quasicharacters χ_1, χ_2 of $F^\times$, and a nonzero intertwining map*

$$L : V \longrightarrow B(\chi_1, \chi_2).$$

Proof. ($\implies$) We first show that $V_N = V/V(N)$ is a finitely generated A-module. By irreducibility, if $v \in V$ with $v \neq 0$, we have V equals the span of gv for $g \in G$. Let N be such that v is fixed by N compact and open. (Such a group exists because V is smooth.) Then, since K is compact,

$$K = \bigcup_{i=1}^{l} k_i N.$$

So, by the Iwasawa decomposition, (3.4.1), $V = \operatorname{span}\{bk_iv : b \in P\}$. So V is a finitely generated P-module, and it follows that V_N is a finitely generated $P/N = A$-module.

Now, by Lemma 3.5.8, there exists an A-invariant subspace W of V_N such that V_N/W is A-irreducible. Since A is abelian, V_N/W must be 1-dimensional, and we identify it with $\mathbb{C}$. This gives an (intertwining) map $\theta : V \to \mathbb{C}$ given by the composition

$$V \longrightarrow V_N \longrightarrow V_N/W \simeq \mathbb{C}.$$

On V_N/W, the action of $p = \left(\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}\right) \in P$ is given by a pair of characters χ_1', χ_2'. Specifically,

$$\pi(p) = \chi_1'(a_1)\chi_2'(a_2). \tag{3.5.2}$$

Let $\chi_1 = \chi_1' |\cdot|^{-1/2}$, and $\chi_2 = \chi_2' |\cdot|^{1/2}$, and notice that a function in $B(\chi_1, \chi_2)$ must be smooth and satisfy

$$f\left(\left(\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}\right) g\right) = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g) = \chi_1'(a_1)\chi_2'(a_2)f(g). \tag{3.5.3}$$

We claim that defining $(Lv)(g) = \theta(\pi(g)v)$ defines an intertwining operator

$$L : V \longrightarrow B(\chi_1, \chi_2).$$

To see this, first note that Lv is smooth (because V is smooth.) Next, we use equations (3.5.2) and (3.5.3) to calculate directly that

$$(Lv)\left(\left(\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}\right) g\right) = \theta\left(\pi\left(\left(\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}\right) g\right) v\right) = \chi_1'(a_1)\chi_2'(a_2)\pi(g)\theta(v) = \chi_1'(a_1)\chi_2'(a_2)(Lv)(g).$$

Finally, note that L is injective by irreducibility. So V is a subrepresentation of $B(\chi_1, \chi_2)$. We will prove the other direction in the next section. □

Remark. The word 'cuspidal' comes from the cusp arising from fixed points of parabolic subgroups, and integrating over a parabolic group gives the constant term.

The following lemma applies in the proof above because $N \simeq F \simeq \varpi^{\mathbb{Z}} \times H$ where ϖ is a uniformizer and H is finite and commutative. (See Neukirch for details.)

Lemma 3.5.8. *Suppose the group $\varpi^{\mathbb{Z}} \times H$, where H is finite and commutative acts on a space W. If W is finitely generated then there exists $W' \subset W$ invariant such that W/W' is irreducible.*

Proof. First, suppose that $H = \{1\}$ acts trivially. Then take $W = \langle v_1, \cdots, v_n\rangle$ with n minimal and $W_1 = \langle v_1, \cdots, sv_{n-1}\rangle$ then W/W_1 is generated by 1 element. So we may assume that W is generated by a single element and write

$$W = \operatorname{span}_{\mathbb{Z}}\{L^n v : n \in \mathbb{Z}\}.$$

There are two cases. The first is that there is no relation among the $L^n v$. So $L^n v$is a basis for W. In this case, the subspace $W' = \{L^n v - L^{n+1} v \mid n \in \mathbb{Z}\}$ is invariant, and W/W' is 1-dimensional. On the other hand, if there is a relation then $p(L)v = 0$ for some p a polynomial. Therefore, there must be an eigenvector and, therefore, a 1-dimensional submodule.

We leave the proof that for nontrivial action of H as an exercise. □

3.6. **Matrix coefficients.** In this section, we prove the other implication in Theorem 3.5.7. Let (π, V) be admissible and $(\widetilde{\pi}, \widetilde{V})$ its contragradient (which is also admissible). Choose $v \in V$, $\widetilde{v} \in \widetilde{V}$ then

$$f_{v,\widetilde{v}}(g) := \langle \pi(g)v, \widetilde{v}\rangle \tag{3.6.1}$$

is called a *matrix coefficient function.* When π is irreducible Z, the center, acts on V by a quasicharacter, say ω. (See Exercise 3.5.6.) Then $\pi(z) = \omega(z)$. This implies $f(zg) = \omega(z) f_{v,\widetilde{v}}(g)$.

Theorem 3.6.1. *If (π, V) is a supercuspidal irreducible admissible representation, then $f_{v,\widetilde{v}}$ is compactly supported on $Z\backslash G$. In other words, there exists $\Omega \subseteq G$ a compact subset such that f vanishes outside of $Z\Omega$.*

Remark. So this means that supercuspidal representations of G_F behave just like those for compact groups which, in turn, mimic representations of finite groups.

We use the following double coset decomposition of G_F in the proof.

Lemma 3.6.2.

$$G_F = \mathrm{GL}_2(F) = \bigcup_{n_1 \geq n_2 \in \mathbb{Z}} K \begin{pmatrix} \varpi^{n_1} & \\ & \varpi^{n_2} \end{pmatrix} K = \bigcup_{n \geq 0} ZK \begin{pmatrix} \varpi^{n} & \\ & 1 \end{pmatrix} K,$$

with ϖ a uniformizer of F.

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_F$. Denote the ring of integers of F by $\mathcal{O}_F$, and let $U = \mathcal{O}_F^\times$, and if $Kg_1K = Kg_2K$, then we will say that g_1 and g_2 are equivalent and denote it by $g_1 \sim g_2$.

Notice that acting by K on the left can be viewed as performing elementary row operations, and acting on the right as column operations. So, permuting rows and/or columns, multiplying a row/column by a unit $u \in U$, and adding a multiple (by an element of $\mathcal{O}_F$) of one row/column to the other transforms g to an equivalent element.

Therefore, we may assume that $v(d)$ is minimal among $v(a), v(b), v(c), v(d)$, and, in fact that $a = \varpi^{n_2}$. That is,

$$g \sim \begin{pmatrix} a & b \\ c & \varpi^{n_2} \end{pmatrix}.$$

Since $n_2 \leq v(b), v(c)$, by adding a multiple of the second row or column, respectively, we can assume that $b = c = 0$. Indeed, letting $x := -c/\varpi^{n_2} \in \mathcal{O}_F$, we have

$$\begin{pmatrix} a & b \\ c & \varpi^{n_2} \end{pmatrix} \sim \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & \varpi^{n_2} \end{pmatrix} = \begin{pmatrix} a' & 0 \\ c & \varpi^{n_2} \end{pmatrix}.$$

A similar column operation (i.e. multiplication by K on the right allows us to assume that $c = 0$. Hence, there is a $u \in U$, such that

$$g \sim \begin{pmatrix} u\varpi^{n_1} & 0 \\ 0 & \varpi^{n_2} \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi^{n_2} & 0 \\ 0 & \varpi^{n_2} \end{pmatrix}.$$

Permuting rows and columns (if necessary), we may assume that $n_1 \geq n_2$. This gives the first decomposition. Factoring out $\varpi^{n_2} I_2$ gives the second. ☐

Proof of Theorem 3.6.1. The lemma implies that if Ω is a compact subset of G, then there exists n_0 such that

$$\bigcup_{n=0}^{n_0} ZK \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} K \supset \Omega.$$

We claim that is suffices to show a matrix coefficient vanishes on $ZK \left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right) K$ for $n \gg 1$. The one direction is easy. In the other direction, if $f_{v,\widetilde{v}}$ vanishes on $ZK \left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right) K$ for $n \gg 1$ then there exists Ω a compact set such that $f = 0$ outside ΩZ. In particular, we may take $\Omega = \bigcup_{n=0}^{n_0} K \left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right) K$.

Next, we reduce to showing that for any $u, \widetilde{u}$ there exists n_0 such that $f_{u,\widetilde{u}}(\left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)) = 0$ for all $n > n_0$. By smoothness, for a given $v, \widetilde{v}$ there exists an $L \subseteq K$ open normal which fixes $v, \widetilde{v}$. Write

$$K = \bigsqcup_{j=1}^{\ell} k_j L.$$

Then for any $k, k' \in K$ there are i and j such that $k' \in k_i L$ and $k \in L k_j$. Then

$$\begin{aligned} f_{v,\widetilde{v}}(k \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} k') =& \langle \pi(k)\pi(\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix})\pi(k')v, \widetilde{v}\rangle \\ =& \langle \pi(\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix})\pi(k')v, \widetilde{\pi}(k^{-1})\widetilde{v}\rangle \\ =& \langle \pi(\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix})\pi(k_i')v, \widetilde{\pi}(k_j^{-1})\widetilde{v}\rangle \\ =:& f_{v_i,\widetilde{v}_j}\left(\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}\right). \end{aligned}$$

So if there exists n_{ij} such that $f_{v_i,\widetilde{v}_j}\left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)) = 0$ vanishes for all $n \geq n_{ij}$, then if we choose $n_0 = \max(n_{ij})$, and $f_{v,\widetilde{v}}(k \left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right) k') = 0$ for $n > n_0$.

So we have reduced to showing that for any $v, \widetilde{v}$ there exists n_0 such that $f_{v,\widetilde{v}}(\left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)) = 0$ for all $n > n_0$. Since V is supercuspidal, Lemma 3.5.3 implies that for any $v \in V(N)$ there exists a $U \subseteq N$ open and compact satisfying

$$\int_U \pi(u)v du = 0.$$

Also, $\widetilde{V}$ smooth implies there is a $U' \subseteq N$ compact and open such that $U'\widetilde{v} = \widetilde{v}$. Choose n_0 such that

$$\begin{pmatrix} \varpi^{n_0} & \\ & 1 \end{pmatrix} U \begin{pmatrix} \varpi^{-n_0} & \\ & 1 \end{pmatrix} \subset U'.$$

To see that this is possible, notice that multiplying by $\begin{pmatrix} \varpi^{n_0} & \\ & 1 \end{pmatrix}$ shrinks U. Indeed,

$$\begin{pmatrix} \varpi^{n_0} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^{-n_0} & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varpi^{n_0}x \\ & 1 \end{pmatrix}. \tag{3.6.2}$$

Then

$$\begin{aligned} 0 &= \langle \int_U \pi\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} \pi(u)vdu, \widetilde{v}\rangle \\ &= \langle \int_U \pi(u')\pi\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} vdu, \widetilde{v}\rangle \end{aligned}$$

for $u' = \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} u \begin{pmatrix} \varpi^{-n} & \\ & 1 \end{pmatrix}$. Notice equation (3.6.2) implies that $du = |\varpi^n|\, du'$. So, if $n > n_0$ then

$$\begin{aligned}
0 =& \int_{\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} U \begin{pmatrix} \varpi^{-n} & \\ & 1 \end{pmatrix}} \langle \pi(u')\pi \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} v, \widetilde{v}\rangle du'\, |\varpi^n| \\
=& |\varpi^n| \int < \pi \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} v, \widetilde{\pi}((u')^{-1})\widetilde{v} > du' \\
=& |\varpi^n| \int \langle \pi \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} v, \widetilde{v}\rangle du' \\
=& f_{v,\widetilde{v}} \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} |\varpi^n|\, \mu(\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} U \begin{pmatrix} \varpi^{-n} & \\ & 1 \end{pmatrix}) \\
=& f_{v,\widetilde{v}} \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} \mu(U).
\end{aligned}$$

□

3.7. **Corollaries to Theorem 3.6.1.** This crucial theorem allows us to deduce some strong conclusions about supercuspidal representations. In this section we assume that (π, V) is an irreducible admissible supercuspidal representation with unitary central character.

3.7.1. *A G_F-invariant Hermitian positive definite pairing on V.* Choose a nonzero $\widetilde{v} \in \widetilde{V}$, and for $v_1, v_2 \in V$ define

$$(v_1, v_2) := \int_{Z\backslash G} \langle \pi(g)v_1, \widetilde{v}\rangle \overline{\langle \pi(g)v_2, \widetilde{v}\rangle} dg = \int_{Z\backslash G} f_{v,\widetilde{v}}(g)\overline{f_{v_2,\widetilde{v}}(g)} dg. \tag{3.7.1}$$

Corollary 3.7.1. *The pairing defined in equation* (3.7.1) *is a G_F-invariant positive definite Hermitian pairing.*

Proof. Notice that we need ω to be unitary in order for this to be well-defined Also, the integral make sense because (by the theorem!) matrix coefficients are compactly supported. It is clearly G_F-invariant, positive and Hermitian, so we just need to show that it is definite. If $(v, v) = 0$, then

$$0 = \langle \pi(g)v, \widetilde{v}\rangle = 0 = \langle v, \widetilde{\pi}(g^{-1})\widetilde{v}\rangle$$

for all g. Then irreducibility of $\widetilde{V}$ implies that $\langle v, \widetilde{v}\rangle = 0$ for all $\widetilde{v}$. Thus $v = 0$. □

This corollary implies that V is unitary. Moreover, we have the following.

Corollary 3.7.2. *With V as above, let $\overline{V}$ be the complex conjugate of V. (As a set $\overline{V} = V$, but that action of $\lambda \in \mathbb{C}$ is given by $\lambda \circ v = \overline{\lambda} v$.) Then the following hold.*

- *The map $\Phi : \overline{V} \to \widetilde{V}$ given by $v \mapsto (\cdot, v)$ is a isomorphism.*
- *Every G_F-invariant, Hermitian, positive definite pairing on V is unique up to scaling.*
- *(Orthogonality Relations)*

If $(\cdot,\cdot)$ is a G_F-invariant Hermitian positive definite pairing then there exists a $d > 0$ such that for any $u, v, w, y \in V$

$$\int_{Z\backslash G} (\pi(g)u, w)\overline{(\pi(g)v, y)}dg = d^{-1}(u, v)\overline{(w, y)}. \tag{3.7.2}$$

Choose $v_0 \in V$ such that $(v_0, v_0) = d$, then

$$\int_{Z\backslash G} \pi(g)v_0\overline{(\pi(g)v_0, u)}dg = u \tag{3.7.3}$$

for any $u \in V$.

Proof. The fact that $\overline{V} \longrightarrow \widetilde{V}$ is an vector space homomorphism follows from the fact that $(\cdot,\cdot)$ is Hermitian. The following computation shows that Φ is an intertwining operator.

$$\widetilde{\pi}(g)\Phi(v)(w) = (\widetilde{\pi})(\cdot, v)(w) = (\pi(g^{-1})w, v) = (w, \pi(g)v) = \Phi(\pi(g))(w)$$

The first two equalities are by definition, and the next follows from G_F-invariance. From this computation, together with Lemma 3.3.4, it follows that Φ is actually an isomorphism. (Notice that we have used admissibility here.)

Let Φ_1, Φ_2 be the maps (as above) resulting from two G_F-invariant, Hermitian positive definite forms. The above says that $\Phi_2^{-1} \circ \Phi_1$ is an isomorphism, so by Schur's lemma, it must be a scalar. The result follows.

To prove equation (3.7.2) of the orthogonality relations, notice that by fixing w and y, the left hand side is a G_F-invariant Hermitian positive definite form, hence a multiple of (u, v). Denote this multiple by $F(w, y)$. Similarly, fixing u and v, we get that this is also equal to $\overline{(w, y)}$ times a constant which we call $G(u, v)$. Thus

$$(u, v)F(w, y) = G(u, v)\overline{(w, y)},$$

where the functions F and G are not identically zero. (That they aren't zero can be seen by taking $w = y \neq 0$ for example.) It follows that the left hand side of equation (3.7.2) is a nonzero multiple of $(u, v)\overline{(w, y)}$.

To prove the second part, let LHS denote the left hand side of equation (3.7.3), and notice that

$$(\mathrm{LHS}, w) = \int (\pi(g)v_0, w)\overline{(\pi(g)v_0, u)}dg = d^{-1}(v_0, v_0)\overline{(w, u)} = \overline{(w, u)} = (u, w).$$

Since this works for any w, the left hand side must be u. □

Remark. The constant $d = d_\pi$ is called the *formal degree* of π. It exists whenever π is *square-integral.* A representation is square-integrable if for some (or, equivalently, all) $u, v \in \pi$, $g \mapsto \langle \pi(g)u, v\rangle$ is an L^2 function. The formal degree depends, obviously, on the choice of Haar measure.

Remark. So we see that this one function allows us to use one element to generate all the others. (This remark is supposed to relate to equation 3.7.3, but I have no idea what it really means! Any explanation someone could give to me would be greatly appreciated.)

Corollary 3.7.3. *Suppose that (π, V) is a supercuspidal irreducible G_F-module, (π', V') is admissible, and there exists a nonzero intertwining operator $A : V' \to V$. Then V is equivalent to a subrepresentation of V'. That is, there exists $B : V \to V'$ nonzero.*

Note that, by irreducibility of V, A must be surjective and B must be injective.

Proof. As in Corollary 3.7.2, choose $v_0 \in V$ such that $(v_0, v_0) = d$ and $v_0' \in V'$ which is a preimage of v_0. Define

$$B(u) := \int_{Z\backslash G} \pi'(g)(v_0')\overline{(\pi(g)v_o, u)}dg.$$

Then B is a homomorphism of G_F-modules, and calculating AB directly, we see that

$$AB(u) = A\left(\int_{Z\backslash G} \pi'(g)(v_0')\overline{(\pi(g)v_o, u)}dg\right) = \int_{Z\backslash G} \pi(g)A(v_0')\overline{(\pi(g)v_o, u)}dg = u$$

by equation (3.7.3). □

3.7.2. *Finishing the proof of Theorem 3.5.7.* Recall that the theorem says that a representation (π, V) is not supercuspidal if and only if there exists an intertwining operator from V to some $B(\chi_1, \chi_2)$. The above corollaries allow us to complete the proof.

Proof of Theorem 3.5.7. ($\Longleftarrow$) Note that by Corollary 3.7.3 we only need to show that V is not a subrepresentation of $B(\chi_1, \chi_2)$. Let e denote the identity element of G_F, and consider

$$L : B(\chi_1, \chi_2) \to \mathbb{C} \quad f \mapsto f(e).$$

Then $L \neq 0$, and the calculation

$$(\rho(n)f - f)(e) = f(en) - f(e) = 0$$

for $n = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in N$ shows that L vanishes on $V(N)$. (Recall that $V(N)$ is generated by elements of the form $\pi(n)v - v$.) However, L can not be identically zero (on V) because if $f \in V$ is nonzero then there exists $g \in G_F$ such that $f(g) \neq 0$. Thus $L(\rho(g)f) = (\rho(g)f)(e) \neq$. So, we conclude that $V_N \neq 0$. But this is a contradiction since V is supercuspidal—by definition $V_N = 0$. □

So we have two types of irreducible admissible representations—those coming from induction and the supercuspidals.

Remark. We will see that there are two ways to define L-factors. We can use Whittacker models or matrix coefficients. These are important for calculations of periods and special values.

3.8. **Composition series of $B(\chi_1, \chi_2)$.** In this section we will determine all of the irreducible quotients of $B(\chi_1, \chi_2)$. To this end we determine the composition series:

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = B(\chi_1, \chi_2)$$

such that V_i/V_{i-1} is irreducible. The length m of a composition series and the irreducible quotients V_i/V_{i-1} (up to ordering) are invariants of V, and such a series determines all the irreducible quotients. We use the exactness of the Jacquet functor to aid our study.

Theorem 3.8.1. *The dimension of $B(\chi_1, \chi_2)_N$ is 2. As a result, the length of the composition series for $B(\chi_1, \chi_2)$ is at most 2.*

We note that length 1 and length 2 both exist. We will see this later. Also, for notational convenience let e be the identity element of G_F and $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof. The second conclusion follows from the first because

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = B(\chi_1, \chi_2)$$

implies

$$0 \subsetneq V_{1,N} \subsetneq V_{2,N} \subsetneq \cdots \subsetneq V_{m,N} = B(\chi_1, \chi_2)_N$$

because the Jacquet functor is exact. But $B(\chi_1, \chi_2)_N$ is two dimensional so there is at most one term in the middle of the second exact sequence.

To prove the first claim we will need a series of conclusions. The Jacquet module depends only on the N-module structure. So we decompose as an N-module first. Recall the Bruhat decomposition

$$G = P \sqcup PwN, \text{ with } w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{3.8.1}$$

Notice that PwN is open in G, since it is the union of translates of N which is open. Define

$$V := \{\phi \in B(\chi_1, \chi_2) : \phi \text{ is supported on } PwN\},$$

a sub P-module. Then

$$1 \longrightarrow V \longrightarrow B(\chi_1, \chi_2) \longrightarrow \mathbb{C} \longrightarrow 1$$

with the map to $\mathbb{C}$ defined by $\phi \mapsto \phi(e)$ is an exact sequence of N-modules. (The action on $\mathbb{C}$ is trivial.) Hence

$$1 \longrightarrow V_N \longrightarrow B(\chi_1, \chi_2)_N \longrightarrow \mathbb{C} \longrightarrow 1$$

is an exact sequence of A-modules. (A does not act trivially on $\mathbb{C}$, but by a character.)

Note that in order to show that $\dim_{\mathbb{C}}(B(\chi_1, \chi_2) = 2$, it suffices to show that V_N has dimension 1 because the space of functions supported on P clearly has dimension 1. We claim that $V \simeq C_c^\infty(F)$ via the map

$$\phi \mapsto f_\phi : x \mapsto \phi\left(w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right). \tag{3.8.2}$$

Recall that the left action of P on ϕ is fixed (see Definition 3.4.1), so ϕ is determined by its values on wN. We check that this is a well-defined bijection.

Note that the action ρ of N on V is by right multiplication, and the action of $N \simeq F$ on $C_c^\infty(F)$ is by addition. Explicitly, if $\rho_t(y)$ denotes the translation action of F, we have

$$\begin{aligned} \rho_t(y) f_\phi(x) \quad &:= f_\phi(x+y) = \phi(w \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}) \\ &= \phi(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}) \\ &= \rho\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) \phi\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \\ &= f_{\rho(\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right))\phi}(x). \end{aligned} \tag{3.8.3}$$

We need to check that $f \in C_c^\infty(F)$. The fact that f is locally constant follows immediately from the above calculation. By definition, ϕ is invariant by some L which we can assume to be contained in N. Then $\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \in L$ if and only if $|y|$ is sufficiently small. So equation (3.8.3) implies that, for all such y, $f_\phi(x+y) = f_\phi(x)$.

We next show that f_ϕ has compact support. That is[5], we must show that there exists m such that $v(x) < m$ implies $f_\phi(x) = 0$. Since

$$w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} & 1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ & x \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix},$$

$$f_\phi(x) = \chi_1(-x^{-1})\chi_2(x)\,|x|^{-2}\,\phi\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}.$$

But ϕ is invariant by $L \subseteq K$, so there exists n such that $v(y) > n$ implies that $\phi\left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right) = \phi(e) = 0$. In other words, if $v(x) < -n$ then $v(x^{-1}) > n$ and

$$f_\phi(x) = \chi_1(-x^{-1})\chi_2(x)\,|x|^{-2}\,\phi(e) = 0.$$

Note that $\phi(e) = 0$ because $e \in P$ and $\phi\,|_P = 0$. We have shown that the map of equation (3.8.2) is well defined.

We next show that it is a bijection. It is injective because ϕ is uniquely determined by its values on x. To prove surjectivity, given $f \in C_c^\infty(F)$ define

$$\phi_f(g) := \begin{cases} 0 & \text{if } g \in P \\ \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(x) & g = \begin{pmatrix} a_1 & y \\ & a_2 \end{pmatrix} w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}. \end{cases} \tag{3.8.4}$$

We note that the decomposition in the second case is unique, and that it clearly implies that ϕ_f satisfies property (i) of Definition 3.4.1. The hard part is to show that ϕ_f satisfies property (ii), i.e. that ϕ is invariant under a compact open subgroup L. Consider the following subgroups of K:

$$\begin{pmatrix} 1+\varpi^n\mathcal{O}_F & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1+\varpi^n\mathcal{O}_F \end{pmatrix}, \begin{pmatrix} 1 & \varpi^n\mathcal{O}_F \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ \varpi^n\mathcal{O}_F & 1 \end{pmatrix}.$$

If f is invariant by each the above groups for large enough n, then f is invariant by

$$K_n := \{g \in K : g \equiv I \pmod{\varpi^n\mathcal{O}_F}\}.$$

Thus is suffices to treat each subgroup separately.

The group $\begin{pmatrix} 1 & \varpi^n\mathcal{O}_F \\ & 1 \end{pmatrix}$ is easy since f is invariant by right translation for large enough n. We need

$$\phi_f\left(g\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\right) = \phi_f(g) \tag{3.8.5}$$

when $|x|$ is small. If $g \in P$ both sides of equation 3.8.5 are 0. If $g = \left(\begin{smallmatrix} a_1 & z \\ & a_2 \end{smallmatrix}\right) w \left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right)$ then

$$\phi_f\left(g\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\right) = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(x+y),$$

while

$$\phi_f(g) = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(y).$$

But $f \in C_c^\infty$. So for x small enough we have equality.

[5]Note that if F is a local nonarchimedean field, there is a valuation v and an absolute value $|\cdot|$ that are related by $|x| = c^{v(x)}$ where $0 < xc < 1$. Therefore, $v(x)$ large corresponds to $|x|$ small and vice versa.

For the lower diagonal group, we need to show that

$$\phi_f(g\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}) = \phi_f(g) \tag{3.8.6}$$

when $|x|$ is small. Again we have $g \in P$ or $g \in PwN$. If $g = \left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) \in P$ then the right hand side of equation (3.8.6) is zero and the left hand side equals

$$\begin{aligned}
\phi_f\left(\left(\begin{smallmatrix} a_1 & y \\ & a_2 \end{smallmatrix}\right) w \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) =& \phi_f\left(\left(\begin{smallmatrix} a_1 & y \\ & a_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x & 1 \\ & x^{-1} \end{smallmatrix}\right) w \left(\begin{smallmatrix} 1 & x^{-1} \\ & 1 \end{smallmatrix}\right)\right) \\
=& \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \chi_1(-x^{-1})\chi_2(x^{-1})\,|x|^2\,\phi_f\left(\left(\begin{smallmatrix} 1 & x^{-1} \\ & 1 \end{smallmatrix}\right)\right) \\
=& \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \chi_1(-x^{-1})\chi_2(x^{-1})\,|x|^2\,f(x^{-1}).
\end{aligned}$$

Since f is compactly supported it vanishes when $\left|x^{-1}\right|$ is big. So for $|x| \ll 1$, the left hand side is also zero. The check involving $g \notin P$ is left as an exercise.

The diagonal cases are also left as an exercise. Note that the identity

$$\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix}\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} = \begin{pmatrix} a_1\alpha & x \\ & a_2 \end{pmatrix} = \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\begin{pmatrix} a_1 & \alpha^{-1}x \\ & a_2 \end{pmatrix}$$

allows one to pull the the extra bit into the character part. A similar calculation does the same for the final group.

The final step of the proof is to show that $C_c^\infty(F)_N$ is 1-dimensional. To do this we consider its dual which consists of linear functionals L on $C_c^\infty(F)$ satisfying

$$L(f(x + \ \cdot\ )) = L(f(\cdot))$$

It is a well-known fact of analysis that every translation invariant linear function this is $Lf = \int_F f(x)dx$ up to scalar. So

$$C_c^\infty(F)_N \simeq (C_c^\infty(F)_N)^\vee = \left\{\lambda \int_F (\ \cdot\ )dx \mid \lambda \in \mathbb{C}\right\} \simeq \mathbb{C}.$$

□

The following theorem completes our classification of the composition series of $B(\chi_1, \chi_2)$.

Theorem 3.8.2. *If χ_1, χ_2 are characters of $F^\times$ and $B(\chi_1, \chi_2)$ the corresponding induced representation.*

(i) *If $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$ then $B(\chi_1, \chi_2)$ is irreducible.*
(ii) *If $\chi_1\chi_2 = |\cdot|^{-1}$, that is $\chi_1 = \chi\,|\cdot|^{-1/2}$ and $\chi_2 = \chi\,||^{1/2}$, then $\mathbb{C}(\chi \circ \det)$ is an invariant subspace of $B(\chi_1, \chi_2)$, and its quotient, denoted $\sigma(\chi_1, \chi_2)$ is irreducible.*
(iii) *If $\chi_1\chi_2^{-1} = |\cdot|$ then there is an irreducible invariant subspace of $B(\chi_1, \chi_2)$ with a 1-dimensional quotient isomorphic to $\mathbb{C}(\chi^{-1} \circ \det)$.*

Proof. Part (i) will be handled in the next section using the interwining operator. Parts (ii) and (iii) are a consequence of Theorem 3.8.1, the following lemma and the observation that, in the case of (ii), $\mathbb{C}(\chi \circ \det) \subset B(\chi_1, \chi_2)$, and it is fixed by the action of G_F. To see this, we calculate directly that

$$(\chi \circ \det)\left(\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} g\right) = \chi(a_1a_2)\chi \circ \det(g),$$

and therefore $\chi \circ \det \in B(\chi_1, \chi_2)$. Also,

$$\rho(g)(\chi \circ \det) = \chi(\det g) \cdot \chi \circ \det .$$

So $\mathbb{C}(\chi \circ \det)$ is invariant by G_F.

Now suppose $\chi_1 \chi_2^{-1} = |\cdot|$. Then $\chi_1^{-1}\chi_2 = |\cdot|^{-1}$, and $\chi_1 = \chi\,|\cdot|^{-1/2}$. By the above, $B(\chi_1^{-1}, \chi_2^{-1}) \supsetneq \mathbb{C}(\chi^{-1} \circ \det)$, and Lemma 3.8.3 therefore implies that $B(\chi_1, \chi_2)$ contains the dual of that one dimensional space. By Theorem 3.8.1, this dual is irreducible, so (again by the lemma) its quotient is isomorphic to $\chi^{-1}(\det(\cdot))\cdot\mathbb{C}$. $\square$

Lemma 3.8.3. $B(\chi_1, \chi_2)^{\sim} \simeq B(\chi_1^{-1}, \chi_2^{-1})$. *Actually there is a G_F-invariant non-degenerate pairing $B(\chi_1, \chi_2) \times B(\chi_1^{-1}, \chi_2^{-1}) \to \mathbb{C}$ given by $(\phi_1, \phi_2) \mapsto \int_K \phi_1\phi_2(k)dk$.*

Note that $\phi_1\phi_2(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) g) = \left|\frac{a_1}{a_2}\right| \phi_1\phi_2(g)$.

Exercise 3.8.4. Prove the lemma. In particular, show that the pairing defined above is G_F-invariant.

3.9. **Intertwining operators.** The goal of this section is to determine if there are any relations among the various $B(\chi_1, \chi_2)$. So we are looking for "intertwining operators" between $B(\chi_1, \chi_2)$ and some $B(\eta_1, \eta_2)$. (Recall that we know by Corollary 3.7.3, that there are no intertwining maps from $B(\chi_1, \chi_2)$ to any irreducible supercuspidal representation V.) In other words, we want to consider $\mathrm{Hom}_G(B(\chi_1, \chi_2), B(\eta_1, \eta_2))$. The following lemma tells us that the existence of $M \in \mathrm{Hom}_G(B(\chi_1, \chi_2), B(\eta_1, \eta_2))$, is equivalent to the existence of a linear functional L on $B(\chi_1, \chi_2)$ satisfying certain properties.

Lemma 3.9.1. *The following are equivalent.*

- *There exists a nonzero $M \in \mathrm{Hom}_G(B(\chi_1, \chi_2), B(\eta_1, \eta_2))$.*
- *There exists a nonzero linear functional $L : B(\chi_1, \chi_2) \to \mathbb{C}$ satisfying*

$$L(\rho \begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} \phi) = \eta_1(a_1)\eta_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} L(\phi). \tag{3.9.1}$$

Proof. Suppose there exists a nonzero $M \in \mathrm{Hom}_G(B(\chi_1, \chi_2), B(\eta_1, \eta_2))$. Then let $L : B(\chi_1, \chi_2) \to \mathbb{C}$ be given by $\phi \mapsto M\phi(e)$. This satisfies (3.9.1). Indeed,

$$\begin{aligned} L(\rho \left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) \phi) &= M(\rho \left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) \phi)(e) \\ &= \rho(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right)) M\phi(e) \\ &= M\phi(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right)) \\ &= \eta_1(a_1)\eta_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} L(\phi). \end{aligned}$$

The second equality is due to the fact that M is intertwining, and the final equality follows since $M\phi \in B(\eta_1, \eta_2)$.

On the other hand, given L satisfying equation (3.9.1), for $\phi \in B(\chi_1, \chi_2)$ define $M(\phi)(g) = L(\rho(g)\phi)$. Then $M\phi \in B(\eta_1, \eta_2)$, and M is a G-homomorphism. $\square$

3.9.1. *Relations among $B(\chi_1, \chi_2)$.* So, as mentioned above, if we want to define an intertwining map, we must define certain good functionals.

Proposition 3.9.2.

$$\mathrm{Hom}_G(B(\chi_1,\chi_2),B(\eta_1,\eta_2)) = \begin{cases} \{0\} & \textit{if } (\eta_1,\eta_2)\neq(\chi_1,\chi_2) \textit{ or } (\chi_2,\chi_1) \\ \mathbb{C} & \textit{otherwise} \end{cases}.$$

In other words, if a linear functional as above exists then it is unique up scalar.

Proof. Let $L : B(\chi_1,\chi_2) \to \mathbb{C}$ be as in Lemma 3.9.1. Then L is a functional on $B(\chi_1,\chi_2)_N$ because $L(\rho\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\phi) = L\phi$ which implies $L(\rho\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\phi - \phi) = 0$, hence L is trivial on $B(\chi_1,\chi_2)(N)$. Recall that $V = \{\phi \in B(\chi_1,\chi_2) : \phi \mid_P = 0\} \simeq C_c^\infty(F)$[6],

$$0 \longrightarrow V_N \xrightarrow{\alpha} B(\chi_1,\chi_2)_N \xrightarrow{\beta} \mathbb{C} \longrightarrow 0, \tag{3.9.2}$$

where $\beta([\phi]) = \phi(e)$, is an exact sequence. □

Lemma 3.9.3. *The action of A, the subgroup of diagonal matrices, is given by $\chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2}$ on the right hand side of (3.9.2), and by $\chi_2(a_1)\chi_1(a_2)\left|\frac{a_1}{a_2}\right|^{1/2}$ on $V_N \simeq \mathbb{C}$.*

Proof. The action on the right hand side is clear from the fact that the following diagram must commute.

$$\begin{tikzcd} {[\phi](\,\cdot\,)} \arrow[r, "\beta"] \arrow[d, "\rho(\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right))"'] & \phi(e) \arrow[d] \\ {[\phi](\,\cdot\,\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right))} \arrow[r, "\beta"] & \phi(\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right)) = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \end{tikzcd}$$

For the second part of the proof, recall that $V \simeq C_c^\infty(F) \to C_c^\infty(F)_N$ via $\phi \mapsto f_\phi \mapsto [f_\phi]$. A must act by a scalar. To determine it, we use $C_c^\infty(F)_N^\vee = \mathbb{C}\Lambda$

[6]The isomorphism is $\phi \mapsto f_\phi : x \mapsto \phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))$.

where Λ denotes the integration operator. Then we have

$$\begin{aligned}
\Lambda([\rho\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right) f_\phi]) &= \int \rho\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right) f_\phi(x)dx \\
&= \int \phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right))dx \\
&= \int \phi(w\left(\begin{smallmatrix} a_1 & a_2x \\ & a_2 \end{smallmatrix}\right))dx \\
&= \int \phi(w\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a_1^{-1}a_2x \\ & 1 \end{smallmatrix}\right))dx \\
&= \int \phi(\left(\begin{smallmatrix} a_2 & \\ & a_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a_1^{-1}a_2x \\ & 1 \end{smallmatrix}\right))dx \\
&= \chi_1(a_2)\chi_2(a_1)\left|\frac{a_2}{a_1}\right|^{1/2} \int f_\phi(a_1^{-1}a_2x)dx \\
&= \chi_1(a_2)\chi_2(a_1)\left|\frac{a_2}{a_1}\right|^{1/2} \int f_\phi(x)d(a_1a_2^{-1}x) \\
&= \chi_1(a_2)\chi_2(a_1)\left|\frac{a_1}{a_2}\right|^{1/2} \int f_\phi(x)dx \\
&= \chi_1(a_2)\chi_2(a_1)\left|\frac{a_1}{a_2}\right|^{1/2} \Lambda([f_\phi]),
\end{aligned}$$

where the second to last equality follows since $d(a_1a_2^{-1}x) = \left|a_1a_2^{-1}\right| dx$. Thus we have $\rho(\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right))[f_\phi] = \chi_1(a_2)\chi_2(a_1)\left|\frac{a_1}{a_2}\right|^{1/2}[f_\phi]$ as desired. $\square$

With the following lemma, we can now prove Proposition 3.9.2 in the case that $\chi_1 \neq \chi_2$. Lemma 3.9.3 implies that $B(\chi_1, \chi_2) \simeq \mathbb{C} \oplus \mathbb{C}$ as an A-module with the given actions. So the following lemma completes the proof in the case $\chi_1 \neq \chi_2$.

Lemma 3.9.4. *Suppose $W = W_1 \oplus W_2$ with W_i 1-dimensional and A acts on W_i by μ_i. If L is a nonzero functional on W such that $L(a \cdot w) = \mu(a)L(w)$ then $\mu_1 \neq \mu_2$ implies that $\mu = \mu_1$ or μ_2.*

Proof. Choose $w_i \in W_i$ such that $L(w_i) \neq 0$. Then $L(a \cdot w_i) = L(\mu_i(a)w_i) = \mu_i(a)Lw_i = \mu(a)Lw_i$. Thus $\mu = \mu_i$. $\square$

In order to complete the proof of Proposition 3.9.2, we still need to treat the case $\chi_1 = \chi_2 = \chi$. In this case, we can prove the proposition directly. It suffices to prove that $B(\chi, \chi)$ is irreducible. Schur's Lemma would then imply that $\mathrm{Hom}_G(B(\chi,\chi), B(\eta_1, \eta_2))$ is nonzero if and only if $\eta_1 = \eta_2 = \chi$, and is zero otherwise.

Exercise 3.9.5. The map from $B(\chi_1, \chi_2)$ to

$$W := \{f \in C^\infty(F) : f(x) = \frac{c}{|x|} \cdot \chi_1(-1)\chi_1^{-1}\chi_2(x) \text{ for some } c \text{ when } |x| \gg 0\}$$

given by $\phi \mapsto f_\phi : x \mapsto \phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))$ is one-to-one.

Solution to Exercise 3.9.5. Notice that this map is the same as that in (3.8.2). Our proof that $V \simeq C_c^\infty(F)$ nicely generalizes. First, notice that since $w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) =$

$\left(\begin{smallmatrix} -x^{-1} & 1 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)$, we have

$$f_\phi(x) = \phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)) = \chi_1(-1)\chi_1^{-1}\chi_2(x)\,|x|^{-1}\,\phi(e)$$

when $|x| \gg 0$. Thus, the map is well defined. We do not include the rest of the proof at this time, but as a hint, refer to the discussion of equation 3.8.2. $\square$

Note that the constant c in the definition of W can be taken to be $\phi(e)$.

Returning to the case $\chi = \chi_1 = \chi_2$ we have $B(\chi,\chi) = B(1,1)\otimes(\chi\circ\det)$, so we may assume that $\chi = 1$. We show that W (as in the exercise) is irreducible as a G_F-module. To do this, we first need to understand the action. By Bruhat decomposition, we basically just need to know how w, $\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right)$ act, since these matrices generate G_F. This is accomplished in the following diagrams.

$$\begin{tikzcd} \phi \arrow[r] \arrow[d, "\rho\left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right)"'] & f_\phi \arrow[d] \\ \phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right)) \arrow[r] & f_\phi(x+y) \end{tikzcd}$$

So N acts by translation.

$$\begin{tikzcd} \phi \arrow[r] \arrow[d, "\rho\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)"'] & f_\phi \arrow[d] \\ \phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)) \arrow[r] & |a|^{-1/2}\,f_\phi(a^{-1}x) \end{tikzcd}$$

since

$$\begin{aligned} \phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)) =& \phi(w\left(\begin{smallmatrix} a & x \\ & 1 \end{smallmatrix}\right)) \\ =& \phi(w\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a^{-1}x \\ & 1 \end{smallmatrix}\right)) \\ =& \phi((\begin{smallmatrix} 1 & \\ & a \end{smallmatrix})\,w\left(\begin{smallmatrix} 1 & a^{-1}x \\ & 1 \end{smallmatrix}\right)) \\ =& |a|^{-1/2}\,f_\phi(a^{-1}x). \end{aligned}$$

Similarly, one sees that $\left(\begin{smallmatrix} 1 & \\ & b \end{smallmatrix}\right)$ acts on $f_\phi(x)$ by $|b|^{1/2}\,f_\phi(bx)$.

$$\begin{tikzcd} \phi \arrow[r] \arrow[d, "\rho(w)"'] & f_\phi \arrow[d] \\ \phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)w) \arrow[r] & |x|^{-1}\,f_\phi(x^{-1}) \end{tikzcd}$$

since

$$\phi(w\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)w) = \phi((\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix})) = |x|^{-1}\,\phi(w\left(\begin{smallmatrix} 1 & x^{-1} \\ & 1 \end{smallmatrix}\right)) = |x|^{-1}\,f_\phi(x^{-1}),$$

where we have used the fact that

$$\left(\begin{smallmatrix} -x^{-1} & * \\ & x \end{smallmatrix}\right)w\left(\begin{smallmatrix} 1 & x^{-1} \\ & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right).$$

We want to show that for any nonzero $f\in W$, $W' = \operatorname{span}\{\rho(g)f\} = W$. This will follow if we can show that $C_c^\infty(F)\subset W'$ and that there exists an $f'\in W'$ such that $f'(x)\neq 0$ for all x sufficiently large. We accomplish this in a series of steps.

1. For the given $f\in W$ we may assume that $f(0)\neq 0$ by applying translation.
2. Let $f_1 = w\circ f$, then $f_1(x) = \frac{c}{|x|}$ when $|x|\gg 0$ and $c\neq 0$.

(3) Let $f_2 = |a|^{1/2}\rho(\left(\begin{smallmatrix}1 & \\ & a\end{smallmatrix}\right))f_1$. So $f_2(x) = |a|\, f_1(ax)$.
(4) Set $f_3 = f_2 - f_1 = 0$. By (2), if $|x| \gg 0$ then $f_3(x) = 0$. Moreover, we may assume that $|a|$ is sufficiently large so that $\int f_4(x)dx \neq 0$.
(5) Suppose f_3 is supported in $A = \varpi^n\mathcal{O}_F$ and is invariant by $B = \varpi^m\mathcal{O}_F$ with $n < m$. Then let

$$f_4 = \sum_{A/B} f_3(x+\alpha) = (\int_F f_3 dx)1_{\varpi\mathcal{O}_F}.$$

Hence $1_{\varpi\mathcal{O}_F} \in W'$. Which in turn gives $1_{x+\varpi^k\mathcal{O}_F} \in W'$ for all x and k. Thus $C_c^\infty(F) \subseteq W'$. Finally, to get the desired function f', just take any $f \in C_C^\infty(F)$ such that $f(0) \neq 0$. Acting by w, we get $f_5 = \frac{f(x^{-1})}{|x|}$ which satisfies $f_5(x) = \frac{f(0)}{|x|}$ if $|x| \gg 0$.

This proves that $B(1,1)$ (and thus $B(\chi,\chi)$) is irreducible, and completes the proof of Proposition 3.9.2.

3.9.2. *An explicit functional $L : B(\chi_1,\chi_2) \to \mathbb{C}$.* The nontrivial case of Proposition 3.9.2 is to show that $\mathrm{Hom}_G(B(\chi_1,\chi_2), B(\chi_2,\chi_1))$ has dimension 1. Recall, that we needed $L : B(\chi_1,\chi_2) \to \mathbb{C}$ such that $L(\rho(\left(\begin{smallmatrix}a_1 & * \\ & a_2\end{smallmatrix}\right))\phi) = \chi_2(a_1)\chi_1(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} L\phi$.

In this section we explicitly describe this functional. Our candidate is $L\phi = \int_F \phi(w\left(\begin{smallmatrix}1 & x \\ & 1\end{smallmatrix}\right))dx$.[7] Formally we have the following calculation

$$\begin{aligned}
L(\rho\left(\begin{smallmatrix}a_1 & y \\ 0 & a_2\end{smallmatrix}\right)\phi) &= \int_F \phi(w\left(\begin{smallmatrix}1 & x \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a_1 & y \\ 0 & a_2\end{smallmatrix}\right))dx \\
&= \int_F \phi(w\left(\begin{smallmatrix}1 & x \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a_1 & \\ & a_2\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & a_1^{-1}y \\ & 1\end{smallmatrix}\right)dx \\
&= \int_F \phi(\left(\begin{smallmatrix}a_2 & \\ & a_1\end{smallmatrix}\right)w\left(\begin{smallmatrix}1 & a_1^{-1}a_2x \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & a_1^{-1}y \\ & 1\end{smallmatrix}\right))dx \\
&=\chi_1(a_2)\chi_2(a_1)\left|\frac{a_2}{a_1}\right|^{1/2}\int_F \phi(w\left(\begin{smallmatrix}1 & a_1^{-1}a_2x+a_1^{-1}y \\ & 1\end{smallmatrix}\right))dx \\
&=\chi_1(a_2)\chi_2(a_1)\left|\frac{a_2}{a_1}\right|^{1/2}\int_F \phi(w\left(\begin{smallmatrix}1 & y \\ 0 & 1\end{smallmatrix}\right))d(a_1^{-1}a_2x - a_2^{-1}y) \\
&=\chi_2(a_1)\chi_1(a_2)\left|\frac{a_1}{a_2}\right|^{1/2}\int_F \phi(w\left(\begin{smallmatrix}1 & x \\ 0 & 1\end{smallmatrix}\right))dx = \chi_2(a_1)\chi_1(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} L\phi.
\end{aligned}$$

The issue here is that the integral may not be well defined, but we will get convergence in some region, and then extend by analytic continuation.

Remark. We have

$$M\phi(g) = L(\rho(g)\phi) = \int_F f(w\left(\begin{smallmatrix}1 & x \\ & 1\end{smallmatrix}\right)g)dx = \int_{N_F} f(wng)dn.$$

This final expression is often used in the literature.

The question of convergence comes down to knowing the behavior of

$$\int_F f_\phi(x)dx = \int_F \phi(w\left(\begin{smallmatrix}1 & x \\ & 1\end{smallmatrix}\right))dx$$

[7]Yannan said that this generalizes to higher rank groups. I'm not sure what he meant exactly.

at infinity. To study this, we will use the one-to-one correspondence of Exercise 3.9.5 given by $\phi \mapsto f_\phi : x \mapsto \phi(w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))$. By this correspondence, there exists n_1 such that when $v(x) \leq n_1$, $\phi(w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)) = \chi_1(-1)\chi_1^{-1}\chi_2(x)\frac{\phi(e)}{|x|}$. Let $\chi_i = \chi_{i,0} |\cdot|^{s_i}$ where $\chi_{i,0}$ is unitary.[8] We will see that the analytic continuation will come with respect to $s_2 - s_1$. So

$$\begin{aligned}
\int_{v(x)\leq n_1} |\phi(w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))|\, dx &= \int_{v(x)\leq n_1} \left|\chi_1^{-1}\chi_2(x)\right| \frac{|\phi(e)|}{|x|} dx \\
&= |\phi(e)| \int_{v(x)\leq n_1} |x|^{Re(s_2-s_1)} \frac{dx}{|x|} \\
&= |\phi(e)| \sum_{n\leq n_1} \int_{\varpi^n \mathcal{O}_F^\times} |\varpi^n|^{Re(s_2-s_1)} \frac{dx}{|x|} \\
&= |\phi(e)| \sum_{n\leq n_1} \int_{\mathcal{O}_F^\times} |\varpi^n|^{Re(s_2-s_1)}\, dx \\
&= |\phi(e)|\, m(\mathcal{O}_F^\times) \sum_{k=n_1}^{\infty} \left|q^{-k}\right|^{Re(s_2-s_1)}
\end{aligned}$$

where $q = |\mathcal{O}_F/\varpi\mathcal{O}_F| > 1$. Thus the integral converges if and only if $Re(s_2 - s_1) > 0$. Since $\{x \in F \mid v(x) > n_1$ is compact, L is well defined when $\chi_1 > \chi_2$ where we say $\chi_1 > \chi_2$ if and only if $\left|\chi_1\chi_2^{-1}\right| = |\cdot|^t$ for some $t > 0$.

In order to get analytic continuation, we define $B(\chi_1, \chi_2, s)$ to be the set of $\phi : G \to \mathbb{C}$ satisfying

- $\phi(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) g) = \chi_1(a_1)\chi_2(a_2) \left|\frac{a_1}{a_2}\right|^{(s+1)/2} \phi(g)$,
- ϕ is right invariant by some L an open subgroup of K.

The above discussion implies that

$$L_{\chi_1,\chi_2,s} : B(\chi_1, \chi_2, s) \to \mathbb{C}$$

is well defined for $\mathrm{Re}(s) \gg 0$. Note that given $\phi_s \in B(\chi_1, \chi_2, s)$, we have $\phi_s |_K$ is smooth and it satisfies

$$\phi(\left(\begin{smallmatrix} a_{1,0} & x \\ & a_{2,0} \end{smallmatrix}\right) k) = \chi_{1,0}(a_{1,0})\chi_{2,0}(a_{2,0})\phi(k) \tag{3.9.3}$$

when $a_{i,0} \in \mathcal{O}_F^\times$ and $x \in \mathcal{O}_F$. Define $B_k(\chi_{1,0}, \chi_{2,0})$ to be the smooth functions on K satisfying (3.9.3). Then $\phi_s \mapsto \phi_s |_k$ is one-to-one. Its inverse is the map

$$\phi \mapsto \phi_s(\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) k) := \chi_1(a_1)\chi_2(a_2) \left|\frac{a_1}{a_2}\right|^{(s+1)/2} \phi(k).$$

Note that by the Bruhat decomposition, ϕ_s is defined on all of G_F.

So given $\phi \in B_k(\chi_{1,0}, \chi_{2,0})$ one has $\phi_s \in B(\chi_1, \chi_2, s)$. Define (formally)

$$\begin{aligned}
L_{\chi_1,\chi_2,s}\phi_s &= \int_F \phi_s(w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))dx \\
&= \int_{|x|\leq 1} \phi_s(w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))dx + \int_{|x|>1} \phi_s(w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))dx.
\end{aligned}$$

[8]If we further impose that $s_i \in \mathbb{R}$ then it is unique.

The first integral converges because it is over a compact set, and therefore entire with respect to s. Thus, we need only worry about

$$
\begin{aligned}
\int_{|x|>1} \phi_s(w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))dx = & \int_{|x|>1} \chi_1(-1)\chi_1^{-1}\chi_2(x)\,|x|^s\,\phi_s\left(\begin{smallmatrix} 1 & \\ x^{-1} & 1 \end{smallmatrix}\right)\frac{dx}{|x|} \\
= & \int_{|x|<1} \chi_1(-1)\chi_1\chi_2^{-1}(x)\,|x|^s\,\phi((\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}))d^\times x \\
= & \ \chi_1(-1)Z(1_{\varpi\mathcal{O}_F}(x)\phi((\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}), \chi_1\chi_2^{-1}, s),
\end{aligned} \tag{3.9.4}
$$

where the local zeta function

$$
Z(\Phi, \chi, s) := \int_{F^\times} \Phi(x)\chi(x)\,|x|^s\,d^\times x \tag{3.9.5}
$$

is defined for any $\Phi \in C_c^\infty(F)$, $\chi : F^\times \to \mathbb{C}$ a character, and $s \in \mathbb{C}$.

Write $\chi = \chi_0\,|\cdot|^{s_0}$ where χ_0 is unitary. Since $\Phi \in C_c^\infty(F)$, there exist integers $n_1 > n_2$ such that $\Phi(x) = 0$ when $v(x) < n_2$ and $\Phi(x) = \phi(0)$ when $v(x) \geq n_1$. Thus

$$
Z(\Phi, \chi, s) = \int_{n_2 \leq v(x) \leq n_1} \Phi(x)\chi(x)\,|x|^s\,d^\times x + \int_{v(x)\geq n_1} \Phi(x)\chi(x)\,|x|^s\,d^\times x.
$$

Again, for the first part we are integrating over a compact set so it's entire with respect to $s \in \mathbb{C}$. For the second part we have

$$
\begin{aligned}
\int_{v(x)\geq n_1} \Phi(x)\chi(x)\,|x|^s\,d^\times x = & \Phi(0)\sum_{n\geq n_1}\int_{\varpi^n\mathcal{O}_F^\times} \chi(x)\,|x|^s\,d^\times x \\
= & \Phi(0)\sum_{n=n_1}^{\infty}\int_{\mathcal{O}_F^\times} \chi(\varpi)^n\,|\varpi|^{ns}\,\chi_0(x)d^\times x \\
= & \Phi(0)\left(\int_{\mathcal{O}_F^\times}\chi_0(x)d^\times x\right)\sum_{n=n_1}^{\infty}(\chi(\varpi)\,|\varpi|^s)^n
\end{aligned}
$$

If $\chi_0 \mid_{\mathcal{O}_F^\times}$ is nontrivial this is zero. If $\chi_0 \mid_{\mathcal{O}_F^\times} = 1$ and $|\chi(\varpi)\varpi|^{\mathrm{Re}(s)} < 1$ then it is equal to

$$
\Phi(0)m(\mathcal{O}_F^\times)\frac{(\chi(\varpi)\,|\varpi|^s)^{n_1}}{1-\chi(\varpi)\,|\varpi|^s}.
$$

We call χ *unramified* if $\chi \mid_{\mathcal{O}_F^\times} = 1$, in which case we conclude that $Z(\Phi, \chi, s)$ is analytic. Otherwise, the integral representation for $Z(\Phi, \chi, s)$ converges whenever $|\chi(\varpi)\varpi|^{Re(s)} < 1$. In either case, it converges to something for which the only singularity appears in the form

$$
L(s, \chi) = \begin{cases} 1 & \text{if } \chi \text{ is unramified,} \\ \frac{1}{1-\chi(\varpi)|\varpi|^s} & \text{if } \chi \text{ is ramified.} \end{cases}
$$

This expression is entire except (in the ramified case) where $\chi(\varpi)\,|\varpi|^s = 1$.

If we take $\Phi = 1_{\varpi\mathcal{O}_F} \cdot \phi((\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}))$, and $\chi_1\chi_2^{-1} = \chi = \chi_0\,|\cdot|^{s_0}$ with χ_0 unitary, then we can plug this result into (3.9.4) to get analytic continuation of $L_{\chi_1,\chi_2,s}$. In summary, $L_{\chi_1,\chi_2,s} : B(\chi_1\,|\cdot|^{s/2}, \chi_2\,|\cdot|^{s/2}) = B(\chi_1, \chi_2, s) \to \mathbb{C}$ is defined for all s except at s such that $\chi_1\chi_2^{-1}\,|\cdot|^s = 1$.

We record the following lemma, whose proof is deduced from the above, for future reference.

Lemma 3.9.6. *Suppose that χ is a character of $F^\times$ with $|\chi| = |\cdot|^{s_0}$. If $\Phi \in C_c^\infty(F^\times)$ then the integral representation (3.9.5) of $Z(\Phi, \chi, s)$ converges whenever $s_0 > -1$. $Z(\Phi, \chi, s)$ is holomorphic whenever $\Phi = 0$ or χ is unramified.*

Note that, although the integral may not converge when $s_0 \leq -1$, this result does provide analytic continuation. In fact, more careful consideration of what we have done above would reveal that if we choose an additive character $\psi : F \to S^1 \subset \mathbb{C}^\times$, and define $\hat{\Phi}(x) = \int_F \Phi(y)\psi(xy)dy$ for $\Phi \in C_c^\infty(F)$ then

$$\frac{Z(\hat{\Phi}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi)\frac{Z(\Phi, \chi, s)}{L(s, \chi)} \tag{3.9.6}$$

where $\epsilon(s, \chi, \psi)$ is an exponential function of s.

We have now defined all of the necessary objects in order to give the following proposition.

Proposition 3.9.7. *Suppose $\chi_1 \geq \chi_2$, and $M_{\chi_1,\chi_2} : B(\chi_1, \chi_2) \to B(\chi_2, \chi_1)$, the interwining operator.*

(1) *If $M(\chi_1, \chi_2)\phi \neq 0$ then ϕ generates $B(\chi_1, \chi_2)$.*
(2) *If $\chi_1 \neq \chi_2$ then $B(\chi_1, \chi_2) \to B(\chi_2, \chi_1) \to B(\chi_1, \chi_2)$ is an automorphism, acting by the scalar*

$$\frac{L(0, \chi_1\chi_2^{-1})L(0, \chi_2\chi_1^{-1})}{\epsilon(0, \chi_1^{-1}\chi_2, \psi)\epsilon(0, \chi_1\chi_2^{-1}, \psi)L(1, \chi_1\chi_2^{-1})L(1, \chi_2\chi_1^{-1})}.$$

Remark. The numerator of the expression in the second statement is nonzero by definition. The denominator will be ∞ exactly when $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$. We will treat this case more carefully in Proposition 3.10.1.

Remark. We see that $M(\chi_1, \chi_2)M(\chi_2, \chi_1) = 0$ when $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$.

An important corollary to Proposition 3.9.7 is the following.

Theorem 3.9.8. *When $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$, $B(\chi_1, \chi_2)$ is irreducible.*

Proof. First, assume that $\chi_1 > \chi_2$. Then whether or not $\chi_1\chi_2^{-1}$ is ramified, $M(\chi_2, \chi_1)M(\chi_1, \chi_2)$ is a nonzero scalar. In particular $M(\chi_1, \chi_2)\phi \neq 0$ for all nonzero $\phi \in B(\chi_1, \chi_2)$. Thus, by (1) of Proposition 3.9.7, $B(\chi_1, \chi_2)$ is irreducible. If $\chi_2 > \chi_1$, we argue similarly to conclude that $B(\chi_1^{-1}, \chi_2^{-1})$ (which equals $B(\chi_1, \chi_2)^\sim$ by Lemma 3.8.3) is irreducible. By duality we conclude that $B(\chi_1, \chi_2)$ is irreducible.

If $|\chi_1| = |\chi_2|$ and $\chi_1 \neq \chi_2$, write $\chi_1 = \chi_{1,0} |\cdot|^t$ and $\chi_2 = \chi_{2,0} |\cdot|^t$ where $\chi_{i,0}$ is unitary. Then $B(\chi_1, \chi_2)$ is irreducible if and only if $B(\chi_{1,0}, \chi_{2,0})$ is irreducible since $B(\chi_{1,0}, \chi_{2,0}) \otimes |\det(g)|^t = B(\chi_1, \chi_2)$. So we may assume χ_1, χ_2 are unitary. Then the pairing

$$B(\chi_1, \chi_2) \times B(\chi_2, \chi_1) \to \mathbb{C}$$

defined by

$$(\phi_1, \phi_2) := \int_K \phi_1\overline{\phi_2}(k)dk$$

makes $B(\chi_1, \chi_2)$ a unitary representation. Suppose there exists a nontrivial submodule U. By unitaricity there exists a G-invariant orthogonal complement $U^\perp$ and $B(\chi_1, \chi_2) = U \oplus U^\perp$. If this were the case, $\mathrm{Hom}_G(B(\chi_1, \chi_2), B(\chi_1, \chi_2))$ would be at least two dimensional contradicting Proposition 3.9.2. Thus $B(\chi_1, \chi_2)$ is irreducible in this case.

Finally, if $\chi_1 = \chi_2$ we have already shown irreducibility directly. $\square$

Proof of Proposition 3.9.7 (1). Since the dual of $B(\chi_1, \chi_2)$ is $B(\chi_1^{-1}, \chi_2^{-1})$, to show that ϕ generates $B(\chi_1, \chi_2)$ it suffices to check that for all $\phi' \in B(\chi_1^{-1}, \chi_2^{-1})$, $\langle \rho(g)\phi, \phi' \rangle = 0$ for all g implies that $\phi' = 0$. Indeed, if ϕ does not generate the whole space, then there must exist $\phi' \neq 0$ such that $\langle \rho(g)\phi, \phi' \rangle = 0$ for all g. To proceed we need the following technical lemma.

Lemma 3.9.9. *When $\chi_1 > \chi_2$ (needed to make sense of the integral)*

$$\lim_{|a|\to 0} \chi_2^{-1}(a)\, |a|^{-1/2} \langle \rho \begin{pmatrix} a & \\ & 1 \end{pmatrix} \phi, \phi' \rangle = \left(\int_F \phi[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}] dx \right) \phi'(w).$$

For now we do not prove the lemma but finish the first part of the proposition. By assumption, we have $M\phi \neq 0$. So let g_0 be such that $M\phi(g_0) \neq 0$. Then $M(\rho(g_0)\phi)(e) = M\phi(g_0) \neq 0$. Applying Lemma 3.9.9, we get

$$\begin{aligned} &\left(\int_F \rho(g_0)\phi[\omega \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}] dx \right) \phi'(\omega) \\ &\qquad = \lim_{|a|\to 0} \chi_2^{-1}(a)\, |a|^{-1/2} \langle \rho(\begin{pmatrix} a & \\ & 1 \end{pmatrix} \rho(g_0)\phi, \phi' \rangle = 0, \end{aligned}$$

since we are assuming $\langle \rho(g)\phi, \phi' \rangle = 0$. On the other hand,

$$M(\rho(g_0)\phi)(e) = \int_F \rho(g_0)\phi[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}] dx \neq 0,$$

so $\phi'(w) = 0$.

To show that $\phi'(g) = 0$ for all $g \in G$, consider $\rho(w^{-1}g)\phi'$. Then $\phi'(g) = \rho(w^{-1}g)\phi'(w)$. G-invariance of the pairing $\langle \cdot, \cdot \rangle$ implies that

$$\langle \rho(h)\phi, \rho(w^{-1}g)\phi' \rangle = 0$$

for all $h \in G$. Thus we can apply the lemma again to obtain $\phi'(g) = 0$. Since g was arbitrary, this completes the proof. $\square$

Note that the lemma says $\langle \rho(\cdot)\phi, \phi' \rangle$ is the product of somethings involving only ϕ and something involving only ϕ'. The ϕ part we know to be nonzero, so the ϕ' part is zero.

Exercise 3.9.10. Show that there exists a constant $c \neq 0$ such that for all $\phi \in B(|\cdot|^{1/2}, |\cdot|^{-1/2})$ we have

$$\int_K \phi(k) dk = c \int_F \phi(w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) dx.$$

A hint as to why this is true is as follows. Let $L\phi = \int_K \phi(k)dk$ and $L'\phi = \int_F \phi[w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)]dx$. We need to show that $L = cL'$. Actually, L is G-invariant and L' is N-invariant (obviously) and P-invariant. So L factors through $B(|\cdot|^{1/2}, |\cdot|^{-1/2})_N = \mathbb{C} \oplus \mathbb{C}$. Lemma 3.9.3 gives that A-acts trivially on one summand and by $\left|\frac{a_1}{a_2}\right|$ on the other. L factors through the part fixed by A as does L'. But the only functionals from $\mathbb{C} \to \mathbb{C}$ are those given by scalars. So then L and L' must be the same up to scalar. [The exercise may also be proved by a manipulation of measures.]

Proof of Lemma 3.9.9. By Lemma 3.8.3 the linear functional

$$L : B(|\cdot|^{1/2}, |\cdot|^{-1/2}) \to \mathbb{C} \quad \text{where } \phi \mapsto \int_K \phi(k) dk$$

is G-invariant. Therefore, the exercise implies that (up to a nonzero constant)

$$\langle \phi, \phi' \rangle = \int_F \phi\phi'[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]dx.$$

Since we are only interested in the vanishing or nonvanishing, we will assume equality. So

$$\begin{aligned}
\langle \rho(\begin{pmatrix} a & \\ & 1 \end{pmatrix})\phi, \phi' \rangle =& \int_F \phi[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & 1 \end{pmatrix}]\phi'[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]dx \\
=& \int \chi_2(a)\,|a|^{-1/2}\,\phi[w \begin{pmatrix} 1 & a^{-1}x \\ & 1 \end{pmatrix}]\phi'[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]dx \\
=& \chi_2(a)\,|a|^{-1/2} \int_F \phi[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]\phi'[w \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix}]d(ax) \\
=& \chi_2(a)\,|a|^{1/2} \int_F \phi[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]\phi'[w \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix}]dx
\end{aligned}$$

There exists δ_1, δ_2 such that if $|x| < \delta_1$ then $\phi'[w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)] = \phi'[w]$, and if $|x| > \delta_2^{-1}$ then $\phi[w \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)] = \chi_1(-1)\chi_1^{-1}\chi_2(x)\frac{\phi(e)}{|x|}$. If $|a|\,\delta_2 < \delta_1$, (recall that in the statement of the lemma we will ultimately be taking the limit as $|a| \to 0$), splitting the above integral as $|ax| > \delta_1$ or $|ax| \le \delta_1$ gives the following two integrals.

$$\mathrm{I} := \chi_2(\mathrm{a})\,|\mathrm{a}|^{1/2} \int_{|\mathrm{ax}| \le \delta_1} \left(\phi[\mathrm{w} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]\mathrm{dx} \right) \phi'(\mathrm{w}),$$

and

$$\begin{aligned}
\mathrm{II} :=& \chi_2(a)\,|a|^{1/2} \int_{|x|>|a^{-1}|\delta_1} \phi[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]\phi'[w \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix}]dx \\
=& \chi_2(a)\,|a|^{1/2} \int_{|x|>|a^{-1}|\delta_1} \chi_1(-1)\chi_1^{-1}\chi_2(x)\frac{\phi(e)}{|x|}\phi'[w \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix}]dx \\
=& \chi_2(a)\,|a|^{1/2} \int_{|x|>\delta_1} \chi_1(-1)\chi_1^{-1}\chi_2(ax)\frac{\phi(e)}{|a^{-1}x|}\phi'[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]\left|a^{-1}\right| dx \\
=& \chi_1(-1)\chi_1(a)\,|a|^{1/2}\,\phi(e) \int_{|x|>\delta_1} \phi'(w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\chi_1^{-1}\chi_2(x)\frac{dx}{|x|}.
\end{aligned}$$

Where we have the second equality when $|a| \ll 1$ and we made the change of variables $x \mapsto a^{-1}x$ in the third line. We also consider one additional integral.

$$\begin{aligned}
\mathrm{III} :=& \chi_2(a)\,|a|^{1/2} \int_{|x|>|a^{-1}|\delta_1} \phi[w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}]dx\phi'(w) \\
=& \chi_2(a)\,|a|^{1/2} \left(\int_{|x|>|a^{-1}|\delta_1} \chi_1(-1)\chi_1^{-1}\chi_2(x)\phi(e)d^\times x \right) \phi'(w) \\
=& \chi_1(-1)\chi_2(a)\,|a|^{1/2}\,\chi_1(-1)\phi(e) \left(\int_{|x|>\delta_1} \chi_1^{-1}\chi_2(a^{-1}x)d^\times x \right) \phi'(w) \\
=& \chi_1(a)\,|a|^{1/2}\,\chi_1(-1)\phi(e) \left(\int_{|x|>\delta_1} \chi_1^{-1}\chi_2(x)d^\times x \right) \phi'(w),
\end{aligned}$$

which converges when $\chi_1 > \chi_2$. So we get
(3.9.7)

$$\begin{aligned}\langle\rho(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right))\phi,\phi'\rangle &= (\mathrm{I}+\mathrm{III})-\mathrm{III}+\mathrm{II}\\ &= \chi_2(a)\,|a|^{1/2}\left(\int_F \phi[w\left(\begin{smallmatrix} 1 & x\\ & 1\end{smallmatrix}\right)]dx\right)\phi'(w)\\ &\quad -\chi_1(-1)\chi_1(a)\,|a|^{1/2}\,\phi'(w)\phi(e)\int_{|x|>\delta_1}\chi_1\chi_2^{-1}d^\times x\\ &\quad +\chi_1(-1)\chi_1(a)\,|a|^{1/2}\,\phi(e)\int_{|x|>\delta_1}\phi'[w\begin{pmatrix} 1 & x\\ & 1\end{pmatrix}]\chi_1^{-1}\chi_2(x)d^\times x.\end{aligned}$$

Dividing by $\chi_2(a)\,|a|^{1/2}$ we get

$$\chi_2^{-1}(a)\,|a|^{-1/2}\,\langle\rho(a)\phi,\phi'\rangle = \int_F \phi[w\begin{pmatrix} 1 & x\\ & 1\end{pmatrix}]dx\phi'(w)+\chi_1\chi_2^{-1}(a)\cdot(*),$$

where $(*)$ doesn't depend on a. Letting $|a|\to 0$ gives the desired result because $\chi_1>\chi_2$ implies that $\chi_1\chi_2^{-1}(a)\to 0$ as $|a|\to 0$. □

It remains to prove part (2) of the proposition.

Proof of Proposition 3.9.7 (2). We know $M_{\chi_2,\chi_1}\circ M_{\chi_1,\chi_2}$ is a scalar. To detect it, we choose a particular function and calculate its image. We have seen one way of producing functions in $B(\chi_1,\chi_2)$. That is, we start with $\phi_0\in B_K(\chi_{1,0},\chi_{2,0})$ where $\chi_{i,0}=\chi_i\mid_{\mathcal{O}_F^\times}$. Then we can extend ϕ_0 to a function $\phi\in B(\chi_1,\chi_2)$. For our purposes here, this method won't be easiest. Instead we consider

$$\phi(g)=\chi_1(\det(g))\,|\det(g)|^{1/2}\int_{F^\times}\Phi[(0,t)g]\chi_1\chi_2^{-1}(t)dt,$$

with Φ a Bruhat-Schwarz function (i.e. $\Phi\in C_c^\infty(F^2)$) and $\chi_1\chi_2^{-1}\,|\cdot|<1$. The integral here is well-defined whenever $\chi_1\chi_2^{-1}>1$ (which we are assuming.) Then one readily checks that $\phi\in B(\chi_1,\chi_2)$. We will use this function for χ_1,χ_2 in the given range. Then by analytic continuation the formula is true for all $\chi_1\neq\chi_2$. □

3.10. **Special representations.** Recall that if $\chi: F^\times\to\mathbb{C}^\times$ is a quasicharacter, we have the special representation Sp_χ which is the unique non-trivial invariant subspace inside $B(\chi\,|\cdot|^{1/2},\chi\,|\cdot|^{-1/2})$. We also have the intertwining operators

$$M_\pm : B(\chi\,|\cdot|^{\pm 1/2},\chi\,|\cdot|^{\mp 1/2})\to B(\chi^{\mp 1}\,|\cdot|^{\mp 1/2},\chi^{\mp 1}\,|\cdot|^{\pm 1/2}).$$

In this section we will see that the special representations have many nice properties. For example, like the supercuspidals, specials are unitary (up to a twist) and have matrix coefficients.

Proposition 3.10.1. *Let $\mathbb{C}_{\chi^{-1}}$ be the unique non-trivial irreducible subspace inside $B(\chi\,|\cdot|^{1/2},\chi\,|\cdot|^{-1/2})$. Then the sequences*

$$0\longrightarrow Sp_\chi\longrightarrow B(\chi\,|\cdot|^{1/2},\chi\,|\cdot|^{-1/2})\xrightarrow{M_+}\mathbb{C}_\chi\to 0,$$

and

$$0\longrightarrow\mathbb{C}_{\chi^{-1}}\longrightarrow B(\chi^{-1}\,|\cdot|^{-1/2},\chi^{-1}\,|\cdot|^{1/2})\xrightarrow{M_-}Sp_\chi\longrightarrow 0.$$

are exact.

$\mathbb{C}_\chi$ is the one dimensional subspace on which G acts by $\chi\circ\det$. See Theorem 3.8.2 for our original treatment of the special representation. In fact, that theorem is nearly enough to prove the proposition.

Proof. $M_+ \neq 0$. Since $\ker M_+$ is an invariant subspace, $\ker M_+ = Sp_\chi$ or $\{0\}$. If $\ker M_+ = \{0\}$ then $B(\chi\,|\cdot|^{1/2}, \chi\,|\cdot|^{-1/2}) \hookrightarrow B(\chi^{-1}\,|\cdot|^{-1/2}, \chi^{-1}\,|\cdot|^{1/2})$ is injective. Since both are of length 2 it has to be an isomorphism. However, the only invariant subspace of the image is $\mathbb{C}_{\chi^{-1}}$ which is 1-dimensional, and the image Sp_χ must be infinite dimensional—a contradiction. Thus $\ker M_+ = Sp_\chi$. Proving exactness of the other sequence is similar. $\square$

Lemma 3.10.2. *The following are equivalent:*

- $\phi \in Sp_\chi$,
- $\int_N \phi(wn)dn = 0$,
- $\int_K \phi(k)\chi^{-1}(\det k)dk = 0$,
- $M_+\phi = 0$.

This lemma gives multiple ways to determining whether a given vector comes from a special representation.

Proof. The equivalence of the first and last properties is obvious from Proposition 3.10.1. The second property is just writing out what $M_+\phi$ is. The third property says that $\langle \phi, \chi \circ \det \rangle = 0$. In other words, $\chi \circ \det$ annihilates ϕ, and only in the case that $\phi \in Sp_\chi$ does it have a nontrivial annihilator. $\square$

3.10.1. *Matrix coefficients.* In this section we take

$$\phi \in Sp_\chi \quad \text{and} \quad \phi' \in B(\chi^{-1}\,|\cdot|^{-1/2}, \chi^{-1}\,|\cdot|^{1/2})$$

such that $\langle Sp_\chi, \phi' \rangle \neq 0$. Recall that the pairing given by $\langle \phi, \phi' \rangle = \int_K \phi\phi'(k)dk$ is G-invariant. As in the case of supercuspidals, we get a matrix coefficient

$$f_{\phi,\phi'}(g) := \langle \rho(g)\phi', \phi \rangle = \int_K \phi(kg)\phi'(k)dk.$$

Note that $f_{\phi,\phi'}(\left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right) g) = \chi^2(a) f_{\phi,\phi'}$, so if χ is unitary $|f_{\phi,\phi'}(g)| = |f_{\phi,\phi'}(zg)|$ for all $z \in Z$.

Proposition 3.10.3. *If χ is unitary then $|f_{\phi,\phi'}(g)|$ is square integrable on $Z\backslash G$.*

Corollary 3.10.4. *When χ is unitary, Sp_χ is unitarizable.*

Proof. Choose $\phi' \in B(\chi^{-1}\,|\cdot|^{-1/2}, \chi^{-1}\,|\cdot|^{1/2})$ such that $\langle Sp_\chi, \phi' \rangle \neq 0$ then define $Sp_\chi \times Sp_\chi \to \mathbb{C}$ by

$$(\phi_1, \phi_2) = \int_{Z\backslash G} \langle \rho(g)\phi_1, \phi' \rangle \overline{\langle \rho(g)\phi_2, \phi' \rangle} dg.$$

This is a positive definite Hermitian, G-invariant pairing. So Sp_χ is a unitary representation with respect to this pairing. $\square$

Remark. Note that the proof of the corollary is nearly same as in the supercuspidal case. Also, since Sp_χ is irreducible the pairing is unique up to a scalar.

Proof of Proposition 3.10.3. We use the decomposition $G = \bigsqcup_{n\geq 0} ZK \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} K$ of Lemma 3.6.2.

Step 1. Recall that equation (3.9.7) gives

$$\begin{aligned}\langle\rho(a)\phi,\phi'\rangle =&\chi_2(a)\,|a|^{1/2}\int_F \phi[w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}]dx\phi'(w)\\ &-\chi_1(-1)\chi_1(a)\,|a|^{1/2}\,\phi'(w)\phi(e)\int_{|x|>\delta_1}\chi_1\chi_2^{-1}d^\times x\\ &+\chi_1(-1)\chi_1(a)\,|a|^{1/2}\,\phi(e)\int_{|x|>\delta_1}\phi'[w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}]\chi_1^{-1}\chi_2(x)d^\times x,\end{aligned}$$

when $|a|$ is sufficiently small and δ_1 is such that $\phi'(w\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)) = \phi'(w)$ whenever $|x| < \delta_1$. By Lemma 3.10.2, the first term is 0. So

$$\langle\rho\begin{pmatrix}a & \\ & 1\end{pmatrix}\phi,\phi'\rangle = \chi_1(a)\,|a|^{1/2}\,c(\phi,\phi') \tag{3.10.1}$$

when $|a|$ is sufficiently small.

Step 2. We evaluate the square integral of the matrix coefficient.

$$\int_{Z\backslash G}|f_{\phi,\phi'}(g)|^2\,dg = \sum_{n\geq 0}\int_{K\left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)K}|f_{\phi,\phi'}(g)|^2\,dg. \tag{3.10.2}$$

Consider the surjective map

$$K\times K\to K\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}K \quad (k_1,k_2)\mapsto k_1\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}k_2.$$

Every fiber is (topologically) isomorphic to

$$K_n := \{k\in K \mid k\equiv\begin{pmatrix}* & *\\ & *\end{pmatrix} \pmod{\varpi^n}\}.$$

This implies that

$$\int_{K\left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)K} f(g)dg = [K:K_n]\int_{K\times K} f(k_1\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}k_2)dk_1dk_2$$

So (3.10.2) becomes

$$\begin{aligned}&=\sum_{n\geq 0}[K:K_n]\int_{K\times K}\left|\langle\rho(k_1)\rho(\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix})\rho(k_2)\phi,\phi'\rangle\right|^2 dk_1dk_2\\ &=\sum_{n\geq 0}[K:K_n]\int_{K\times K}\left|\langle\rho(\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix})\rho(k_2)\phi,\rho(k_1^{-1})\phi'\rangle\right|^2 dk_1dk_2.\end{aligned}$$

Let $L\subseteq K$ be a open normal subgroup fixing ϕ and ϕ'. We need only consider the coset representatives of k_1^{-1} and k_2. Explicitly, $K=\bigsqcup k_\alpha L=\bigsqcup Lk_\alpha$. So then $\rho(k_2)\phi$ is one of $\rho(k_\alpha)\phi =: \phi_\alpha$ and $\rho(k_1^{-1})\phi'$ is one of $\rho(k_\beta^{-1})\phi' =: \phi'_\beta$. Then by (3.10.1), there exists n_0 such that if $n\geq n_0$

$$\begin{aligned}|\langle\rho\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\phi_\alpha,\phi'_\beta\rangle| =&|\chi_1(\varpi^n)|\,|\varpi^n|^{1/2}\,c(\phi_\alpha,\phi'_\beta)\\ =&|\chi(\varpi^n)|\,|\varpi|^n\,c(\phi_\alpha,\phi'_\beta)\\ \leq& c\cdot|\varpi|^n\end{aligned}$$

where $c = \max\{|c(\phi_\alpha, \phi'_\beta|\}$. Putting everything together, we have

$$\begin{aligned}\int_{Z\backslash G} |f_{\phi,\phi'}(g)|^2 dg &= \int_{0\le n\le n_0} + \int_{n\ge n_0} \\ &\le \int_{0\le n\le n_0} + c^2 \sum_{n>n_0} [K:K_n]\, |\varpi|^{2n} \\ &= \int_{0\le n\le n_0} + c^2 \sum_{n>n_0} (1+|\varpi|)\, |\varpi|^n\,,\end{aligned}$$

which converges. In the final equality, we have used the result of Exercise 3.10.5 $\square$

Exercise 3.10.5. Show that $[K:K_n] = |\varpi|^{-n}(1+|\varpi|)$.

Solution. We will prove that

$$K = \bigcup_{x\in\mathcal{O}/\varpi^n} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} K^n \cup \bigcup_{y\in\mathcal{O}/\varpi^{n-1}} \begin{pmatrix} 1 & \\ y\varpi & 1 \end{pmatrix} wK^n,$$

and that the cosets are disjoint. From this the result clearly follows.

Let $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K$. First, assume that $v(c) = 0$. Then

$$k = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & d\det k/c \\ & 1 \end{pmatrix}\begin{pmatrix} c & \\ & \det k/c \end{pmatrix}.$$

In other words $k = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) wb$ where $b \in B_K = B\cap K$. If $k' \in K_n$ is such that $\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) wk' = \left(\begin{smallmatrix} 1 & x' \\ & 1 \end{smallmatrix}\right)$ then $k' = \left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right)$ for some $y \in \varpi^n\mathcal{O}$. Since $\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) w \left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & x-y \\ & 1 \end{smallmatrix}\right) w$ it is obvious that the set of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K$ such that $v(c) > 0$ is given by the disjoint union

$$\bigcup_{x \mod \varpi^n} \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) wK_n.$$

Now suppose that $v(c) > 0$. In this case,

$$k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} 1 & \\ ac & 1 \end{pmatrix}\begin{pmatrix} 1 & ba/\det k \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & \det k/a \end{pmatrix}.$$

(Note that since $k \in K$ and $v(c) \ge 1$, the element a must be a unit.) Therefore $k = \left(\begin{smallmatrix} 1 & \\ y\varpi & 1 \end{smallmatrix}\right) b$, $b \in B_K$. If $k' \in K_n$ is such that $\left(\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right) k' = \left(\begin{smallmatrix} 1 & \\ y' & 1 \end{smallmatrix}\right)$ then $k' = \left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)$ for some $x \in \varpi^n$. Hence $y\varpi$ is unique modulo ϖ^n. $\square$

3.11. **Unramified representations and spherical functions.** In this section (π, V) will be an irreducible admissible representation. We will study $V^K = \{v \mid kv = v \text{ for all } k \in K\}$. Let $\mathcal{H}(G,K) := \mathcal{H}_K = \{f \in C_c^\infty(G) \mid f \text{ is } K \text{ bi-invariant}\}$ be the *Hecke algebra of maximum level.* Recall that we first encountered this object in (3.3.1), and in Lemma 3.3.2 we saw that V^K is an irreducible $\mathcal{H}_K$-module.

Lemma 3.11.1. *$\mathcal{H}_K$ is commutative.*

Proof. Using $G = \cup_{n\ge 0} ZK \left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right) K$, it is clear that if $f \in \mathcal{H}_K$ then f is determined by its values on diagonal matrices. For $f \in C_c^\infty(G)$ define ${}^tf(g) = f({}^tg)$. Then it is easily verified from the definition of convolution that ${}^t(f_1 * f_2) = {}^tf_2 * {}^tf_1$. On the other hand, by the comment above, if $f \in \mathcal{H}_K$ then ${}^tf = f$. Let $f_1, f_2 \in \mathcal{H}_K$. Then

$$f_1 * f_2 = {}^t(f_1 * f_2) = {}^tf_2 * {}^tf_1 = f_2 * f_1$$

since $f_1 * f_2$ is also in $\mathcal{H}_K$. $\square$

Since V^K is an irreducible $\mathcal{H}_K$-module, we have the following corollary.

Corollary 3.11.2. $\dim V^K \leq 1$.

Definition 3.11.3. *If $\dim V^K = 1$ then we say that (π, V) is unramified.*

If V is unramified, then $\widetilde{V}$ (the contragradient representation) is also unramified. In this case, choose $e_0 \in V^K$ and $\widetilde{e_0} \in \widetilde{V}^K$ such that $\langle e_0, \widetilde{e_0}\rangle = 1$. The vector e_0 is called the spherical vector*.*

The matrix coefficient

$$\omega(g) := \langle \pi(g)e_0, \widetilde{e_0}\rangle$$

is called the spherical function *of (π, V). It is K bi-invariant.*

Since $\mathcal{H}_K$ acts on V^K by scalars, if $f \in \mathcal{H}_K$ $\pi(f)e_0 = \lambda_\pi(f)e_0$. The algebra homomorphism $f \mapsto \lambda_\pi(f)$ mapping $\mathcal{H}_K$ to $\mathbb{C}$ is called the spherical homomorphism*.*

λ_π and $\omega_\pi(g)$ determine each other. To see this, let $f \in \mathcal{H}_K$. Then

$$\begin{aligned}\int f(g)\omega_\pi(g)dg = \int f(g)\langle \pi(g)e_0, \widetilde{e_0}\rangle dg = \langle \int f(g)\pi(g)e_0 dg, \widetilde{e_0}\rangle \\ =\langle \pi(f)e_0, \widetilde{e_0}\rangle = \lambda_\pi(f)\langle e_0, \widetilde{e_0}\rangle = \lambda_\pi(f).\end{aligned}$$

Conversely, $\omega_\pi(g) = \frac{1}{m(KgK)}\lambda_\pi(1_{KgK})$.

It is also true that ω_π determines π. Let U be the space of functions on G spanned by right translation of ω_π. Then $V \to U$ by $v \mapsto \langle \pi(g)v, \widetilde{e_0}\rangle$ is surjective by definition. By irreducibility of V, it must be an isomorphism.

Proposition 3.11.4. (i) *If (π, V) is unramified and irreducible, and $\omega_\pi(g)$ is the spherical function of π then*

$$\int_K \omega_\pi(g_1 k g_2)dk = \omega_\pi(g_1)\omega_\pi(g_2). \tag{3.11.1}$$

(ii) *If ω is a K bi-invariant function on G satisfying (3.11.1) then there exists a unique unramified representation such that $\omega(g)$ is the spherical function of this representation.*

Remark. Uniqueness follows since ω_π determines π.

Proof. (i) Define $P : V \to V^K$ by $v \mapsto \int_K \pi(k)v dk$. Then $Pv = \lambda e_0$. We calculate

$$\lambda = \langle Pv, \widetilde{e_0}\rangle = \langle \int_K \pi(k)v dk, \widetilde{e_0}\rangle = \int \langle \pi(k)v, \widetilde{e_0}\rangle dk = \langle v, \widetilde{e_0}\rangle.$$

The last equality follows from the K invariance of $\widetilde{e_0}$ and the fact that K has measure 1.

Now,

$$\begin{aligned}\int_K \omega(g_1 k g_2)dk = \int_K \langle \pi(g_1)\pi(k)\pi(g_2)e_0, \widetilde{e_0}\rangle dk \\ =\langle \pi(g_1)\int_K \pi(k)\pi(g_2)e_0 dk, \widetilde{e_0}\rangle \\ =\langle \pi(g_1)\langle \pi(g_2)e_0, \widetilde{e_0}\rangle e_0, \widetilde{e_0}\rangle \\ =\langle \pi(g_2)e_0, \widetilde{e_0}\rangle\langle \pi(g_1)e_0, \widetilde{e_0}\rangle \\ =\omega_\pi(g_2)\omega_\pi(g_1).\end{aligned}$$

(ii) Denote by U the the space of functions on G spanned by right translation of $\omega(g)$. Clearly U is smooth. We show it is irreducible. Suppose $0 \subsetneq U' \subseteq U$ and choose $f \in U'$ with $f \neq 0$. Then we claim that

$$\int_K f(g_1 kh)dk = \omega(g)f(h). \tag{3.11.2}$$

In other words, since this integral is really just a finite linear combination of right translations of f, $\omega \in U'$. But ω generates U, so $U' = U$. To prove the claim, write

$$f = \sum_i \lambda_i \pi(h_i)\omega = \sum_i \lambda_i \omega(\cdot h_i).$$

By the assumption that ω satisfies (3.11.1), we have

$$\int_K \omega(gkhh_i)dk = \omega(g)\omega(hh_i).$$

So this gives the claim. But then U is irreducible and smooth, hence, by Proposition 3.3.3, it is admissible.

Now, we prove that ω is the spherical function of U. Since it is right invariant by K, it is certainly a spherical vector. So consider

$$L_e : U \to \mathbb{C} \quad \text{via} \quad f \mapsto f(e).$$

Then L_e is a smooth vector in U^*, and so $L_e \in \widetilde{U}$. Actually, L_e is invariant by K and hence $L_e \in (U^\wedge)^K$, and $(\omega, L_e) = \omega(e) = 1$. The spherical function of U is thus

$$\omega_U(g) = \langle \omega(g), L_e \rangle = \omega(e \cdot g) = \omega(g).$$

We conclude that U is unramified. □

Remark. By this proposition, the spherical function determines the representation. Later we will use this to get the L-factor.

Theorem 3.11.5. *Let (π, V) be an irreducible unramified representation of $\mathrm{GL}_2(F)$. Then π is either of the form $\chi(\det g)$ for some $\chi : F^\times \to \mathbb{C}^\times$ unramified or of the form $B(\chi_1, \chi_2)$ for some unramified χ_1, χ_2 with $\chi_1\chi_2 \neq |\cdot|^{\pm 1}$.*

Remark. In particular, supercuspidal and special representations are always ramified.

Remark. We will see later that an irreducible automorphic representation factors as a product of local representations such that all but finitely many are unramified.

The proof of Theorem 3.11.5 is a bit technical. We begin with the following lemma.

Lemma 3.11.6. *If (π, V) is unramified then ω_π is not compactly supported.*

By Theorem 3.6.1, matrix coefficients of supercuspidal representations are compactly supported. So the following is an immediate consequence.

Corollary 3.11.7. *Every supercuspidal representation is ramified.*

Proof of Lemma 3.11.6. We use that $\omega = \omega_\pi$ satisfies (3.11.1). Also, the decomposition $G = \bigsqcup_{n \geq 0} ZK \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} K$ implies that ω is determined by its values on diagonal elements. The central character χ of π controls the influence of Z. Taking $g_i = \begin{pmatrix} \varpi^{n_i} & \\ & 1 \end{pmatrix}$ for $i = 1, 2$, we may assume that $0 < n_1 \leq n_2$.

Using (3.11.1), we want to give a relation between the $\omega(\left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right))$ and χ. To do this, we need to calculate $\int_K \omega(g_1 k g_2)dk$. We will decompose this integral into a finite sum. The first step is, for $k = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in K$, to determine the type of

$$k_{12} := \begin{pmatrix}\varpi^{n_1} & \\ & 1\end{pmatrix} k \begin{pmatrix}\varpi^{n_2} & \\ & 1\end{pmatrix} = \begin{pmatrix}\varpi^{n_1+n_2}a & \varpi^{n_1}b\\ \varpi^{n_2}c & d\end{pmatrix}.$$

This will depend, since $ad - bc \in \mathcal{O}_F^\times$, on whether ad or $bc \in \mathcal{O}_F^\times$. In fact it will depend solely on the valuation of d, so we define the set

$$X_m = \{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in K \mid v(d) = m\}.$$

We first analyze the type of k_{12} when $k \in X_0$. In this case, we can use d to kill the upper right and lower left entries giving that $k_{12} \sim \left(\begin{smallmatrix}\varpi^{n_1+n_2} & \\ & 1\end{smallmatrix}\right)$. (Recall that multiplication by K on the left or right corresponds to row or column operations respectively. For a reminder of how this works see the proof of Lemma 3.6.2.)

If $0 < v(d) = m$, note that $bc \in \mathcal{O}_F^\times$. So $k_{12} \sim \left(\begin{smallmatrix} & \varpi^{n_1}\\ \varpi^{n_2} & d\end{smallmatrix}\right)$. Then there are two subcases. First, if $m \geq n_1$, then $k_{12} \in \left(\begin{smallmatrix} & \varpi^{n_1}\\ \varpi^{n_2} & \end{smallmatrix}\right) \sim \left(\begin{smallmatrix}\varpi^{n_1} & \\ & \varpi^{n_2}\end{smallmatrix}\right)$. If $0 \leq m < n_1$. Then $k_{12} \sim \left(\begin{smallmatrix}\varpi^{n_1+n_2-m} & \\ & \varpi^{m}\end{smallmatrix}\right)$.

Plugging this into (3.11.1), we get $\omega(g_1)\omega(g_2)$

$$\begin{aligned}
&= \int_K \omega(g_1 k g_2)dk\\
&= \int_{X_0} \omega(\begin{pmatrix}\varpi^{n_1+n_2} & \\ & 1\end{pmatrix})dk + \sum_{m=1}^{n_1-1}\int_{X_m} \omega(\begin{pmatrix}\varpi^{n_1+n_2-m} & \\ & \varpi^m\end{pmatrix})dk \qquad (3.11.3)\\
&\quad + \sum_{n\geq n_1}\int_{X_n} \omega(\begin{pmatrix}\varpi^{n_1} & \\ & \varpi^{n_2}\end{pmatrix})dk.
\end{aligned}$$

We have shown that ω is constant on each X_i, so we now need to calculate the measures of the sets X_i. Recall that the (multiplicative) Haar measure on G is given by $dg = \frac{d^+g}{|\det(g)|^2}$. Since $|\det k| = 1$ for all $k \in K$, to calculate measures of subsets of K, it suffices to use the additive measure. Indeed,

$$m^\times(X_i) = \frac{m^+(X_i)}{m^+(K)}.$$

For the remainder of the proof we let m denote the additive measure on K, so that $m(\varpi^n \mathcal{O}_F^\times) = |\varpi|^n$. So

$$\begin{aligned}
m(K) =& m(\{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in K \mid ad \in \mathcal{O}_F^\times\}) + m(\{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in K \mid ad \in \varpi\mathcal{O}_F^\times\})\\
=& m(\{a, d \in \mathcal{O}_F^\times\})m(\{bc \in \varpi\mathcal{O}_F\}) + m(\{a,b,c,d \in \varpi\mathcal{O}_F^\times \mid ad \not\equiv bc \pmod{\varpi}\})\\
&+ m\{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in K \mid ad \in \varpi\mathcal{O}_F, bc \in \mathcal{O}_F^\times\}\\
=& (1-|\varpi|)^2(2|\varpi| - |\varpi|^2) + (1-|\varpi|)^3(1-2|\varpi|) + (1-|\varpi|)^2(2|\varpi| - |\varpi|^2)\\
=& (1-|\varpi|^2)(1+|\varpi|),
\end{aligned}$$

using $m(\{ad \in \varpi\mathcal{O}_F\}) = m\{a \in \varpi\mathcal{O}_F\} + m(b \in \varpi\mathcal{O}_F\} - m\{a, b \in \varpi\mathcal{O}_F\} = 2|\varpi| - |\varpi|^2$.

Similarly, one can show that $m(X_0) = (1-|\varpi|)^2$, and it is easy to see that $m(X_m) = (1-|\varpi|)^3|\varpi|^m$ if $m > 0$.

Plugging everything into (3.11.3), we get

$$\begin{aligned}\omega(\begin{pmatrix} \varpi^{n_1} & \\ & 1 \end{pmatrix})\omega(\begin{pmatrix} \varpi^{m_2} & \\ & 1 \end{pmatrix}) &= \frac{1}{1+|\varpi|}\omega(\begin{pmatrix} \varpi^{n_1+n_2} & \\ & 1 \end{pmatrix}) \\ &+ \sum_{m=1}^{n_1-1} \frac{(1-|\varpi|)\,|\varpi|^m}{1+|\varpi|}\omega(\begin{pmatrix} \varpi^{n_1+n_2-m} & \\ & \varpi^m \end{pmatrix}) \\ &\qquad + \frac{|\varpi|^{n_1}}{1+|\varpi|}\chi(\varpi^{n_1})\omega(\begin{pmatrix} \varpi^{n_2-n_1} & \\ & 1 \end{pmatrix}).\end{aligned} \tag{3.11.4}$$

If ω is compactly supported modulo Z, $n_0 = \max\{n \mid \omega\left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right) \neq 0\}$ is well defined. If $n_0 > 0$, letting $n_1 = 1$ and $n_2 = n_0 + 1$ gives a contradiction. If $n_0 = 0$, then letting $n_1 = n_2 = 1$ gives a contradiction. □

Remark. By (3.11.4), ω is determined by its values on $\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)$.

Lemma 3.11.8. *Every special representation Sp_χ is ramified.*

Proof. We claim that $B(\chi_1, \chi_2)$ is unramified if and only if χ_1, χ_2 are unramified. Suppose $B(\chi_1, \chi_2)$ is unramified. Let $\phi \in B(\chi_1, \chi_2)^K$. Then up to constant multiple, ϕ is the unique function satisfying

$$\phi(\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix} k) := \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \tag{3.11.5}$$

For all $k \in K$. In particular, if $a \in \mathcal{O}^\times$ then $\phi(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)) = \chi_1(a) = 1$. Similarly, $\phi(\left(\begin{smallmatrix} 1 & \\ & a \end{smallmatrix}\right)) = \chi_2(a) = 1$. Hence χ_1, χ_2 are unramified. On the other hand, if χ_1, χ_2 are unramified, then (3.11.5) gives a well-defined element of $B(\chi_1, \chi_2)$ that is fixed by K. This proves the claim.

Since $Sp_\chi \subseteq B(\chi\,|\cdot|^{1/2}, \chi\,|\cdot|^{-1/2})$, it follows from the claim that Sp_χ is ramified whenever χ is ramified. Assume that χ is unramified. Then it suffices to show that the vector ϕ in $B(\chi\,|\cdot|^{1/2}, \chi\,|\cdot|^{-1/2},)$ does not belong to Sp_χ. Lemma 3.10.2 says that $\int_K \phi(k)\chi^{-1}(\det k)dk = 0$. On the other hand, χ unramified implies that $\chi^{-1} \circ \det|_K = 1$, and so $\int_K \phi(k)\chi^{-1}(\det k)dk = \phi(e) \neq 0$. □

Proof of Theorem 3.11.5. By the lemmas, we need only verify the theorem in the cases of $\mathbb{C}(\chi \circ \det)$ and $B(\chi_1, \chi_2)$ irreducible. First, $\chi \circ \det|_K = 1$ if and only if $\chi|_{\mathcal{O}_F^\times} = 1$. So $\mathbb{C}\chi \circ \det$ is unramified if and only if χ is unramified. The case of $B(\chi_1, \chi_2)$ was proved in the course of the proof of Lemma 3.11.8. □

Exercise 3.11.9. Using the spherical vector of (3.11.5), show that

$$\omega(\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}) = \frac{q^{-n/2}}{1+\frac{1}{q}}\left(\alpha_1^n \frac{1 - \frac{\alpha_1^{-1}\alpha_2}{q}}{1-\alpha_1^{-1}\alpha_2} + \alpha_2^n \frac{1 - \frac{\alpha_1\alpha_2^{-1}}{q}}{1-\alpha_1\alpha_2^{-1}}\right)$$

where $\alpha_i = \chi_i(\varpi)$ and $|\varpi| = \frac{1}{q}$.

Solution. We use that

$$\omega(\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}) = \int_K \phi(k\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix})dk$$

and, as in the proof of Lemma 3.11.6, we look at the "type" of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} = \begin{pmatrix} a\varpi^n & b \\ c\varpi^n & d \end{pmatrix}$$

when $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K$. Since ϕ is right invariant by K, by "type," we are allowed to act on $\left(\begin{smallmatrix} a\varpi^n & b \\ c\varpi^n & d \end{smallmatrix}\right)$ by multiplying on the right by elements of K. This corresponds to column operations. One verifies that

$$\begin{pmatrix} a\varpi^n & b \\ c\varpi^n & d \end{pmatrix} \sim \begin{cases} \begin{pmatrix} \varpi^{n-m} & * \\ 0 & \varpi^m \end{pmatrix} & \text{if } m = v(d) < n \\ \begin{pmatrix} 1 & * \\ 0 & \varpi^n \end{pmatrix} & \text{if } m = v(d) \geq n. \end{cases}$$

Using the results on $m^\times(X_n)$ from above and the fact that $\phi \in B(\chi_1, \chi_2)$, it follows that

$$\int_K \phi(k \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix})dk = \frac{1}{1+\frac{1}{q}} \times$$
$$\left(\alpha_1^n q^{-n/2} + (1-\frac{1}{q})\alpha_1^n q^{-n/2} \sum_{m=1}^{n-1} (\alpha_1^{-1}\alpha_2)^m + \alpha_2^n q^{n/2}(1-\frac{1}{q}) \sum_{m=n}^{\infty} q^m \right).$$

Simplifying this formula gives the result. □

We finish this section with two calculations that may be useful to us later. First, suppose $B(\chi_1, \chi_2)$ is irreducible and unramified with spherical function ω_{χ_1,χ_2}. Let ϕ be the spherical vector in $B(\chi_1, \chi_2)$ such that $\phi(e) = 1$ and ϕ' the spherical vector in $B(\chi_1^{-1}, \chi_2^{-1})$ such that $\phi'(e) = 1$. Then

$$\omega_{\chi_1,\chi_2}(g) = \langle \pi(g)\phi, \phi' \rangle = \int_K \phi(kg)\phi'(k)dk = \int_K \phi(kg)dk. \tag{3.11.6}$$

The second calculation is of the spherical homomorphism $\mathcal{H}(G, K) \to \mathbb{C}$:

$$\begin{aligned} f \mapsto \lambda_\pi(f) &= \int_G f(g)\omega(g)dg = \int_{G\times K} f(g)\phi(kg)dkdg \\ &= \int_{G\times K} f(k^{-1}g)\phi(g)dkdg = \int_{G\times K} f(g)\phi(g)dkdg \\ &= \int_G f(g)\phi(g)dg \\ &= \int_{NAK} f\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right)k\right)\phi(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right)k)\left|\frac{a_1}{a_2}\right|^{-1} dxd^\times a_1 d^\times a_2 dk \\ &= \int f(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right))\chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{-1/2} dxd^\times a_1 d^\times a_2 dk. \end{aligned}$$

We have used the fact that $G = NAK$ and $dg = \left|\frac{a_1}{a_2}\right|^{-1} dxd^\times a_1 d^\times a_2 dk$.

3.12. **Unitary representations.** Let $G_F = \mathrm{GL}_2(F)$ with F a local field, V a vector space over $\mathbb{C}$ with a Hermitian pairing $\langle \cdot, \cdot \rangle$. Note that the pairing makes V a *topological* vector space.

Definition 3.12.1. *A* unitary representation of *G on V (or, equivalently a* unitary G_F-module*) is an action of G on V such that $\langle \pi(g)v_1, \pi(g)v_2 \rangle = \langle v_1, v_2 \rangle$. (i.e. the action of G_F preserves the pairing.)*

Definition 3.12.2. *Let (π, V) be a unitary G_F-module. It is called* topologically irreducible *if the only invariant closed subspaces of V are $\{0\}$ and V itself.*

Theorem 3.12.3. *Let (π, V) be an irreducible unitary G-module. Consider*

$$V^{\infty} := \{v \in V : v \text{ is fixed by some open subgroup } L \subset G\}.$$

Then $(\pi|_{V^\infty}, V^{\infty})$ is irreducible admissible.

With this one may ask which irreducible admissible G-modules are unitarizable. In other words, when can one put a positive-definite g-invariant Hermitian pairing on the space? In group theory a basic question is given a group H find its unitary dual. We have already classified all irreducible admissible G-modules.

Exercise 3.12.4. Determine all unitarizable irreducible representations of $\mathrm{GL}_2(F)$.

As a hint, recall that every irreducible admissible representation is isomorphic to one of the following: supercuspidal, irreducible $B(\chi_1, \chi_2)$ (principal series), special representations Sp_χ or $\mathbb{C}_\chi$ with G acts b $\chi(\det(\cdot))$. Show $\mathbb{C}_\chi$ is unitarizable if and only if χ is unitarizable. In Section 3.7.1 we saw that supercusipdals are unitarizable, and in Corollary 3.10.4 that Sp_χ are as well. It remains to determine when a principal series representation $B(\chi_1, \chi_2)$ are unitarizable. If it is unitarizable then $B(\chi_1, \chi_2)^\sim = \overline{B(\chi_1, \chi_2)}$ since for unitarizable representations we have a non-degenerate perfect pairing $V \times \overline{V} \to \mathbb{C}$. Moreover,

$$\overline{B(\chi_1, \chi_2)} = B(\overline{\chi_1}, \overline{\chi_2}) \quad \text{and} \quad B(\chi_1, \chi_2)^\sim = B(\chi_1^{-1}, \chi_2^{-1})$$

forces that

$$(\chi_1^{-1}, \chi_2^{-1}) = (\overline{\chi_1}, \overline{\chi_2}) \quad \text{or} \quad (\chi_1^{-1}, \chi_2^{-1}) = (\overline{\chi_2}, \overline{\chi_1}).$$

In the first case, $\chi_j^{-1} = \overline{\chi_j}$ implies that χ_j is unitary, and in the other case $\chi_1 = \overline{\chi_2}^{-1}$. One needs to determine if these necessary conditions are sufficient. In the first case it is sufficient, but in the second it is not.

The homework is to use the natural pairing you get from $B(\chi_1, \chi_2)^\sim = B(\overline{\chi_1}, \overline{\chi_2})$. Under what conditions will it be positive definite? To check this when $\chi_1 = \overline{\chi_2}^{-1}$ let $\sigma : \overline{B(\chi_1, \chi_2)} \to B(\chi_1^{-1}, \chi_2^{-1})$ be the isomorphism and check when $(f_1, f_2) = \langle f_1, \sigma \overline{f_2} \rangle$ is positive definite.

3.13. **Whittaker and Kirillov models.** The following gives some motivation: Let f be a function on G_F and $\psi : F \to \mathbb{C}^\times$ an additive character[9]. Then we can consider the ψ-coefficient

$$f_{\psi,N} = \int_F f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi^{-1}(x)dx,$$

or the functional

$$L_\psi : f \to \int_F f\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \psi^{-1}(x)dx$$

which (formally) satisfies $L(\rho\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) f) = \psi(x)Lf$. *WARNING: this is a naive notion since the integral need not converge.*

Definition 3.13.1. *Let (π, V) be an irreducible admissible G_F-module, $\psi : F \to \mathbb{C}$ an additive character. The ψ-*Whittaker functional *is a nonzero linear functional $L : V \to \mathbb{C}^\times$ such that $L(\pi\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) v) = \psi(x)Lv$.*

[9]We use the terminology *quasicharacter* when a character is not necessarily unitary, and *character* if it is.

Remark. In the global theory, we will consider spaces of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. In this case the nilpotent radical $N(\mathbb{Q})\backslash N(\mathbb{A}) \simeq F\backslash \mathbb{A}_F$ is a compact group, so the corresponding integrals as given above will always converge. In other words, there is a good Fourier expansion theory. But locally N is not compact so we don't literally have a Fourier expansion.

Basic Observation. If $L_\psi : V \to \mathbb{C}$ is a Whittaker functional then we have the following:

(1) For $v \in V$ one associates $W_v(g) = L_\psi(\pi(g)v)$. Then

$$W_v(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) = L_\psi(\pi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)v) = \psi(x)L(\pi(g)v) = \psi(x)W_v(g).$$

(2) For $v \in V$ one can associate $\varphi_v : F^\times \to \mathbb{C}$ by $\alpha \mapsto L_\psi(\pi\left(\begin{smallmatrix} \alpha & \\ & 1 \end{smallmatrix}\right)v)$. Then φ_v is locally constant because π is smooth, and

$$\begin{aligned}\varphi_{\pi\left(\begin{smallmatrix} a & x \\ & 1 \end{smallmatrix}\right)v}(\alpha) =& L_\psi(\pi\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\pi\begin{pmatrix} a & x \\ & 1 \end{pmatrix}v) \\ =& L_\psi(\pi\begin{pmatrix} \alpha a & \alpha x \\ & 1 \end{pmatrix}v) \\ =& L_\psi(\pi\begin{pmatrix} 1 & \alpha x \\ & 1 \end{pmatrix}\pi\begin{pmatrix} \alpha a & \\ & 1 \end{pmatrix}v) \\ =& \psi(\alpha x)\varphi_v(a\alpha).\end{aligned}$$

We will see that (1) corresponds to a *Whittaker model* and (2) corresponds to a *Kirillov model*.

Let $B_F = \{\left(\begin{smallmatrix} a & x \\ & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(F)\}$. Then B_F has an action ξ_ψ on $C^\infty(F^\times)$, the space of locally constant functions on $F^\times$, given by

$$\xi_\psi(\begin{pmatrix} a & x \\ & 1 \end{pmatrix}\varphi)(\alpha) = \psi(\alpha x)\varphi(a\alpha),$$

which matches the action of B_F in (2).

Lemma 3.13.2. *Let $S(F^\times) = C_c^\infty(F^\times)$ be the space of Bruhat Schwarz functions (i.e. the space of locally constant compactly supported functions of $F^\times$.) Then $\xi_\psi \mid_{S(F^\times)}$ is irreducible.*

Definition 3.13.3. *Suppose that (π, V) is irreducible and admissible.*

(1) *A* ψ-Whittaker model *of V is a subspace $W(\pi, \psi)$ of*

$$W(\psi) := \{W : G \to \mathbb{C} \mid f(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)g) = \psi(x)f(g) \text{ for all } x \in F, g \in G\}$$

such that the right translation action of G on $W(\pi, \psi)$ is equivalent to π.

(2) *A* Kirillov model *is a subspace $V' \subset \{$locally constant functions on $F^\times\}$ and an action π' of G on V' such that $(\pi', V') \simeq (\pi, V)$ and $\pi' \mid_{B_F} = \xi_F$.*

Remark. We have the following correspondence between these notions.

$$\left\{\begin{matrix}\text{Kirillov}\\ \text{models}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Whittaker}\\ \text{functionals}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Whittaker}\\ \text{models}\end{matrix}\right\} \tag{3.13.1}$$

Whittaker functionals and Whittaker models are linked by $L_\psi \mapsto W(\pi, \psi) = \{W_v \mid W_v(g) = L_\psi(\pi(g)v)\}$. Conversely, if $v \in V \simeq W(\pi, \psi)$ and W_v corresponds to v then we define the Whittaker functional $L_\psi : V \to \mathbb{C}$ by $v \mapsto W_v(e)$.

Whittaker functionals and Kirillov Models are linked via $L_\psi \mapsto V' = \{\varphi_v(\alpha) = L_\psi(\pi\left(\begin{smallmatrix}\alpha & \\ & 1\end{smallmatrix}\right) v)\}$. In the other direction, if $V' \simeq V$ with $\varphi_v \leftrightarrow V$ then we define the Whittaker functional $L_\psi : V \to \mathbb{C}$ by $v \mapsto \varphi_v(e)$.

Our main goal of this section is to prove that a ψ-Whittaker functional exists and is unique up to scalar. (See Theorem 3.13.4.)

Remark. Theorem 3.13.4 is not true in general for GL_n when $n > 2$ because existence is not guaranteed. However, if a Whittaker functional does exist, it is always unique.

Proof of Lemma 3.13.2. Recall that $F^\times = \cup_{n\in\mathbb{Z}} \varpi^n \mathcal{O}_F^\times$. Let V be a nonzero B_F-invariant subspace of $S(F^\times)$. We need to show $V = S(F^\times)$. Denote the set of characters of $\mathcal{O}_F^\times$ by X. Since translates of $\{\varphi_\nu = \nu \cdot 1_{\mathcal{O}_F^\times} \mid \nu \in X\}$ form a basis of $S(F^\times)$, it suffices to show that $\varphi_\nu \in V$ for every $\nu \in X$.

Choose a nonzero $\varphi_0 \in V$. Then $\mathcal{O}_F^\times \varphi_0$ (the action is: $(\alpha\cdot\varphi_0)(x) = \varphi_0(\alpha x)$) is finite dimensional because it is a smooth action. We conclude that $\mathcal{O}_F^\times\varphi_0$ is a direct sum of one dimensional $\mathcal{O}_F^\times$-modules because $\mathcal{O}_F^\times$ is commutative. Let φ be a non-zero element in this summand. Then $\varphi(\epsilon\cdot a) = \nu(\epsilon)\varphi(a)$ for some character $\nu \in X$.

Next, we show that for any $\mu \in X \setminus \{\nu\}$, $\mu 1_{\mathcal{O}_F^\times} \in V$. As motivation for our method, observe that if $\varphi \in S(F^\times)$ then

$$\varphi_{[\mu]} := \int \varphi(\epsilon\alpha)\mu^{-1}(\epsilon)d\epsilon$$

satisfies $\varphi_{[\mu]}(\epsilon\alpha) = \mu(\epsilon)\varphi(\alpha)$. For this reason we call $\varphi_{[\mu]}$ the *μ-component* of φ.

Given φ as before, let

$$\widetilde{\varphi} = \int_{\mathcal{O}_F^\times} \mu^{-1}(\epsilon)\xi_\psi\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix}\xi_\psi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\varphi d\epsilon.$$

Note that $\widetilde{\varphi} \in V$. Moreover,

$$\xi_\psi\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix}\xi_\psi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\varphi(\alpha) = \psi(\alpha\epsilon x)\varphi(\alpha\epsilon).$$

So

$$\widetilde{\varphi}(\alpha) = \int_{\mathcal{O}_F^\times} \mu^{-1}(\epsilon)\psi(\alpha\epsilon x)\varphi(\alpha\epsilon)d\epsilon = \int \nu\mu^{-1}(\epsilon)\psi(\alpha\epsilon x)d\epsilon\varphi(\alpha).$$

This implies that $\widetilde{\varphi}(\epsilon\alpha) = \mu(\epsilon)\widetilde{\varphi}(\alpha)$. We want to choose x so that the support of $\widetilde{\varphi}$ is exactly $\mathcal{O}_F^\times$. That is, so that it is 0 outside of $\mathcal{O}_F^\times$ and not equal to zero on $\mathcal{O}_F^\times$. This, in turn, implies that $\widetilde{\varphi} = \lambda\cdot\mu 1_{\mathcal{O}_F^\times}$ for some $\lambda \in \mathbb{C}$. So we have to calculate the Gauss sum

$$\int_{\mathcal{O}_F^\times} \nu\mu^{-1}(\epsilon)\psi(\alpha\epsilon x)d\epsilon.$$

Finally, by this same argument $\mu 1_{\mathcal{O}_F} \in V$ so we are done. $\square$

For homework let $\psi : F \to \mathbb{C}$ of conductor $\varpi^n\mathcal{O}_F$. That is the smallest group such that $\psi\mid_{\varpi^m\mathcal{O}_F} = 0$. Let $\chi : F^\times \to \mathbb{C}$ of conductor $1 + \varpi^n\mathcal{O}_F^\times$ then

$$\int_{\mathcal{O}_F^\times} \psi(\epsilon)\chi(\epsilon)d\epsilon \begin{cases} = 0 & m \neq n \\ \neq 0 & m = n \end{cases}.$$

So just choose x so that $\nu\mu^{-1}$ has conductor $1+\varpi^n\mathcal{O}_F^\times$ for $n\geq 1$ and ψ has conductor $\varpi^m\mathcal{O}_F$ such that $v(x)=m-n$ then

$$\int \nu\mu^{-1}(\epsilon)\psi(\alpha x\epsilon)d\epsilon \begin{cases} =0 & \text{outside } \mathcal{O}_F^\times \\ \neq 0 & \text{on } \mathcal{O}_F^\times \end{cases}.$$

Theorem 3.13.4. *Suppose (π,V) is an infinite dimensional irreducible admissible G_F-module, and $\psi: F\to\mathbb{C}^\times$ a nontrivial character. Then a ψ-Whittaker functional exists and is unique up to scalars.*

Let us explain why the hypotheses of this theorem are necessary. First, if ψ is trivial, L is a functional on V_N, the Jacquet module, which has dimension 0 or 1, as V is, respectively, supercuspidal or not.

If V is finite dimensional then $V\simeq\mathbb{C}_\chi$ (i.e. V is one dimensional with G acting by $\chi\circ\det$.) If ψ is non-trivial, then there is no ψ-Whittaker functional because $\pi\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)$ acts trivially.

To prove this we follow a similar method as we did when studying the Jacquet module, but the proof is harder. By definition, if L is a ψ-Whittaker functional then

$$L(v-\psi^{-1}(x)\pi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}v)=0 \tag{3.13.2}$$

for $v\in V$ and $x\in F$. So if

$$V_\psi' := \operatorname{span}\{v-\psi^{-1}(x)\pi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}v : v\in V, x\in F\},$$

and $X=V/V_\psi'$ then the ψ-Whittaker functional L corresponds to a function on X. As a consequence, Theorem 3.13.4 is equivalent to the following.

Theorem 3.13.5. $\dim(X)=1$.

Lemma 3.13.6. *Suppose (π,V) and ψ are as in Theorem 3.13.4, and V_ψ' as above. Then $v\in V_\psi'$ if and only if there exists $U\subseteq F$ open and compact such that $\int_U \psi^{-1}(x)\pi\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)v dx=0$.*

The proof of this lemma is nearly identical to that of Lemma 3.5.3 which characterizes when a vector belongs to $V(N)$.

Proposition 3.13.7. *Consider the map $V\to C_0^\infty(F^\times,X)$, space of locally constant functions with values in X, given by $v\mapsto\varphi_v$ where $\varphi_v(\alpha)=[\pi\left(\begin{smallmatrix}\alpha & \\ & 1\end{smallmatrix}\right)v]$.*

(i) *This map is injective.*
(ii) *For all $a\in F^\times$ and $x\in F$*

$$\varphi_{\pi\left(\begin{smallmatrix}a & x\\ & 1\end{smallmatrix}\right)v} = \xi_\psi\begin{pmatrix}a & x\\ & 1\end{pmatrix}\varphi_v.$$

Recall that the action ξ_ψ is given by

$$\xi_\psi\begin{pmatrix}a & x\\ & 1\end{pmatrix}\varphi(\alpha)=\psi(\alpha x)\varphi(a\alpha).$$

Also, note that once we know that $\dim X=1$, the mapping $v\mapsto\varphi_v$ will give the Kirillov model.

Lemma 3.13.8. *If (π, V) is irreducible admissible and infinite dimensional, and $v \in V$ is fixed by $N := \{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in F\}$ then $v = 0$.*

Proof. Suppose v is fixed by N. We will show that $N' = \{\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right) \mid x \in F\}$ also fixes v, and, therefore, that SL$_2$ does as well. By smoothness, v is fixed by some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ with $c \neq 0$. Then

$$\begin{pmatrix} 1 & -ac^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -c^{-1}d \\ & 1 \end{pmatrix} = \begin{pmatrix} & -adc^{-1} + b \\ c & \end{pmatrix}$$

fixes v. Call $b_0 = -adc^{-1} + b \neq 0$. So

$$\begin{pmatrix} & b_0 \\ c & \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} & b_0 \\ c & \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \\ b_0^{-1}cx & 1 \end{pmatrix}$$

fixes v for all x. Hence letting x run over F, we see that N' fixes v. As N and N' generates SL$_2(F)$, we get the claim.

This implies that

$$\pi(g)v = \pi \begin{pmatrix} \det(g) & \\ & 1 \end{pmatrix} v = \pi \begin{pmatrix} \delta\varpi^L & \\ & 1 \end{pmatrix} v$$

where $\det(g) = \delta\varpi^\ell$ with $\delta \in \mathcal{O}_F^\times$ we get $g \cdot v =$. This gives an action of $F^\times$ on v. We claim that the resulting space, $F \cdot v$, is finite dimensional space. To see this, suppose $\alpha = \gamma\beta^2$ then

$$\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} v = \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} \beta & \\ & \beta \end{pmatrix} v = \omega(\beta) \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} v$$

with ω the central character of π. Hence, the space $F^\times \cdot v$ (which is the same as that spanned by $\pi(g)v$ for all $g \in G$) is the same as that spanned by $\pi\left(\begin{smallmatrix} \alpha & \\ & 1 \end{smallmatrix}\right)v$ for α running through a set of representatives of $F^\times/(F^\times)^2$. This is a finite set, so the space must be finite dimensional. Unless $v = 0$ this would contradict the assumption that V is irreducible and infinite dimensional. $\square$

Lemma 3.13.9. *Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character with conductor $\mathfrak{p}^m = \varpi^m\mathcal{O}_F$ meaning it is trivial over $\mathfrak{p}^m$ but not over $\mathfrak{p}^{m-1}$. Suppose $f \in C^\infty(\mathfrak{p}^\ell, Y)$ where Y is a finite dimensional vector space over $\mathbb{C}$ and $\ell \in \mathbb{Z}$. If $n > \ell$ the following are equivalent.*

(i) *f is constant over the cosets of $\mathfrak{p}^n$ in $\mathfrak{p}^\ell$.*
(ii) *$\int_{\mathfrak{p}^\ell} \psi(-ax)f(x)dx = 0$ for all a outside $\mathfrak{p}^{m-n}$.*

Remark. This technical lemma is linked to Fourier analysis.

Proof. Consider f as a function supported on $\mathfrak{p}^\ell$. Then $f \in S(F, Y)$, and its *Fourier transform* is

$$\hat{f}(a) := \int_F \psi(-ax)f(x)dx = \int_{\mathfrak{p}^L} \psi(-ax)f(x)dx.$$

So, to prove that (i) implies (ii), we need to show that if f is constant on cosets of $\mathfrak{p}^n$ in $\mathfrak{p}^\ell$, then $\hat{f}(a) = 0$ whenever $a \notin \mathfrak{p}^{n-m}$. Let α_i be coset representatives. Then, assuming (i),

$$\begin{aligned}\hat{f}(a) &= \sum \int_{\alpha_i+\mathfrak{p}^n} \psi(-ax)f(x)dx = \sum \psi(-a\alpha_i) \int_{\mathfrak{p}^n} \psi(-ax)f(\alpha_i+x)dx \\ &= \sum \psi(-a\alpha_i)f(\alpha_i).\end{aligned}$$

Since the conductor of $\psi(-a\ \cdot)$ is $\mathfrak{p}^{m-v(a)}$, whenever $a \notin \mathfrak{p}^{m-n}$, $\psi(-a\alpha_i) = 0$ for all i. Hence $\hat{f}(a) = 0$ as desired.

To prove that (ii) implies (i), we use the *Fourier inversion formula*: $\hat{\hat{\varphi}}(x) = \varphi(-x)c(\psi)$ where $c(\psi)$ is a positive constant depending on ψ and the measure. So

$$f(x) = \frac{1}{c(\psi)}\hat{\hat{f}}(-x) = \frac{1}{c(\psi)}\int_F \hat{f}(y)\psi(xy)dy = \frac{1}{c(\psi)}\int_{\mathfrak{p}^{m-n}} \hat{f}(y)\psi(xy)dy.$$

So, if $z \in \mathfrak{p}^n$ and $y \in \mathcal{O}_F$ then $\psi(yz) = 1$, and therefore

$$\begin{aligned}f(y+z) &= \frac{1}{c(\psi)}\int_{\mathfrak{p}^{m-n}} \hat{f}(y)\psi(xy+yz)dy \\ &= \frac{1}{c(\psi)}\int_{\mathfrak{p}^{m-n}} \hat{f}(y)\psi(xy)dy = f(x)\end{aligned}$$

as desired. □

Proof of Proposition 3.13.7. Recall that we want to show that the map $v \mapsto \varphi_v(\alpha) = [\pi\left(\begin{smallmatrix}\alpha & \\ & 1\end{smallmatrix}\right)v] \in X$ is injective. Suppose $\varphi_v = 0$ and consider the function $f(x) = \pi\left(\begin{smallmatrix}1 & x \\ & 1\end{smallmatrix}\right)v \in C^\infty(F, V)$. We will show $f(x) = v$. Then Lemma 3.13.8 implies $v = 0$. We show f is constant by looking over every coset and applying Lemma 3.13.9.

We use an inductive argument. We obtain the base step by smoothness: there exists an n_0 such that $\pi\left(\begin{smallmatrix}1 & x \\ & 1\end{smallmatrix}\right)v = v$ for all $x \in \mathfrak{p}^{n_0}$. That is, f is constant over cosets of $\mathfrak{p}^{n_0}$. We will argue that if f is constant over cosets of $\mathfrak{p}^n$, then it is constant over cosets of $\mathfrak{p}^{n-1}$, implying f is constant (since $F^\times = \cup_{n<n_0}\mathfrak{p}^n$ and $f(0) = v$).

To get the inductive step, suppose f is constant over cosets of $\mathfrak{p}^n$. Let $\ell \ll 0$ we will argue that f is constant over cosets of $\mathfrak{p}^{n-1}$ in $\mathfrak{p}^\ell$. By Lemma 3.13.9, it suffices to show that $\int_{\mathfrak{p}^\ell} f(x)\psi(-ax)dx = 0$ for all a outside $\mathfrak{p}^{m-n-1}$. By the induction hypothesis, this is true for a outside $\mathfrak{p}^{m-n}$. So we need to consider what happens if $a \in \varpi^{m-n}\mathcal{O}_F^\times$.

Although this set is infinite, we will show that we only need to check finitely many values. Observe that there exists n_1 such that $\pi\left(\begin{smallmatrix}1+x & \\ & 1\end{smallmatrix}\right)v = v$ for all $x \in \varpi^{n_1}\mathcal{O}_F$. So for $b \in 1 + \varpi^{n_1}\mathcal{O}_F$, $\pi(b)f = f$, and therefore

$$\pi(b)\int_{\mathfrak{p}^\ell} \psi(-ax)f(x)dx = \int_{\mathfrak{p}^\ell} \psi(-\frac{a}{b}x)f(x)dx.$$

We conclude that $\hat{f}(a) = 0$ if and only if $\hat{f}(a/b) = 0$. Hence we only need to check $\hat{f}(a) = 0$ for $1 + \varpi^{n_0}\mathcal{O}_F$ orbits of $\varpi^{m-n}\mathcal{O}_F^\times$ and this is a finite set.

Let $a_1, \cdots, a_d$ be $1 + \varpi^{n_0}\mathcal{O}_F$ coset representatives of $\varpi^{m-n}\mathcal{O}_F^\times$. Need to check $\hat{f}(a_j) = 0$ for $j = 1, \ldots, d$. Note that $\varphi_v = 0$ implies that $\varphi_v(a_j) = 0 \in X$ which is equivalent to

$$\pi\begin{pmatrix} a_j & \\ & 1\end{pmatrix}v \in V'_\psi.$$

By Lemma 3.13.6, this holds if and only if

$$\int_U \psi^{-1}(x)\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\pi\begin{pmatrix} a_j & \\ & 1 \end{pmatrix} v dx = 0$$

for some U. We may take $U = \mathfrak{p}^{\ell_j}$ for some $\ell_j \in \mathbb{Z}$. This integral is equal to

$$\pi\begin{pmatrix} a_j & \\ & 1 \end{pmatrix}|a_j|\int_{\mathfrak{p}^{\ell_j}a_j}\psi(-a_j x)\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} v dx,$$

and so whenever $\ell < v(a_j) + \ell_j$,

$$\int_{\mathfrak{p}^L}\psi(-a_j x) f(x) dx = 0.$$

The verification of (ii) is left as an easy exercise. $\square$

Given that $V \hookrightarrow C^\infty(F^\times, X)$, we will often identify V with its image. The following proposition describes the image.

Proposition 3.13.10. *Let $V_0 = S(F^\times, X)$ the space of functions $f : F^\times \to X$ that are locally constant and have compact support. Then $V \supseteq V_0$. More precisely, $V = V_0 + \pi(w)V_0$.*

Since $G = P \sqcup PwP$, and we know that the action of P is by ξ_ψ and ω, the central character of π, the point will be to analyze the action of w. After doing this, it will follow almost immediately that $\dim X = 1$.

We have the following facts:

Observation 1: If Y is a finite dimensional vector space and $\varphi \in S(F^\times, Y)$ then $\mathcal{O}_F^\times \cdot \varphi$ is finite dimensional, and we may write

$$\varphi = \sum_\mu \varphi_{[\mu]}$$

where the sum runs through all characters $\mu : \mathcal{O}_F^\times \to \mathbb{C}^\times$ and

$$\varphi_{[\mu]} = \int_{\mathcal{O}_F^\times}\mu^{-1}(\epsilon)\varphi(\epsilon\alpha) d\epsilon$$

is the *μ-component* of φ. (The action of $\mathcal{O}_F^\times$ on φ is via $\xi_\psi((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}))$.) Note that since $\varphi \in S(F^\times, Y)$, the integral is actually a finite sum. Moreover,

$$\varphi_{[\mu]}(\epsilon a) = \mu(\epsilon)\varphi(a) \quad \text{for all } \epsilon \in \mathcal{O}_F^\times.$$

Observation 2: Suppose $\varphi \in S(F^\times, Y)$ and $\varphi(\epsilon a) = \nu(\epsilon)\varphi(a)$. Then all $\mu \neq \nu$ can be obtained via

$$\varphi' = \int_{\mathcal{O}_F^\times}\mu^{-1}(\epsilon)\xi_\psi\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\varphi d\epsilon.$$

Indeed, for suitable x, $\varphi' = c 1_{\mathcal{O}_F^\times}\mu \cdot \varphi(1)$ where c is a fixed constant. (Note that we used this trick in the proof of Lemma 3.13.2.)

Observation 3: Given $\varphi \in V$, there exists n_0 such that $\varphi(\alpha) = 0$ whenever $v(\alpha) < n_0$. (That is, φ vanishes at infinity.) To see that this is true, note that there exists n such that if $v(x), v(a-1) < n$[10] then $\pi((\begin{smallmatrix} a & x \\ & 1 \end{smallmatrix}))\varphi = \varphi$. In particular, this implies that $\varphi(\alpha x) = \varphi(x)$, and looking at the action of $(\begin{smallmatrix} a & x \\ & 1 \end{smallmatrix})$ on φ,

$$\varphi(\alpha) = \psi(\alpha x)\varphi(\alpha)$$

if $x \in \mathfrak{p}^n$. Now let m be the greatest integer such that $\psi_{\mathfrak{p}^n} \neq 1$. (Since ψ is non-trivial this makes sense.) One can show that $n_0 = n - m$ works.

Proof of Proposition 3.13.10. We first show that $S(F^\times, X) \subset V$.

$$S(F^\times, X) = \sum_{u \in X} S(F^\times, \mathbb{C}u)$$

implies that it suffices to show that $S(F^\times, \mathbb{C}u) \subset V$ for all $u \in X$. Suppose $\phi \in V$ is a preimage of $u \in V/V'_\psi$. Then $\phi(1) = u$, but ϕ need not be Bruhat-Schwartz. However, $\phi' = \phi - \pi((\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}))\phi$ is. Indeed,

$$\phi'(\alpha) = \phi(\alpha) - \pi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\phi(\alpha) = \phi(\alpha) - \psi(\alpha x)\phi(\alpha) = (1 - \psi(\alpha x))\phi(\alpha).$$

Thus, ϕ' vanishes near 0 and, by observation 3, near ∞ as well. In particular, $\phi'(1) = (1 - \psi(x))\phi(1)$, so choosing x such that $\psi(x) \neq 1$ forces $\phi'(1)$ to be a nonzero multiple of $\phi(1) = u$. In summary, by replacing ϕ with $(1 - \psi(x))^{-1}\phi'$, we may assume that $\phi \in S(F^\times, \mathbb{C}u)$, and $\phi(1) = u$.

For this choice of ϕ, write

$$\phi = \sum_\nu \phi_{[\nu]}$$

where the sum is taken over all characters $\nu : \mathcal{O}_F^\times \to \mathbb{C}^\times$, and

$$\phi_{[\nu]} = \int_{\mathcal{O}_F^\times} \nu(\epsilon)^{-1}\pi(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix})\phi d\epsilon$$

is an element of $V \cap V_0$. Choose one component $\varphi_{[\nu]}$. By observation 2, we get $1_{\mathcal{O}_F^\times}\mu \cdot \varphi(1) \in V \cap V_0$ for all $\mu \neq \nu$. Using any of these, we similarly get $1_{\mathcal{O}_F^\times}\nu \cdot \varphi(1) \in V \cap V_0$. Since these generate $S(F^\times, \mathbb{C}u)$ this proves (1).

Using $G = P \cup NwP$, we have

$$\begin{aligned} V &= span\{\pi(g)\varphi \mid g \in G, v \in V_0\} \\ &= V_0 + span\{\pi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\pi(w)\varphi \mid x \in F\}. \end{aligned}$$

Since

$$\pi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\pi(w)\varphi = -[\pi(w)\varphi - \pi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\pi(w)\varphi] + \pi(w)\varphi,$$

the bracketed term is in V_0 by the same trick as above. Hence $V = V_0 + \pi(w)V_0$. $\square$

Remark. Determining the action of w will help us to get the local functional equation later.

Given $\varphi \in C^\infty(F^\times, X)$, the idea is to understand the interaction between the values of φ on $\varpi^n\mathcal{O}_F^\times$ (note that $F^\times = \sum_n \varpi^n\mathcal{O}_F^\times$) and characters of $\mathcal{O}_F^\times$.

[10] This means that $a \in U^{(n)} = 1 + \mathfrak{p}^n$.

Remark. If $f \in C^\infty(\mathcal{O}_F^\times, \mathbb{C})$ then

$$f = \sum_{\mu:\mathcal{O}_F^\times \to \mathbb{C}^\times} f_{[\mu]} = \sum_\mu c_\mu \mu$$

where $c_\mu = \int_{\mathcal{O}_F^\times} f(\epsilon)\mu^{-1}(\epsilon)d\epsilon$.

More precisely, let ν be such a character. Then set

$$\widehat{\varphi}_n(\nu) = \int_{\mathcal{O}_F^\times} \nu(\epsilon)\varphi(\varpi^n\epsilon)d\epsilon,$$

and consider the formal power series

$$\widehat{\varphi}(\nu, t) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}_n(\nu)t^n.$$

We call this the *Mellin series* associated to φ or the *Mellin transform* of φ.

We record some facts about the Mellin series

- If $\varphi \in V$ then $\widehat{\varphi}(\nu, t)$ has finitely many nonzero negative terms since φ vanishes near infinity.
- If $\varphi \in V_0$ then φ also vanishes near zero so the sum is actually finite.
- $\widehat{\varphi}(\nu, t) = 0$ for all ν implies that $\varphi = 0$.
- Let $\omega : F^\times \to \mathbb{C}^\times$ be a quasicharacter. If

  $$\langle \varphi, \omega \rangle := \int_{F^\times} \varphi(a)\omega(a)d^\times a$$

  is absolutely convergent then

  $$\begin{aligned}\langle \varphi, \omega \rangle &= \sum_{n=-\infty}^{\infty} \int_{\varpi^n\mathcal{O}_F^\times} \varphi(a)\omega(a)d^\times a \\ &= \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}_F^\times} \varphi(\varpi^n\epsilon)\omega(\varpi^n\epsilon)d^\times \epsilon \\ &= \sum_n \omega(\varpi)^n \int_{\mathcal{O}_F^\times} \varphi(\varpi^n\epsilon)\omega(\epsilon)d^\times\epsilon \\ &= \widehat{\varphi}(\omega\mid_{\mathcal{O}_F^\times}, \omega(\varpi)).\end{aligned}$$

For $\varphi \in V$, set $\pi(g)\widehat{\varphi}(\nu, t) = \widehat{\pi(g)\varphi}(\nu, t)$.

Proposition 3.13.11. (i) *If $\varphi \in V$,*

$$\pi(\begin{pmatrix} \delta\varpi^\ell & \\ & 1 \end{pmatrix})\widehat{\varphi}(\nu, t) = t^{-\ell}\nu^{-1}(\delta)\widehat{\varphi}(\nu, t),$$

and

$$\pi(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\widehat{\varphi}(\nu, t) = \sum_n t^n \left\{ \sum_\mu \widehat{\varphi}_n(\mu)\eta(\mu^{-1}\nu, \varpi^n x) \right\},$$

where $\eta(\mu, y) = \int_{\mathcal{O}_F^\times} \mu(\epsilon)\psi(y\epsilon)d\epsilon$.

(ii) *For* $\varphi \in V_0$,
$$\pi(w)\widehat{\varphi}(\nu, t) = C(\nu, t)\widehat{\varphi}(\nu^{-1}\omega^1 \mid_{\mathcal{O}_F^\times}, t^{-1}\omega(\varpi)^{-1})$$
where ω *is the central character of* π *and* $C(\nu, t) = \sum_n C_n(\nu)t^n$ *with* $C_n(\nu) : X \to X$ *linear operators. For all* u, $C_n(\nu)u = 0$ *for* $n \ll 0$.

Proof. The proof is a straightforward manipulation of the definitions. Since
$$\pi(\begin{pmatrix} \delta\varpi^\ell & \\ & 1 \end{pmatrix})\widehat{\varphi}(\nu, t) = \pi(\widehat{\begin{pmatrix} \delta\varpi^\ell & \\ & 1 \end{pmatrix})\varphi}(\nu, t) = \sum_n \pi(\widehat{\begin{pmatrix} \delta\varpi^\ell & \\ & 1 \end{pmatrix})\varphi}_n t^n,$$
we need to calculate
$$\begin{aligned}
\pi(\widehat{\begin{pmatrix} \delta\varpi^\ell & \\ & 1 \end{pmatrix})\varphi}_n &= \int_{\mathcal{O}_F^\times} \pi\begin{pmatrix} \delta\varpi^\ell & \\ & 1 \end{pmatrix}\varphi(\epsilon\varpi^n)\nu(\epsilon)d\epsilon \\
&= \int_{\mathcal{O}_F^\times} \varphi(\delta\epsilon\varpi^{\ell+n})\nu(\epsilon)d\epsilon \\
&= \nu(\delta)^{-1}\int_{\mathcal{O}_F^\times} \varphi(\epsilon\varpi^{\ell+n})d\epsilon \\
&= \nu(\delta)^{-1}\widehat{\varphi}_{\ell+n}(\nu).
\end{aligned}$$
Plugging this into the above equation, we get
$$\pi(\begin{pmatrix} \delta\varpi^\ell & \\ & 1 \end{pmatrix})\widehat{\varphi}(\nu, t) = \sum_n \nu(\delta)^{-1}\widehat{\varphi}_{n+\ell}t^n = t^{-\ell}\nu(\delta)^{-1}\sum_n \widehat{\varphi}_n t^n.$$
The second formula of (i) is similarly proved. We leave it as an exercise.

It remains to prove (ii). As above,
$$\pi(w)\widehat{\varpi}(\nu, t) = \widehat{\pi(w)\varphi}(\nu, t) = \sum_n \widehat{\pi(w)\varphi}_n(\nu)t^n,$$
so we compue the individual terms:
$$\begin{aligned}
\widehat{\pi(w)\varphi}_n(\nu) &= \int_{\mathcal{O}_F^\times} \pi(w)\varphi(\epsilon\varpi^n)\nu(\epsilon)d\epsilon \\
&= \int_{\mathcal{O}_F^\times} \pi(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix})\pi(w)\varphi(\varpi^n)\nu(\epsilon)d\epsilon \\
&= \int_{\mathcal{O}_F^\times} \pi(\begin{pmatrix} \epsilon & \\ & \epsilon \end{pmatrix})\pi(\begin{pmatrix} 1 & \\ & \epsilon^{-1} \end{pmatrix})\pi(w)\varphi(\varpi^n)\nu(\epsilon)d\epsilon \\
&= \int_{\mathcal{O}_F^\times} \pi(w)\pi(\begin{pmatrix} \epsilon^{-1} & \\ & 1 \end{pmatrix})\varphi(\varpi^n)(\omega\nu)(\epsilon)d\epsilon \\
&= \sum_\mu \int_{\mathcal{O}_F^\times} (\omega\nu\mu^{-1})(\epsilon)\pi(w)\pi(\begin{pmatrix} \epsilon^{-1} & \\ & 1 \end{pmatrix})\varphi_{[\mu]}(\varpi^n)d\epsilon \\
&= \pi(w)\varphi_{[\omega\nu]}(\varpi^n).
\end{aligned}$$
In the third to last line we used that $\varphi = \sum_\mu \varphi_{[\mu]}$ and in the second to last line that $\int_{\mathcal{O}_F^\times}(\omega\nu\mu^{-1})(\epsilon)d\epsilon$ is zero if $\omega\nu\mu^{-1}$ is not the trivial character and 1 otherwise. So
$$\pi(w)\widehat{\varphi}(\nu, t) = \sum_n \pi(w)\varphi_{[\omega\nu]}(\varpi^n)t^n.$$

The right hand side of this equation will be zero when $\varphi_{[\omega\nu]} = 0$. This happens if and only if $\widehat{\varphi}((\omega\nu)^{-1}, t) = 0$. For $u \in X$, set

$$\varphi_{u,\mu} := 1_{\mathcal{O}_F^\times}\mu \cdot u,$$

and define $C_n(\nu)\pi(w)\varphi_{u,\omega\nu}(\varpi^n)$.
Claim: For $\varphi \in V_0$,

$$\pi(w)\widehat{\varphi}(\nu, t) = C(\nu, t)\widehat{\varphi}((\omega\nu)^{-1}, t^1\omega(\varpi)^{-1}). \tag{3.13.3}$$

One only needs to check this claim for translates of $\varphi_{u,\omega\nu}$ because these span V_0. Actually, one can show that (3.13.3) is true for φ implies that it is true for $\pi(\left(\begin{smallmatrix}\pi^\ell & \\ & 1\end{smallmatrix}\right))\varphi$. To do this, one uses (i). Need to check:

$$\pi(w)\widehat{\pi(\begin{pmatrix}\pi^\ell & \\ & 1\end{pmatrix})\varphi}(\nu, t) = C(\nu, t)\pi(\begin{pmatrix}\pi^\ell & \\ & 1\end{pmatrix})\widehat{\varphi}((\omega\nu)^{-1}, t^{-1}\omega(\varpi)^{-1}).$$

This is routine and is left as an exercise. □

Remark. The operators $C_n(\nu)$ determine the structure of the representation.

Corollary 3.13.12. (i) *If $X_1 \subseteq X$ is a subspace that is invariant by all $C_n(\nu)$ then $V_1 = span\{\pi(g)\varphi \mid \varphi \in S(F^\times, X_1)\}$ is G-invariant.*
(ii) *If $T : X \to X$ is a linear map that commutes with every $C_n(\nu)$ then for all $\varphi \in V$, $T\varphi(\alpha) = T(\varphi(\alpha))$ is a G-intertwining map.*

Note that since $V \supseteq S(F^\times, X_1)$ is irreducible, (i) implies that there is no subspace of X invariant by all $C_n(\nu)$, and (ii) implies that if T commutes with all $C_n(\nu)$ it must be a scalar multiple of the identity map on X.

Proof. (i) We begin with two observations:
Observation 1: Suppose that $\varphi \in C^\infty(F^\times, X)$. Then φ takes values in X_1 if and only if $\varphi_n(\nu) \in X$ for all n, ν. Equivalently, $\widehat{\varphi}(\nu, t)$ is a series with coefficients in X_1 for all ν. This is clear by noting that φ is spanned by characters. i.e. $\varphi(\varpi\epsilon) = \sum \varphi_n(\nu)\nu^{-1}(\epsilon)$.
Observation 2: If φ takes values in X_1 then $\pi((\begin{smallmatrix}* & * \\ & *\end{smallmatrix}))$ takes values in X_1. This is true because acting by upper triangular matrices only changes φ by a character.

Since $G = P \sqcup NwP$, observation 2 implies that it suffices to show that for $\varphi \in S(F^\times, X_1)$, $\pi((\begin{smallmatrix}1 & x \\ & 1\end{smallmatrix}))\pi(w)\varphi$ takes values in X_1, and since $(\begin{smallmatrix}1 & x \\ & 1\end{smallmatrix})$ acts by character, just need to check that $\pi(w)\varphi$ takes values in X_1. But

$$\widehat{\pi(w)\varphi}(\nu, t) = C(\nu, t)\widehat{\varphi}((\nu\omega)^{-1}, t^{-1}\omega(\varpi)^{-1})$$

is a series with coefficients in X_1 for all ν, so by observation 1, $\pi(w)\varphi$ does take values in X_1. This proves (1).

(ii) We first show that $\varphi \in V$ implies that $T\varphi \in V$. If $\varphi \in V_0$ then it is clear that $T\varphi \in V_0$, so we suppose that $T\varphi = \pi(w)\varphi_0$ for some $\varphi_0 \in V_0$. Then one can show that $T\varphi = \pi(w)T\varphi_0 \in V$. To do so, one compares the Mellin transforms of both sides:

$$\begin{aligned}\widehat{T\varphi}(\nu,t) &= \sum_n \widehat{T\varphi}_n(\nu)t^n = \sum_n \left(\int_{\mathcal{O}_F^\times} T\varphi(\varpi^n\epsilon)\nu(\epsilon)d\epsilon\right)t^n \\ &= \sum_n T\left(\int_{\mathcal{O}_F^\times} \varphi(\varpi^n\epsilon)\nu(\epsilon)d\epsilon\right)t^n \\ &= \sum_n T\widehat{\varphi}_n(\nu)t^n = T\widehat{\varphi}(\nu,t) \\ &= T\widehat{\pi(w)\varphi_0}(\nu,t) = TC(\nu,t)\widehat{\varphi}((\omega\nu)^{-1},t^{-1}\omega\varpi^{-1}) \\ &= C(\nu,t)T\widehat{\varphi_0}((\omega\nu)^{-1},t^{-1}\omega\varpi^{-1}) = \pi(w)T\varphi_0(\nu,t).\end{aligned}$$

This is true for all ν, so observation 1 implies $T\varphi = \pi(w)T\varphi_0$.

To check that T is intertwining we need to check that if $\varphi \in V$,

$$T(\pi(g)\varphi) = \pi(g)(T\varphi) \tag{3.13.4}$$

for all $g \in G$. Note that it suffices to verify (3.13.4) for $g = \left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}a & \\ & 1\end{smallmatrix}\right)$ or w. In the first two cases the comparison is easy. The hard part is to check $T\pi(w)\varphi = \pi(w)T\varphi$. We split this into cases.

Case 1: $\varphi \in V_0$ follows because, by above,

$$\begin{aligned}\widehat{T\pi(w)}\varphi(\nu,t) &= T\widehat{\pi(w)\varphi}(\nu,t) \\ &= TC(\nu,T)\widehat{\varphi}((\omega\nu)^{-1},t^{-1}\omega\varpi^{-1}) \\ &= C(\nu,T)\widehat{T\varphi}((\omega\nu)^{-1},t^{-1}\omega\varpi^{-1}) \\ &= \widehat{\pi(w)\varphi}(\nu,t).\end{aligned}$$

Case 2: $\varphi = \pi(w)\varphi_0$. Then $T\pi(w)\varphi = T\pi(w)\pi(w)\varphi_0 = T\pi(-e)\varphi_0 = \omega(-1)T\varphi_0$. On the other hand, $\pi(w)T\varphi = \pi(w)T\pi(w)\varphi_0 = \omega(-1)T\varphi_0$. $\square$

Proposition 3.13.13. *$C_n(\nu)C_p(\rho) = C_p(\rho)C_n(\nu)$ for all n, p, ν, ρ.*

Proof of Theorem 3.13.5. The proposition and corollary imply that $C_n(\nu)$ is a scalar for all ν, n. If $\dim X \neq 1$, then for any subspace $0 \subsetneq X_1 \subsetneq X$, X_1 is invariant under all $C_n(\nu)$. Thus $V_1 = \mathit{span}\{\pi(g)\varphi \mid \varphi \in S(F^\times, X_1)\}$ is a nontrivial G-invariant subspace of V. A contradiction. $\square$

To prove the proposition, we use the identity

$$w\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}w = -\begin{pmatrix}1 & -1\\ & 1\end{pmatrix}w\begin{pmatrix}1 & -1\\ & 1\end{pmatrix}$$

which implies that for all $\varphi \in V_0 = S(F^\times, X)$

$$\pi(w)\pi(\begin{pmatrix}1 & 1\\ & 1\end{pmatrix})\pi(w)\varphi = \omega(-1)\pi(\begin{pmatrix}1 & -1\\ & 1\end{pmatrix})\pi(w)\pi(\begin{pmatrix}1 & -1\\ & 1\end{pmatrix})\varphi. \tag{3.13.5}$$

We compare the Mellin series of both sides of (3.13.5). The ν, t series are of the form

$$\sum_n t^n \sum_{p,\rho}\left(\quad * \quad \right)\widehat{\varphi}_p(\rho).$$

By comparing the coefficients $*$ from the two sides of (3.13.5), one gets the following.

Lemma 3.13.14. *Suppose ψ has conductor $\mathfrak{p}^{-\ell}$.*

(1) *If $\nu\rho\omega\,|_{\mathcal{O}_f^\times} \neq 1$ has conductor $\mathfrak{p}^m$ then*

$$\begin{aligned}&\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\rho, \varpi^p)C_{p+n}(\sigma)\\ &=\eta((\sigma\rho\omega\,|_{\mathcal{O}_f^\times})^{-1})\omega(\varpi^{m+\ell})(\nu\rho\omega)(-1)C_{n-m-\ell}(\nu)C_{p-m-\ell}(\rho)\end{aligned}$$

(2) *Write $\widetilde{\nu} = (\nu\omega\,|_{\mathcal{O}_F^\times})-1$. Then*

$$\begin{aligned}&\sum_\sigma \eta(\sigma^{-1}\eta, \varpi^n)\eta(\sigma^{-1}\rho, \varpi^p)C_{p+n}(\sigma)\\ &=\omega(\varpi)^p\omega(-1)\delta_{n,p} + (|\varpi| - 1)^{-1}\omega\varpi^{\ell+1}C_{n-1-\rho}(\nu)C_{p-1-\ell}(\rho)\\ &\quad - \sum_{r=-2-\ell}^{-\infty} \omega(\varpi)^{-r}C_{n+r}C_{p+r}.\end{aligned}$$

The proposition will follow from this lemma.

3.14. ***L*-factors attached to local representations.** As before, we assume that (π, V) is an irreducible admissible infinite dimensional representation of G and that $\psi : F \to \mathbb{C}^\times$ is a nontrivial character. In this section we explore some of the consequences of Theorem 3.13.4.

Corollary 3.14.1. *A Kirillov model exists and is unique. A ψ-Whittaker model exists and is unique.*

See Definition 3.13.3 for details as to how these models are defined.

3.14.1. *Motivation and review of* GL_1 *theory.* These two models have extensive use. We will use them to study the local L-factors. The idea in defining them is as follows. (π, V) irreducible admissible infinite dimensional with Whittaker model $W(\pi, \psi)$. For $W \in W(\pi, \psi)$, consider

$$\Psi(g, s, W) = \int_{F^\times} W\left[\begin{pmatrix} a & \\ & 1\end{pmatrix} g\right] |a|^{s-\frac{1}{2}}\, d^\times a.$$

Then there exists an L-factor $L(\pi, s) = p(q^{-s})^{-1}$ where p is a polynomial with constant term 1 and q is the order of the residue field of F, such that

$$\Phi(g, s, W) = \frac{\Psi(g, s, W)}{L(s, \pi)}$$

is holomorphic for all W and g. In other words, the L-factor is the nonholomorphic part of the Whittaker integral.

This mirrors the GL_1 theory where start with $\chi : F^\times \to \mathbb{C}^\times$ and a Bruhat-Schwartz function Φ to get

$$Z(\Phi, \chi, s) = \int_{F^\times} \Phi(x)\chi(x)\,|x|^s\, d^\times x.$$

We defined an L-factor $L(s, \chi)$ such that

$$\frac{Z(\Phi, \chi,)}{L(s, \chi)}$$

is holomorphic. Moreover, (3.9.6) gave a functional equation

$$\frac{Z(\Phi,\chi,s)}{L(s,\chi)} = \epsilon(s,\chi,\Phi)\frac{Z(\widehat{\Phi},\chi,1-s)}{L(1-s,\chi^{-1})}.$$

Remark. The L-factor is of interest because it does not depend on χ (in the GL_1 case) or W (in the GL_2 case.)

3.14.2. *The contragradient representation.* In this section we describe the difference between the Whittaker model of π and that of $\widetilde{\pi}$.

Proposition 3.14.2. *Let ω be the central quasicharacter of π. Then $\widetilde{\pi} = \pi \otimes \omega^{-1}$. Also,*

$$W(\widetilde{\pi},\psi) = \left\{W(g)\omega^{-1}(\det g) \mid W \in W(\pi,\psi)\right\}.$$

Proposition 3.14.3. *Let (π, V) be in its Kirillov form. Then*

$$V' = span\left\{v - \pi\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} v \mid v \in V, x \in F\right\} = S(F^\times)$$

Proof. $V' \subseteq S(F^\times)$ because, as we saw in the proof of Proposition 3.13.10, $v - \pi\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right) v$ is compactly supported. So we need to show that $S(F^\times) \subseteq V'$. Actually we can show that for all $\varphi \in S(F^\times)$,

$$\varphi \in span\left\{\widetilde{\varphi} - \pi\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right)\widetilde{\varphi} \mid \widetilde{\varphi} \in S(F^\times), x \in F\right\}.$$

For $\varphi \in S(F^\times)$, set $\varphi(0)$. Now φ can be considered as an element of $S(F)$, and so it has a Fourier transform

$$\widetilde{\varphi}(y) = \int_F \varphi(x)\psi(-xy)dx$$

which is also in $S(F)$. By Fourier inversion,

$$\varphi(x) = c \cdot \int_F \widetilde{\varphi}(-y)\psi(xy)dy.$$

(The constant c depends on ψ and the choice of measure.)

Let n_1, n_2 be such that $\mathrm{supp}(\varphi) \subseteq \mathfrak{p}^{n_1}$ and $\mathrm{supp}(\widetilde{\varphi}) \subseteq \mathfrak{p}^{n_2}$. When $x \in \mathfrak{p}^{n_1}$, $\varphi(-y)\psi(xy)$ is constant over cosets of $\mathfrak{p}^{n_3}$ for some $n_3 > n_1, n_2$. (Note that the conductor of $\psi(xy)$ is the conductor of ψ times $\mathfrak{p}^{-v(x)}$.) Let $\{y_i\}$ be coset representatives of $\mathfrak{p}^{n_3}$ in $\mathfrak{p}^{n_2}$. Whenever $x \in \mathfrak{p}^{n_1}$

$$\varphi(x) = \sum_i c \cdot \int_{y_i + \mathfrak{p}^{n_3}} \widehat{\varphi}(-y)\psi(xy)dy = \sum_i c\widehat{\varphi}(-y_i)\psi(xy_i).$$

Setting $\varphi_0 = 1_{\mathfrak{p}^{n_1}}$ and $\lambda_i = c\widehat{\varphi}(-y_i)$, we have an equality of functions

$$\varphi(x) = \sum_i \lambda_i \psi(xy)\varphi_0(x) = \sum_i \lambda_i \pi\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\varphi_0(x)$$

for all $x \in F$. Since $\varphi(0) = 0$ and $\psi(0) = 1$, in particular, we have $\sum \lambda_i = 0$. Thus

$$\varphi = \sum \lambda_i (\pi\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\varphi_0 - \varphi_0).$$

Since $\varphi_0 \in S(F^\times)$, this completes the proof. $\square$

As an immediate consequence of the definition of the Jacquet module, we get the following.

Corollary 3.14.4. *π is supercuspidal if and only its Kirillov model is realized in $S(F^\times)$.*

Remark. Suppose $\psi : F \to \mathbb{C}^\times$ is nontrivial. If $L : S(F^\times) \to \mathbb{C}$ satisfies $L(\xi_\psi \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \varphi) = \psi(x)L\varphi$ for all φ, then $L = 0$.

For (π, V) as above, by $(\pi \otimes \chi, V)$ we mean the representation of G with underlying space V and action $(\pi \otimes \chi)(g)v = \chi(\det g)\pi(g)v$. The following lemma describes the difference between the Kirillov and Whittaker models of π and $\pi \otimes \chi$.

Lemma 3.14.5. *Denote $\pi \otimes \chi$ by π'. Let V and $W(\pi, \psi)$ be the Kirillov and ψ-Whittaker models of π. Then*

(i) *The corresponding models of π' are*

$$V' = \{\chi\varphi \mid \varphi \in V\},$$

$$W' = W(\pi', \psi) = \{\chi \circ \det \cdot W \mid W \in W(\pi, \psi)\}.$$

(ii) $C'(\nu, t) = C(\nu\chi \mid_{\mathcal{O}_F^\times}, \chi(\varpi)t)$

Proof. If $v \in V$ then the Kirillov model for π is given by

$$v \mapsto \varphi_v : \alpha \mapsto L(\pi \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} v).$$

On the other hand, the Kirillov model for π' is

$$v \mapsto \varphi_v : \alpha \mapsto L(\pi' \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} v) = L(\chi(\alpha)\pi \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} v) = \chi(\alpha)L(\pi \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} v).$$

Similarly, for the Whittaker models, for π,

$$v \mapsto W_v : g \mapsto L(\pi(g)v),$$

and for π',

$$v \mapsto W_v : g \mapsto L(\pi'(g)v) = L(\chi(\det g)\pi(g)v) = \chi \circ \det g \cdot L(\pi(g)v).$$

The verification of (ii) is straightforward. It is left as an exercise. □

Before we prove the proposition, we give some preparation on pairings. If $\varphi, \varphi' \in C^\infty(F^\times)$ we can formally define

$$\langle \varphi, \varphi' \rangle = \int_{F^\times} \varphi(\alpha)\varphi'(-\alpha)d^\times\alpha.$$

If we assume that one of the functions is Schwartz-Bruhat then the integral is well-defined, and moreover, for $b = \left(\begin{smallmatrix} a & x \\ & 1 \end{smallmatrix}\right)$,

$$\begin{aligned} \langle \xi_\psi(b)\varphi, \xi_\psi(b)\varphi' \rangle &= \int_{F^\times} \xi_\psi(b)\varphi(\alpha)\xi_\psi(b)\varphi'(-\alpha)d^\times\alpha \\ &= \int_{F^\times} \psi(\alpha x)\varphi(a\alpha)\psi(-\alpha x)\varphi'(-a\alpha)d^\times\alpha \\ &= \langle \varphi, \varphi' \rangle. \end{aligned}$$

The final equality follows by changing variables and using that $d^\times(a\alpha) = d^\times\alpha$.

Proof of Proposition 3.14.2. Write $\pi' = \pi \otimes \omega^{-1}$. One needs to provide a non-degenerate G-invariant pairing from the underlying spaces of π and π' to $\mathbb{C}$. We use the Kirillov model. Let V, V' be the Kirillov models of π and π' respectively. So

$$V = S(F^\times) + \pi(w)S(F^\times), \qquad V' = S(F^\times) + \pi'(w)S(F^\times).$$

For $\varphi = \varphi_1 + \pi(w)\varphi_2$ and $\varphi' = \varphi_1' + \pi(w)\varphi_2'$, define

$$\beta(\varphi, \varphi') := \langle \varphi_1, \varphi_1' \rangle + \langle \varphi_1, \varphi_2 \rangle + \langle \varphi_2, \varphi_1' \rangle + \langle \varphi_2, \varphi_2' \rangle.$$

One can verify that β does not depend on the choice of decomposition of φ and φ'. That β is G-invariant follows from the comments above and the following lemma. The description of the Whittaker space of $\widetilde{\pi}$ is straightforward. $\square$

Lemma 3.14.6. *For $\varphi, \varphi' \in S(F^\times)$, one has the following.*

(1) $\langle \pi(w)\varphi, \varphi' \rangle = \omega(-1)\langle \varphi, \pi'(w)\varphi' \rangle$.
(2) *If $\pi(w)\varphi$ or $\pi(w)\varphi'$ belongs to $S(F^\times)$ then $\langle \pi(w)\varphi, \pi'(w)\varphi' \rangle = \langle \varphi, \varphi' \rangle$.*

The proof is left as an exercise, or one can read it in the book[5] of Jacquet and Langlands.

3.14.3. *Local L-factors: the supercuspidal case.* The main theorem of the local theory is the following.

Theorem 3.14.7. *Let (π, V) be an irreducible admissible infinite dimensional representation of G with central character ω, and suppose $W(\pi, \psi)$ its ψ-Whittaker model for a nontrivial character $\psi : F \to \mathbb{C}^\times$. Then the functions*

$$\Psi(g, s, W) := \int_{F^\times} W(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g)\, |a|^{s-\frac{1}{2}}\, d^\times a,$$

$$\widetilde{\Psi}(g, s, W) := \int_{F^\times} W(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g)\, |a|^{s-\frac{1}{2}}\, \omega^{-1}(a) d^\times a$$

satisfy the following properties.

(i) *There exists $s_0 \in \mathbb{R}$ such that $\Psi(g, s, W)$ and $\widetilde{\Psi}(g, s, W)$ absolutely converge whenever $Re(s) > s_0$ for all $g \in G$ and $W \in W(\pi, \psi)$.*
(ii) *There exists a unique Euler factor $L(s, \pi)$ such that*

$$\Phi(g, s, W) = \frac{\Psi(g, s, W)}{L(s, \pi)}$$

is holomorphic in s for all $g \in G$ and $W \in W(\pi, \psi)$.
(iii) *Set*

$$\widetilde{\Phi}(g, s, W) = \frac{\widetilde{\Psi}(g, s, W)}{L(s, \widetilde{\pi})}.$$

Then there exists a unique exponential function $\epsilon(s, \pi, \psi)$ such that

$$\widetilde{\Phi}(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g, 1 - s, W) = \epsilon(s, \pi, \psi)\Phi(g, s, W).$$

We first observe that the uniqueness of $L(s, \pi)$ is easy. Indeed, being an Euler factor implies that $L(s, \pi) = (p(q^{-s}))^{-1}$ where p is a polynomial with constant term 1 and $q = |\varpi|^{-1}$. If $L'(s, \pi) = (p'(q^{-s}))^{-1}$ is another such function then

$$\frac{L(s, \pi)}{L'(s, \pi)} \quad \text{and} \quad \frac{L'(s, \pi)}{L(s, \pi)}$$

would both be holomorphic and hence $L'(s,\pi) = CL(s,\pi)$ for some constant C. The condition that constant terms of p and p' be 1 forces $C = 1$.

We also observe that for $W \in W(\pi,\psi)$, $\pi(g)W(h) = W(hg)$. So

$$\begin{aligned}\Psi(g,s,W) &= \int_{F^\times} W(\begin{pmatrix} a & \\ & 1\end{pmatrix} g)\, |a|^{s-\frac{1}{2}}\, d^\times a \\ &= \int_{F^\times} (\pi(g)W)(\begin{pmatrix} a & \\ & 1\end{pmatrix})\, |a|^{s-\frac{1}{2}}\, d^\times a \\ &= \Psi(e,s,\pi(g)W),\end{aligned}$$

and we therefore only need to check the theorem for $g = e$ and W varying over $W(\pi,\psi)$.

The functional equation (part (iii) of the theorem) deals with the action of $w = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$, and this was our main concern when dealing with the Mellin transform series. In fact, the Mellin series will be our main tool for proving the functional equation. As a reminder, if $\varphi \in C^\infty(F^\times)$ and $\nu : \mathcal{O}_F^\times \to \mathbb{C}^\times$ is a character then

$$\widehat{\varphi}(\nu,t) := \sum \widehat{\varphi}_n(\nu) t^n \quad \text{where} \quad \widehat{\varphi}_n(\nu) = \int_{\mathcal{O}_F^\times} \varphi(\epsilon\varpi^n)\nu(\epsilon) d\epsilon.$$

It is an important fact that φ is completely determined by $\{\widehat{\varphi}(\nu,t)\}_\nu$.

As we have shown before, in (3.13.1), the Kirillov and Whittaker models of π are in one to one correspondence:

$$\begin{array}{ccc} W(\pi,\psi) & \longleftrightarrow & V \\ W & \longleftrightarrow & \varphi \end{array}$$

where $W(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)) = \varphi(a)$. So we can write

$$\Psi(e,s,W) = \int_{F^\times} W(\begin{pmatrix} a & \\ & 1\end{pmatrix})\, |a|^{s-\frac{1}{2}}\, d^\times a = \int_{F^\times} \varphi(a)\, |a|^{s-\frac{1}{2}}\, d^\times a. \tag{3.14.1}$$

For any quasicharacter $\chi : F^\times \to \mathbb{C}^\times$, whenever

$$\widehat{\varphi}(\chi) := \int_{F^\times} \varphi(a)\chi(a) d^\times a$$

is absolutely convergent, we have

$$\begin{aligned}\widehat{\varphi}(\chi) &= \sum_n \int_{\mathcal{O}_F^\times} \varphi(\varpi^n a)\chi(\varpi^n a) d^\times a \\ &= \sum_n \left(\int_{\mathcal{O}_F^\times} \varphi(\varpi^n a)\chi(a) d^\times a \right) \chi(\varpi)^n \\ &= \sum_n \widetilde{\varphi}_n(\chi\,|_{\mathcal{O}_F^\times})\chi(\varpi)^n \\ &= \widehat{\varphi}(\chi\,|_{\mathcal{O}_F^\times}, \chi(\varpi)).\end{aligned}$$

In particular, if $\Psi(e,s,W)$ is absolutely convergent then (3.14.1) becomes

$$\Psi(e,s,W) = \widehat{\varphi}(|\cdot|^{s-\frac{1}{2}}) = \widehat{\varphi}(1, q^{\frac{1}{2}-s})$$

where $q = |\varpi|^{-1}$. Similarly, when it is absolutely convergent, we have

$$\begin{aligned}
\widetilde{\Psi}(e,s,W) &= \int_{F^\times} W(\begin{pmatrix} a & \\ & 1 \end{pmatrix}) |a|^{s-\frac{1}{2}} \omega^{-1}(a) d^\times a \\
&= \int_{F^\times} \varphi(a) |a|^{s-\frac{1}{2}} \omega^{-1}(a) d^\times a \\
&= \widehat{\varphi}(|\cdot|^{s-\frac{1}{2}} \omega^{-1}) \\
&= \widehat{\varphi}(\omega_0^{-1}, q^{\frac{1}{2}-s} \mathfrak{z}_0^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\Psi}(w,1-s,W) &= \int_{F^\times} (\pi(w)W)(\begin{pmatrix} a & \\ & 1 \end{pmatrix}) |a|^{\frac{1}{2}-s} \omega^{-1}(a) d^\times a \\
&= \int_{F^\times} (\pi(w)\varphi)(a) |a|^{s-\frac{1}{2}} \omega^{-1}(a) d^\times a \\
&= \widehat{\pi(w)\varphi}(|\cdot|^{\frac{1}{2}-s} \omega) \\
&= \widehat{\pi(w)\varphi}(\omega_0^{-1}, q^{s-\frac{1}{2}} \mathfrak{z}^{-1}).
\end{aligned}$$

For notational convenience we are letting $\omega_0 = \omega \mid_{\mathcal{O}_F^\times}$ and $\mathfrak{z}_0 = \omega(\varpi)$.

From the above, we will see that the fact that whenever $\varphi \in S(F^\times)$

$$\widehat{\pi(w)\varphi}(\nu,t) = C(\nu,t)\widehat{\varphi}((\nu\omega_0)^{-1}, (t\mathfrak{z}_0)^{-1}) \tag{3.14.2}$$

will be crucial to proving the theorem. In fact, when π is supercuspidal, we will see that everything is much easier because $V = S(F^\times)$ and so $\pi(w)\varphi \in S(F^\times)$ and all of the integrals of type (3.14.1) converge absolutely. We will also see that the term $C(\nu,t)$ in (3.14.2) is particular simple for the choice of ν we will be concerned with.

Proof of Theorem 3.14.7 when π is supercuspidal. Let V be the Kirillov model for π and let $\varphi \in V$ be the function that corresponds to $W \in W(\pi,\psi)$. Since π is supercuspidal $V = S(F^\times)$. Thus,

$$\Psi(e,s,W) = \int_{F^\times} \varphi(a) |a|^{s-\frac{1}{2}} d^\times \quad \text{and} \quad \widetilde{\Psi}(e,s,W) = \int_{F^\times} \varphi(a) |a|^{s-\frac{1}{2}} \omega^{-1}(a) d^\times$$

converge absolutely for all s and W. This is statement (i) of the theorem.

So, by our comment above and the fact that for any character $\nu : F^\times \to \mathbb{C}$ and any $\varphi \in V$, $\widehat{\varphi}(\nu,t)$ is a finite sum of powers of t,

$$\Psi(e,s,W) = \widehat{\varphi}(1, q^{\frac{1}{2}-s})$$

is holomorphic for all choices of φ. So, we can take $L(s,\pi) = 1$, and we have proved (ii).

Since $\widetilde{\pi}$ is also supercuspidal, $L(s,\widetilde{\pi}) = 1$ as well, and

$$\Phi(e,s,W) = \frac{\Psi(e,s,W)}{L(s,\pi)} = \int_{F^\times} \varphi(a) |a|^{s-\frac{1}{2}} d^\times = \widehat{\varphi}(1, q^{\frac{1}{2}-s}).$$

Similarly, since $\widehat{\pi(w)\varphi} \in S(F^\times)$,

$$\begin{aligned}
\widetilde{\Phi}(w,1-s,W) = \frac{\widetilde{\Psi}(w,1-s,W)}{L(1-s,\widetilde{\pi})} &= \int_{F^\times} \pi(w)\varphi(a) |a|^{\frac{1}{2}-s} \omega^{-1}(a) d^\times \\
&= \widehat{\pi(w)\varphi}(\omega_0^{-1}, \mathfrak{z}_0^{-1} q^{s-\frac{1}{2}}) = C(\omega_0^{-1}, \mathfrak{z}_0^{-1} q^{s-\frac{1}{2}}) \widehat{\varphi}(1, q^{\frac{1}{2}-s}).
\end{aligned}$$

The final equality is just (3.14.2). Taking $\epsilon(s,\pi,\psi) = C(\omega_0^{-1}, \mathfrak{z}_0^{-1} q^{s-\frac{1}{2}})$ gives the functional equation.

To complete the proof, it will suffice to show that $C(\nu,t)$ is a multiple of a single power of t. That $\epsilon(s,\pi,\psi)$ is an exponential function would follow immediately. First, we show that $C(\nu,t)$ is a finite sum. Indeed, one can choose φ such that $\widehat{\varphi}((\nu\omega_0)^{-1},t) = 1$. This implies that $C(\nu,t) = \widehat{\pi(w)\varphi}(\nu,t)$. Since $\pi(w)\varphi \in S(F^\times)$, this must be a finite sum. Two applications of (3.14.2) immediately gives

$$C(\nu,t)C(\nu^{-1}\omega_0^{-1}, \mathfrak{z}_0^{-1}t^{-1}) = \omega(-1) \tag{3.14.3}$$

If $C(\nu,t)$ has more than one term, then it must have a zero different from $t=0$. On the other hand $C(\nu^{-1}\omega_0^{-1}, \mathfrak{z}_0^{-1}t^{-1})$ has no pole besides at $t=0$. Putting these together contradicts (3.14.3). $\square$

3.14.4. *Local L-factors: principal series representations.* In this section we prove Theorem 3.14.7 in the case of $B(\mu_1,\mu_2)$ irreducible. Recall

$$B(\mu_1,\mu_2) = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{c} f(\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} g) = \mu_1(a_1)\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} f(g) \\ f \text{ is right invariant by some } L \subset K \text{ open.} \end{array} \right\}$$

We will use a different description of this space (the so-called *Godement sections.*) Let $\Phi \in S(F^2)$. Consider

$$f_\Phi(g) := \mu_1(\det g) \left|\det g\right|^{1/2} \int_{F^\times} \mu_1\mu_2^{-1}(t) \left|t\right| \Phi((0,t)g) d^\times t.$$

Write $\left|\mu_1\mu_2^{-1}\right| = \left|\cdot\right|^{s_0}$ for $s_0 \in \mathbb{R}$. Then, since the integral part of f_Φ is a zeta integral of GL$_1$-type, Lemma 3.9.6 is absolutely convergent whenever $s_0 > -1$.

Lemma 3.14.8. *Let μ_1 and μ_2 and s_0 be as above. If $s_0 > -1$ then*

$$B(\mu_1,\mu_2) = \{f_\Phi \mid \Phi \in S(F^2)\}.$$

Remark. The mapping $\Phi \mapsto f_\Phi$ is certainly not injective, so given $f \in B(\mu_1,\mu_2)$, there are multiple choices of Φ such that $f_\Phi = f$.

Proof. We first show that $f_\Phi \in B(\mu_1,\mu_2)$. Since Φ is Bruhat-Schwartz, there exists $L \subset K$ fixing Φ. It is easy to see that this same choice of L fixes f_Φ. We check the transformation property directly:

$$\begin{aligned} f_\Phi(\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} g) &= \mu_1(\det g)\left|\det g\right|^{1/2} \mu_1(a_1a_2)\left|a_1a_2\right|^{1/2} \\ &\quad \times \int_{F^\times} \mu_1\mu_2^{-1}(t)\left|t\right| \Phi((0,a_2t)g) d^\times t \\ &= \mu_1(\det g)\left|\det g\right|^{1/2} \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \int_{F^\times} \mu_1\mu_2^{-1}(t)\left|t\right| \Phi((0,t)g) d^\times t \\ &= \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f_\Phi(g). \end{aligned}$$

(In the second step we replaced t with $a_2^{-1}t$ and simplified.)

Next, we show that for all $f \in B(\mu_1,\mu_2)$ there exists Φ such that $f_\Phi = f$. Given $f \in B(\mu_1,\mu_2)$, set

$$\Phi(x,y) = \begin{cases} 0 & \text{if } (x,y) \notin (0,1)K \quad (K = \mathrm{GL}_2(\mathcal{O}_F)) \\ \mu_1^{-1}(\det k) f(k) & \text{if } (x,y) = (0,1)k \quad \text{for some } k \in K. \end{cases}$$

We first show that Φ is well defined. That is, if $(x,y) = (0,1)k_1 = (0,1)k_2$ then we must show that

$$\mu_1^{-1}(\det k_1)f(k_1) = \mu_1^{-1}(\det k_2)f(k_2).$$

Since $k_1k_2^{-1} = \left(\begin{smallmatrix} a & x \\ & 1 \end{smallmatrix}\right)$,

$$\begin{aligned}\mu_1^{-1}(\det k_1)f(k_1) &= \mu_1(\det k_1)f(k_1k_2^{-1}k_2) \\ &= \mu_1^{-1}(\det k_1)\mu_1(\det k_1k_2^{-1})\left|\det k_1k_2^{-1}\right|^{1/2} f(k_2) \\ &= \mu_2^{-1}(\det k_2)f(k_2).\end{aligned}$$

(Notice that $k \in K$ implies that $\det k \in \mathcal{O}_F^\times$ hence $|\det k| = 1$.)

We now show that $\Phi \in S(F^2)$. Since $(x,y) \in (0,1)K$ if and only if $x,y \in \mathcal{O}^\times$ and at least one of them is a unit, it is clearly compactly supported. If μ is unramified it is easy to see that Φ is fixed by the same L as that which fixes f. In any case $L \cap \mathrm{SL}_2(\mathcal{O}_F)$ works.

Finally, we need to show that $f_\Phi = f$. The Iwasawa decompostion $G = PK$ implies that $f \in B(\mu_1,\mu_2)$ is determined by its values on K. So we only need to show that $f(k) = f_\Phi(k)$ if $k \in K$. Indeed, if $k \in K$,

$$\begin{aligned}f_\Phi(k) &= \mu_1(\det k)\int_{F^\times} \mu_1\mu_2(t)\,|t|\,\Phi((0,t)k)d^\times t \\ &= \mu_1(\det k)\int_{\mathcal{O}_F^\times} \mu_1\mu_2(t)\mu^{-1}(t\det k) f\left[\begin{pmatrix} 1 & \\ & t\end{pmatrix}k\right]d^\times t \\ &= \int_{\mathcal{O}_F^\times} f(k)d^\times t = f(k).\end{aligned}$$

In the last step we used that $f(\left(\begin{smallmatrix} 1 & \\ & t \end{smallmatrix}\right)k) = \mu_2(t)f(k)$, and we used that the measure is normalized so that the measure of $\mathcal{O}_F^\times$ is 1. $\square$

Our next task is to explicitly define the Whittaker functions. If $\Phi \in S(F^2)$ then we define

$$\Phi^\sim(x,y) := \int_F \Phi(x,u)\psi(-yu)du.$$

This is the Fourier transform in the second variable, so, by Fourier inversion, the mapping $S(F^2) \to S(F^2)$ via $\Phi \mapsto \Phi^\sim$ is a bijection. Additionally, $[\rho(g)\Phi]^\sim$ satisfies (3.14.4)

$$\begin{aligned}\left(\rho\left[\begin{pmatrix} 1 & n \\ & 1\end{pmatrix}g\right]\Phi\right)^\sim(x,y) &= \int_F \rho\left[\begin{pmatrix} 1 & n \\ & 1\end{pmatrix}g\right]\Phi(x,u)\psi(-yu)du \\ &= \int_F \rho(g)\Phi\left[(x,u)\begin{pmatrix} 1 & -n \\ & 1\end{pmatrix}\begin{pmatrix} 1 & n \\ & 1\end{pmatrix}\right]\psi(-yu)du \\ &= \psi(nxy)[\rho(g)\Phi]^\sim.\end{aligned}$$

The second equality comes after making the change of variables $u \mapsto u - nx$. Thus, if we define

$$F_\Phi(g,t) := [\rho(g)\Phi]^\sim(t,t^{-1})$$

then it satisfies

$$F_\Phi(\begin{pmatrix} 1 & n \\ & 1\end{pmatrix}g,t) = \psi(n)F_\Phi(g,t).$$

So it's not surprising that this will be used to define a Whittaker function.

It is also clear that

$$\left[\rho\left(\begin{pmatrix} a & \\ & b\end{pmatrix} g\right)\Phi\right]^{\sim}(x,y) = [\rho(g)\Phi]^{\sim}(ax, b^{-1}y). \tag{3.14.5}$$

To remove the dependence of $F_\Phi(g)$ on t, we define

$$W_\Phi(g) := \mu_1(\det g)\,|\det g|^{1/2}\int_{F^\times} \mu_1\mu_2^{-1}(t)F_\Phi(g,t)d^\times t$$

which, by (3.14.4), satisfies $W_\Phi(\left(\begin{smallmatrix} 1 & n \\ & 1\end{smallmatrix}\right) g) = \psi(n)W_\Phi(g)$. Moreover, it is easy to see that

$$\rho(h)f_\Phi = f_{\mu_1(\det h)|\det h|^{1/2}\rho(h)\Phi}, \tag{3.14.6}$$

which implies that

$$\rho(h)W_\Phi = W_{\mu_1(\det h)|\det h|^{1/2}\rho(h)\Phi}. \tag{3.14.7}$$

Therefore, the map $f_\Phi \mapsto W_\Phi$ is an intertwining map from $B(\mu_1,\mu_2)$ to $W(\psi)$ *if* it is well defined.

If the map

$$\mathcal{A} : B(\mu_1,\mu_2) \longrightarrow W(\psi) \quad \text{via} \quad f_\Phi \mapsto W_\Phi$$

is injective and well defined then the above would provide the Whittaker model for $B(\mu_1,\mu_2)$. The following proposition gives this fact.

Proposition 3.14.9. *The intertwining map $\mathcal{A}$ is well defined and injective. Moreover, $\rho(g)f_\Phi = f_{\mu_1(\det g)|\det g|^{1/2}\rho(g)\Phi}$, and so $\mathcal{A}$ commutes with the right translation action of G.*

This proposition and Lemma 3.14.8 immediately give the following.

Corollary 3.14.10. *Suppose μ_1,μ_2 are quasicharacters of $F^\times$, and $s_0 \in \mathbb{R}$ satisfies $\left|\mu_1\mu_2^{-1}\right| = |\cdot|^{s_0}$ as above. If $s_0 > -1$ and $s_0 \neq 1$ then $W(\mu_1,\mu_2,\psi)$, the image of $\mathcal{A}$, is the ψ-Whittaker model of $B(\mu_1,\mu_2)$.*

Remark. Since $B(\mu_1,\mu_2) \simeq B(\mu_2,\mu_1)$, the requirement that $s_0 > -1$ in the corollary does not really pose a problem for us. Even so, via analytic continuation, W_Φ can be made to be well defined even when $s_0 < -1$. (See Lemma 3.9.6 and the accompanying discussion.)

Lemma 3.14.11. *If $s_0 > -1$ then*

$$f_\Phi\left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right] = \int_{F^\times} W_\Phi\left[\begin{pmatrix} a & \\ & 1\end{pmatrix}\right]\mu_2^{-1}(a)\,|a|^{1/2}\,\psi(ax)d^\times a. \tag{3.14.8}$$

Proof. It suffices to show that (3.14.8) holds for[11] $\Phi^\sim$. That is

$$f_{\Phi^\sim}\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) = \int_{F^\times} W_{\Phi^\sim}\begin{pmatrix} a & \\ & 1\end{pmatrix}\mu_2^{-1}(a)\,|a|^{-1/2}\,\psi(ax)d^\times a. \tag{3.14.9}$$

[11]When the measure is appropriately chosen $(\Phi^\sim)^\vee = \Phi$ and $(\Phi^\sim)^\sim = \Phi^-$ where $\Phi^-(x,y) = \Phi(x,-y)$.

This will follow by turning it into a double integral then switching the order of integration. Since $(0,t)\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) = (0,t)\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) = (t, tx)$, the left hand side of (3.14.9) is

$$\begin{aligned} f_{\Phi^\sim}\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) &= \int_{F^\times} \mu_1\mu_2^{-1}(t)\,|t|\,\Phi^\sim(t,tx)d^\times t \\ &= \int_{F^\times\times F} \mu_1\mu_2^{-1}(t)\,|t|\,\Phi(t,u)\psi(-utx)dud^\times t, \end{aligned}$$

which is absolutely convergent.

Working with the right hand side of (3.14.9), we use the facts shown above that $[\rho\left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right)\Phi]^\sim(x,y) = \psi(nxy)\Phi^\sim(x,y)$ and $[\rho\left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)\Phi]^\sim(x,y) = |b|^{-1}\Phi^\sim(ax, b^{-1}y)$. Hence

$$\begin{aligned} W_{\Phi^\sim}\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) &= \mu_1(a)\,|a|^{1/2}\int_{F^\times} \mu_1\mu_2^{-1}(t)(\rho\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\Phi^\sim)^\sim(t,t^{-1})d^\times t \\ &= \mu_1(a)\,|a|^{1/2}\int_{F^\times} \mu_1\mu_2^{-1}(t)(\Phi^\sim)^\sim(at,t^{-1})d^\times t \\ &= \mu_1(a)\,|a|^{1/2}\int_{F^\times} \mu_1\mu_2^{-1}(t)\Phi(at,-t^{-1})d^\times t \end{aligned} \tag{3.14.10}$$

Plugging this into the right hand side of (3.14.9) gives

$$\begin{aligned} &\int_{F^\times} \mu_2(a)^{-1}\,|a|^{-1/2}\,\mu_1(a)\,|a|^{1/2}\int_{F^\times} \mu_1\mu_2^{-1}(t)\Phi(at,-t^{-1})\psi(ax)d^\times tdt \\ &= \int_{F^\times\times F} \mu_1\mu_2^{-1}(at)\Phi(at,-t^{-1})\psi(ax)d^\times tda \\ &= \int_{F^\times\times F} \mu_1\mu_2^{-1}(t)\Phi(t,-at^{-1})\psi(ax)d^\times tda \\ &= \int_{F^\times\times F} \mu_1\mu_2^{-1}(t)\Phi(t,a)\psi(-atx)\,|t|\,d^\times tda. \end{aligned}$$

This is the same as the left hand side (with order of integration switched.) Note that we used the transformations $t \mapsto a^{-1}t$ and $a \mapsto -at$. As mentioned above, the Fourier transform is a bijection on $S(F^2)$ and so (3.14.9) implies (3.14.8). $\square$

Proof of Proposition 3.14.9. To prove that $\mathcal{A}$ is well defined, suppose that $f_\Phi = 0$. Then the right hand side of (3.14.8) is zero for all x. This implies that for all x

$$\langle W_{\Phi^-}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\mu_2^{-1}(a)\,|a|^{1/2}, \psi_x(a)\rangle = 0,$$

which means that $W_\Phi(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)) = 0$ for almost all a. But, since it's locally constant, this implies it's zero for all a. In particular $W_\Phi(e) = 0$. We want to show that $0 = W_\Phi(g) = \rho(g)W_\Phi(e) = W_{\mu_1(\det g)|\det g|^{1/2}\rho(g)\Phi}(e)$. By our argument above, this is zero provided

$$f_{\mu_1(\det g)|\det g|^{1/2}\rho(g)\Phi} = 0$$

But this is obviously true because

$$f_{\mu_1(\det g)|\det g|^{1/2}\rho(g)\Phi} = \rho(g)f_\Phi = 0$$

by assumption.

Conversely, if $W_{\Phi^{\sim}} = 0$ then (3.14.8) says that f_Φ is trivial on wN, and hence on the entire Bruhat cell PwN. But this is dense in G, and so $f_\Phi = 0$. Thus $\mathcal{A}$ is injective. $\square$

Proof of Theorem 3.14.7 for irreducible $B(\mu_1,\mu_2)$. Corollary 3.14.10 says that $W(\mu_1,\mu_2;\psi)$ is the Whittaker model of $B(\mu_1,\mu_2)$. For $W = W_\Phi$ we use (3.14.10) to compute

$$\begin{aligned}\Psi(e,s,W) &= \int_{F^\times} W_\Phi\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)|a|^{s-1/2}\,d^\times a\\ &= \int_{F^\times\times F^\times} \mu_1(a)\,|a|^s\,\Phi^\sim(at,t^{-1})\mu_1\mu_2^{-1}(t)d^\times t d^\times a\\ &= \int_{F^\times\times F^\times} \mu_1(at)\mu_2^{-1}(t)\,|a|^s\,\Phi^\sim(at,t^{-1})d^\times t d^\times a\\ &= \int_{F^\times\times F^\times} \mu_1(a)\mu_2^{-1}(t)\,|a|^s\left|t^{-1}\right|^s\Phi^\sim(a,t^{-1})d^\times a d^\times t\\ &= \int_{F^\times\times F^\times} \mu_1(a)\mu_2(t)\,|a|^s\,|t|^s\,\Phi^\sim(a,t)d^\times a d^\times t,\end{aligned}$$

using the transformations $a \mapsto at^{-1}$ and $t\mapsto t^{-1}$. If[12] $\Phi^\sim = \Phi_1\otimes\Phi_2$, and we define $Z(\Phi,\mu_1,\mu_2) := \int_{F^\times\times F^\times}\Phi(x,y)\mu_1(x)\mu_2(y)d^\times x d^\times y$ then

$$\Phi(e,s,W) = Z(\Phi^\sim,\mu_1\,|\cdot|^s,\mu_2\,|\cdot|^s) = Z(\Phi_1,\mu_1,s)Z(\Phi_2,\mu_2,s).$$

Hence $L(s,B(\mu_1,\mu_2)) = L(s,\mu_1)L(s,\mu_2)$.

To get the functional equation, it suffices to check it on a spanning set. One easily verifies that

$$\widetilde{\Psi}(e,s,W) = \int W\begin{pmatrix} a & \\ & 1\end{pmatrix}|a|^{s-\frac12}(\mu_1\mu_2)^{-1}(a)d^\times a = Z(\Phi^\sim,\mu_2^{-1}\,|\cdot|^s,\mu_1^{-1}\,|\cdot|^s).$$

Therefore,

$$\widetilde{\Psi}(w,s,W) = \widetilde{\Psi}(e,s,\rho(w)W) = Z((\rho(w)\Phi)^\sim,\mu_2^{-1}\,|\cdot|^s,\mu_1^{-1}\,|\cdot|^s)$$

To simplify this expression we evaluate

$$\begin{aligned}(\rho(w)\Phi)^\sim(x,y) &= \int_F \rho(w)\Phi(x,u)\psi(-uy)du\\ &= \int_F \Phi[(x,u)w]\psi(-uy)du\\ &= \int_F \Phi[(u,x)]\psi(ux)du.\end{aligned}$$

When $\Phi^\sim = \Phi_1\otimes\Phi_2$ we have $\Phi = \Phi_1\otimes\Phi_2^\vee$, so we get

$$(\rho(w)\Phi)^\sim(x,y) = \Phi_2^\vee(x)\int_F \Phi_1(u)\psi(ux)du = \Phi_2^\vee\otimes\Phi_1^\vee(x,y).$$

Plugging this into the above expression, we have

$$\begin{aligned}\widetilde{\Psi}(w,s,W) =& Z([\rho(w)\Phi]^\sim,\mu_2^{-1}\,|\cdot|^s,\mu_1^{-1}\,|\cdot|^s)\\ =& Z(\Phi_2^\vee\otimes\Phi_1^\vee,\mu_2^{-1}\,|\cdot|^s,\mu_1^{-1}\,|\cdot|^s)\\ =& Z(\Phi_2^\vee,\mu_2^{-1},s)Z(\Phi_1^\vee,\mu_1^{-1},s).\end{aligned}$$

[12]Note that $S(F^2)$ is spanned by functions of the type $\Phi_1\otimes\Phi_2(x,y) = \Phi_1(x)\Phi_2(y)$.

So

$$\Phi(e,s,W) = Z(\Phi_1,\mu_1,s)Z(\Phi_2,\mu_2,s)$$

and

$$\widetilde{\Phi}(w,1-s,W) = Z(\Phi_1^\vee,\mu_1^{-1},1-s)Z(\Phi_2^\vee,\mu_2^{-1},1-s),$$

and the functional equation comes now from the GL_1 theory:

$$\begin{aligned}\frac{\widetilde{\Psi}(w,1-s,W)}{L(1-s,\widetilde{\pi})} =& \frac{Z(\Phi_1^\vee,\mu_1^{-1},1-s)Z(\Phi_2^\vee,\mu_2^{-1},1-s)}{L(1-s,\mu_1^{-1})L(1-s,\mu_2^{-1})}\\ =&\epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi)\frac{Z(\Phi_1,\mu_1,s)Z(\Phi_2,\mu_2,s)}{L(s,\mu_1)L(s,\mu_2)}\\ =&\epsilon(s,\pi,\psi)\frac{\Psi(e,s,W)}{L(s,\pi)}.\end{aligned}$$

That $\epsilon(s,\pi,\psi)$ is exponential is obvious. □

Remark. Recall that $\Phi^\vee(x) = \int_F \Phi(u)\psi(ux)dx$ and $\Phi^\sim(x) = \int_F \Phi(u)\psi(-ux)dx$. In Tate's thesis he uses $\Phi^\sim$ which gives ψ^{-1} instead of ψ, and so the functional equation has a slightly different look there.

3.14.5. *Local L-factors: special representations.* We still need to prove the theorem for $Sp(\chi)$ which is a submodule of $B(\mu_1,\mu_2)$ when $\mu_1 = \chi\,|\cdot|^{1/2}$ and $\mu_2 = \chi\,|\cdot|^{-1/2}$. As in the case of principal series representations we want to have an explicit description of the Whittaker model of Sp_χ. Using the map $\mathcal{A}$ of Proposition 3.14.9 in this case, we still have a correspondence

$$B(\mu_1,\mu_2) \longleftrightarrow W(\mu_1,\mu_2;\psi) \subset W(\psi).$$

The goal (accomplished in the following lemma) is to describe the corresponding space for $\mathrm{Sp}_\chi \subset B(\mu_1,\mu_2)$.

Lemma 3.14.12. $W(Sp_\chi,\psi) = \{W_\Phi \in W(\chi\,|\cdot|^{1/2}, \chi\,|\cdot|^{-1/2}, \psi) \mid \int_{F\times F}\Phi(x,y)dxdy = 0\}$.

Note that $\int_{F\times F}\Phi(x,y)dxdy = 0$ if and only if $\int_F \Phi^\sim(x,0)dx = 0$, so the lemma provides two ways of characterizing the Whittaker functions of Sp_χ.

Proof. Recall that by Lemma 3.10.2, Sp_χ is the annihlator of $\chi^{-1}(\det g)$ under the pairing $B(\mu_1,\mu_2)\times B(\mu_1^{-1},\mu_2^{-1}) \to \mathbb{C}$. So $W_\Phi \in W(Sp_\chi,\psi)$ if and only if $\langle f_\Phi, \chi^{-1}(\det\cdot)\rangle = 0$. Since pairing is given by

$$\langle f_1,f_2\rangle := \int_{\mathrm{GL}_2(\mathcal{O}_F)} f_1(k)f_2(k)dk = c\int_F f_1f_2(w\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right))dx,$$

we have

$$\begin{aligned}
\langle f_\Phi, \chi^{-1}(\det g)\rangle &= \int_F f_\Phi(w \left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right))\chi^{-1}(\det(w \left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right)))dx \\
&= \int_F \int_{F^\times} \mu_1\mu_2^{-1}(t)\,|t|\,\Phi[(0,t)w \left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right)]d^\times t dx \\
&= \int_{F\times F^\times} \mu_1\mu_2^{-1}(t)\,|t|\,\Phi(-t,-tx)d^\times t dx \\
&= \int_F \int_{F^\times} |t|^2\,\Phi(-t,-tx)d^\times t dx \\
&= \int_F \int_{F^\times} |t|\,\Phi(-t,-x)d^\times t dx \\
&= \int_{F\times F^\times} \Phi(-t,-x)dt dx \\
&= \int_{F\times F} \Phi(t,x)dt dx.
\end{aligned}$$

Note that we used $|t|\,d^\times t = dt$. □

Proof of Theorem 3.14.7 for special representations. Now that we have identified the Whittaker functions we are in good shape. We first show that $L(s, Sp_\chi) = L(s,\mu_1) = L(s,\chi\,|\cdot|^{1/2}) = L(s+\frac{1}{2},\chi)$.

Notice that

$$\begin{aligned}
\Psi(e,s,W) =& Z(\Phi^\sim, \mu_1\,|\cdot|^s, \mu_2\,|\cdot|^s) \\
&= \int_{F^\times\times F^\times} \Phi^\sim(x,y)\chi(x)\,|x|^{s+1/2}\,\chi(y)\,|y|^{s-1/2}\,d^\times x d^\times y
\end{aligned}$$

is always holomorphic when χ is ramified. (See Lemma 3.9.6). So, if χ is ramified $L(s, Sp_\chi) = 1 = L(s+1/2,\chi)$.

Now we assume that χ is unramified and $\int_F \Phi^\sim(x,0)dx = 0$.

There exists $\mathfrak{p}^n$ such that $\Phi^\sim(x,y) = \Phi^\sim(x,0)$ for $y\in\mathfrak{p}^n$. Then

$$\begin{aligned}
\Psi(e,s,W) = &\int_{F^\times}\int_{v(y)<n} \Phi^\sim(x,y)\chi(x)\,|x|^{1/2+s}\,\chi(y)\,|y|^{-1/2+s}\,d^\times x d^\times y \\
&+ \int_{F^\times}\int_{\mathfrak{p}^n} \Phi^\sim(x,0)\chi(x)\,|x|^{1/2+s}\,\chi(y)\,|y|^{-1/2+s}\,d^\times x d^\times y.
\end{aligned}$$

The first integral is $L(s+1/2,\chi)$ times a holomorphic function, so write the second as

$$\int_{F^\times} \Phi^\sim(x,0)\chi(x)\,|x|^{1/2+s}\,d^\times x \int_{\mathfrak{p}^n} \chi(y)\,|y|^{s-1/2}\,d^\times y = A\cdot B.$$

Now B has a (simple) pole precisely when $\chi\,|\cdot|^{s-1/2} = 1$. At this s,

$$A = \int_{F^\times} \Phi^\sim(x,0)\,|x|\,d^\times x = \int_F \Phi^\sim(x,0)dx = 0.$$

So the product AB is always holomorphic. Hence $L(s, Sp_\chi) = L(s+1/2,\chi)$.

To complete the proof, we must show that there exists an exponential function $\epsilon(s, Sp_\chi, \psi)$ such that

$$\frac{\widetilde{\Psi}(w, 1-s, W)}{L(1-s, \widetilde{Sp_\chi})} = \epsilon(s, Sp_\chi, \psi)\frac{\Psi(e, s, W)}{L(s, Sp_\chi)}.$$

Recall $\widetilde{Sp_\chi} = Sp_{\chi^{-1}}$, so $L(s, \widetilde{Sp_\chi}) = L(s+1/2, \chi^{-1})$.

In the course of proving the theorem in the case of principal series representations, we showed that

$$\frac{\widetilde{\Psi}(w, 1-s, W)}{L(1-s, \mu_1^{-1})L(1-s, \mu_2^{-1})} = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi)\frac{\Psi(e, s, W)}{L(s, \mu_1)L(s, \mu_2)}$$

for any $W \in W(\mu_1, \mu_2; \psi)$. In this case, we only need to worry about those W satisfying Lemma 3.14.12. This will allow us to move some unnecessary L-factors into the ϵ-factor.

Since $L(s, Sp_\chi) = L(s, \mu_1)$ and $L(s, Sp_{\chi^{-1}}) = L(s, \mu_2^{-1})$, defining

$$\epsilon(s, Sp_\chi, \psi) := \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi)\frac{L(1-s, \mu_1^{-1})}{L(s, \mu_2)}$$

gives the correct functional equation.

To complete the proof, we need only verify that this choice of ϵ-factor is exponential. But we know that if χ is ramified, $\frac{L(1-s,\mu_1^{-1})}{L(s,\mu_2)} = 1$. Otherwise,

$$\frac{L(1-s, \mu_1^{-1})}{L(s, \mu_2)} = \frac{L(1/2-s, \chi^{-1})}{L(s-1/2, \chi)} = \frac{1-\chi^{-1}(\varpi)\,|\varpi|^{1/2-s}}{1-\chi(\varpi)\,|\varpi|^{s-1/2}} = -\chi^{-1}(\varpi)\,|\varpi|^{1/2-s}.$$

So $\epsilon(s, Sp_\chi, \psi)$ is an exponential function. $\square$

We finish this section with an exercise relating the ϵ-factors for different choices of characters ψ.

Exercise 3.14.13. If $\psi : F \to \mathbb{C}^\times$ is a nontrivial character, then it is a fact that every other character of F is of the form $\psi'(x) = \psi_b(x) = \psi(bx)$ for some $b \in F$. Use this to prove that

(1) $W(\pi, \psi') = \{W(\left(\begin{smallmatrix} b & \\ & 1\end{smallmatrix}\right) g) \mid W \in W(\pi, \psi)\}$.
(2) $\epsilon(s, \pi, \psi') = \omega(b)\,|b|^{2s-1}\,\epsilon(s, \pi, \psi)$.
(3) $\epsilon(s, \pi, \psi)\epsilon(1-s, \widetilde{\pi}, \psi) = \omega(-1)$.

Solution. (1) Define the map

$$\begin{array}{rcl} W(\pi, \psi) & \longrightarrow & W(\pi, \psi') \\ W(\cdot) & \mapsto & W(\left(\begin{smallmatrix} b & \\ & 1\end{smallmatrix}\right)\cdot) \end{array}$$

We calculate directly that W', the image of W under this map, satisfies

$$\begin{aligned} W'(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) &= W(\begin{pmatrix} b & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) \\ &= W(\begin{pmatrix} 1 & bx \\ & 1 \end{pmatrix}\begin{pmatrix} b & \\ & 1 \end{pmatrix} g) \\ &= \psi(bx)W(\begin{pmatrix} b & \\ & 1 \end{pmatrix} g) = \psi'(x)W'(g). \end{aligned}$$

So the map is well defined. Obviously, the right regular action is the same on both sides, so, by irreducibility, the map must be an isomorphism.

(2) Let W' and W be as above. (Note that in order for ψ' to be nontrivial, b must be invertible.) Then

$$\begin{aligned}\Psi(g,s,W') &= \int_{F^\times} W(\begin{pmatrix} b & \\ & 1\end{pmatrix}\begin{pmatrix} a & \\ & 1\end{pmatrix} g)\,|a|^{s-\frac{1}{2}}\,d^\times a\\ &= \int_{F^\times} W(\begin{pmatrix} a & \\ & 1\end{pmatrix} g)\,\left|ab^{-1}\right|^{s-\frac{1}{2}}\,d^\times a\\ &= |b|^{\frac{1}{2}-s}\,\Psi(g,s,W).\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Psi}(g,s,W') &= \int_{F^\times} W(\begin{pmatrix} b & \\ & 1\end{pmatrix}\begin{pmatrix} a & \\ & 1\end{pmatrix} g)\,|a|^{s-\frac{1}{2}}\,\omega^{-1}(a)d^\times a\\ &= \int_{F^\times} W(\begin{pmatrix} a & \\ & 1\end{pmatrix} g)\,\left|ab^{-1}\right|^{s-\frac{1}{2}}\,d^\times\omega^{-1}(ab^{-1})a\\ &= \omega(b)\,|b|^{\frac{1}{2}-s}\,\widetilde{\Psi}(g,s,W).\end{aligned}$$

Plugging this into the functional equation for ψ', we have

$$\omega(b)\,|b|^{s-\frac{1}{2}}\,\widetilde{\Phi}(wg,1-s,W') = \epsilon(s,\pi,\psi')\Phi(wg,s,W') = |b|^{\frac{1}{2}-s}\,\epsilon(s,\pi,\psi')\Phi(g,s,W).$$

From the functional equation for ψ this immediately implies that

$$\epsilon(s,\pi,\psi) = \omega(b)^{-1}\,|b|^{1-2s}\,\epsilon(s,\pi,\psi').$$

(3) Recall that Proposition 3.14.2 says that the Whittaker model of $\widetilde{\pi}$ is given by

$$W(\widetilde{\pi},\psi) = \left\{W(g)\omega^{-1}(\det g) \mid W\in W(\pi,\psi)\right\}.$$

Write $\widetilde{W}(\cdot) = \omega(\det\cdot)W(\cdot)$. Note that

$$\begin{aligned}\widetilde{\Psi}(g,s,W) &= \int_{F^\times} W(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g)\omega^{-1}(a)\,|a|^{s-\frac{1}{2}}\,d^\times a\\ &= \omega(\det g)\int_{F^\times} W(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g)\omega^{-1}(a\det g)\,|a|^{s-\frac{1}{2}}\,d^\times a\\ &= \omega(\det g)\int_{F^\times} \widetilde{W}(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g)\,|a|^{s-\frac{1}{2}}\,d^\times a\\ &= \omega(\det g)\Psi(g,s,W).\end{aligned}$$

Since the central character of $\widetilde{\pi}$ is ω^{-1}, we also have that

$$\begin{aligned}\widetilde{\Psi}(g,s,\widetilde{W}) &= \int_{F^\times} \widetilde{W}(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g)\omega(a)\,|a|^{s-\frac{1}{2}}\,d^\times a\\ &= \int_{F^\times} \widetilde{W}(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g)\omega(a)\,|a|^{s-\frac{1}{2}}\,d^\times a\\ &= \omega^{-1}(\det g)\Psi(g,s,W).\end{aligned}$$

The functional equations for π and $\widetilde{\pi}$ give

$$\epsilon(s,\pi,\psi) = \frac{L(s,\pi)}{L(1-s,\widetilde{\pi})}\cdot\frac{\widetilde{\Psi}(w,1-s,W)}{\Psi(e,s,W)}$$

and

$$\epsilon(1-s,\widetilde{\pi},\psi) = \frac{L(1-s,\widetilde{\pi})}{L(s,\pi)} \cdot \frac{\widetilde{\Psi}(-e,s,\widetilde{W})}{\Psi(w,1-s,W)}$$

respectively. Combining these equations and the above gives the result. □

4. Classification of local representations: F archimedean

Harish-Chandra is the mathematician most responsible for developing the theory of representations of Lie groups. His original idea was the notion of a $(\mathfrak{g}, K)$-module, which is a completely algebraic object, but, as we will see, can be used to classify unitary representations of $\mathrm{GL}_2(\mathbb{R})$. The notion closely resembles the nonarchimedean theory.

4.1. **Two easier problems.**

To aid us in the classification of Lie groups, we consider two easier problems. First, we look at the linearization of our action. This corresponds to looking at the action of a finite dimensional vector space, the Lie algebra whose modules are much easier to classify. Second, we restrict our attention to a maximal compact subgroup. Again, it is much easier to classify the representations of such groups.

4.1.1. *Some background on Lie groups.* Let G be a Lie group. (A group in the category of differentiable real manifolds.) In this section we discuss (briefly!) modules of its Lie algebra, $Lie(G)$, the tangent space to the identity. It is a fact that there is an embedding $\rho : G \to \mathrm{GL}_n(\mathbb{R})$ for some n. So we can think of G as a subgroup/manifold of GL_n. The tangent space of $\mathrm{GL}_n(\mathbb{R})$, denoted $\mathfrak{gl}_n$, consists of all $n \times n$ matrices with real coefficients. There is an exponential map $\exp : \mathfrak{gl}_n \to \mathrm{GL}_n$ given by

$$\exp(X) = e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!},$$

and $X \in Lie(G)$ if and only if $e^X \in G$. Also, $\mathfrak{gl}_n$ acts on smooth functions $f : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}$ via

$$Xf(g) = \left.\frac{d}{dt}\right|_{t=0} f(ge^{tX}).$$

Suppose G acts smoothly on V, a finite dimensional vector space over $\mathbb{C}$. Then there is an induced action of $Lie(G)$ on V. For $X \in Lie(G)$, $X \cdot v = \frac{d}{dt}\mid_{t=0} \pi(e^{tx})v$. This definition agrees with how, whenever $\varphi : M \to N$ is a map of manifolds, there is linear map between the tangent space at any point and that of its image. In our case, $\{e^{tX}\}$ is a line in G with tangent vector X at the identity. Then $\{e^{tX}v\}$ is the image of the line in V, and $X \cdot v$ is the induced linear map.

4.1.2. *Background on representation theory of compact groups.* Let K be a compact group, $\hat{K}$ the collection of equivalence classes of finite dimensional continuous representations of K. An element of $\hat{K}$ will be called a *K-type*. For a K-type (W_γ, γ) where γ is the action and W_γ the underlying space, consider the γ-isotopic component $V(\gamma) = \sum_{\varphi \in \mathrm{Hom}_K(W_\gamma, V)} \varphi(W_\gamma)$ of V. We record the following facts:

(1) If π is a continuous action of K on a Hilbert space then one can change the pairing so that the topology of H is unchanged and the action of K is unitary. To do this, let $\langle u, v\rangle_{new} := \int_K \langle \pi(k)u, \pi(k)v\rangle_{old} dk$. We can assume then that *all* continuous K-modules are unitary.

(2) Matrix coefficients of finite dimensional unitary representations of K span a dense subspace of $L^p(K)$, $p \geq 1$. This is the Peter-Wyel theorem for $p = 1$.
(3) All irreducible unitary representations of K are finite dimensional.
(4) A unitary representation of K on a Hilbert space V is the Hilbert space direct sum of finite dimensional irreducible subrepresentations of K . In other words, $V = \overline{\oplus_{\gamma \in \hat{K}} V(\gamma)}$. The bar denotes topological closure. So given a V how do we pick out the γ-component?
(5) Let σ_i $(i = 1, \cdots, n)$ be inequivalent finite dimensional unitary representations of K then define the elementary idempotent

$$\xi(k) := \sum_{i=1}^{n} \dim(\sigma_i) \operatorname{tr}(\sigma_i(k)^{-1}). \tag{4.1.1}$$

Then $\xi * \xi = \xi$. If $\gamma \in \hat{K}$, let $\xi_\gamma = \dim(\gamma) \operatorname{tr}(\gamma(\cdot)^{-1})$. If $v \in V$, $v = \sum v_\delta$ with $v_\delta \in V(\delta)$, and $v_\delta = \pi(\xi_\delta)v$, where

$$\pi(f)v = \int_K f(k)\pi(k)v dk.$$

A readable reference for this is Bump's book [2].

4.2. **From G-modules to $(\mathfrak{g}, K)$-modules.**

4.2.1. $(\mathfrak{g}, K)$*-modules.* In the previous sections, we discussed $\mathfrak{g}$-modules ($\mathfrak{g}$ a Lie algebra) and K-modules (K a compact group.) Harish-Chandra's idea was to classify G-modules by first considering $(\mathfrak{g}, K)$*-modules* where $K \subset G$ is a maximal compact subgroup and $\mathfrak{g} = Lie(G)$. We now discuss this notion for $G = \mathrm{GL}_2(\mathbb{R})$, and $\mathfrak{g} = \mathfrak{g}l_{2,\mathbb{R}}$ its Lie algebra. $K = O(2, \mathbb{R})$ is a maximal compact subgroup of G (all others being conjugate to it.) Note that both G and K have two connected components: $\mathrm{GL}_2^+(\mathbb{R})$ and $SO(2, \mathbb{R})$ are the identity components of each.

Remark. To prove that K is maximal use Iwasawa decomposition $G = PK$. If $K' \supset K$ is a group then there exists $\left(\begin{smallmatrix} a_1 & x \\ & a_2 \end{smallmatrix}\right) \in K'$. We may assume $a_1, a_2 > 0$. If $a_1 a_2 \neq 1$ by taking powers of this element K' is not compact. If $a_1, a_2 = 1$ and $x \neq 0$ we may again take powers to see that $\left(\begin{smallmatrix} 1 & xn \\ & 1 \end{smallmatrix}\right) \in K'$ and so it is not compact. So K' compact implies the only parabolic element in it is the identity. Hence $K' = K$

Definition 4.2.1. *A* $(\mathfrak{g}, K)$-module *is a complex vector space V together with an action π of $\mathfrak{g}$ and K such that*

(1) *every vector is* K-finite. *That is, the space spanned by $\{\pi(k)v : k \in K\}$ is finite dimensional, and the action of K on any finite dimensional invariant subspace of V is continuous.*
(2) *For $X \in Lie(K) = \mathfrak{s}o(2, \mathbb{R})$, $\pi(X)v = \frac{d}{dt}\big|_{t=0} \pi(e^{tX})v$. (Notice that $\pi(e^{tx})v$ lives in a finite dimensional space, so we are just doing calculus.)*
(3) *For $k \in K$, $Y \in \mathfrak{g}$, $\pi(\mathrm{Ad}(k)Y)v = \pi(k)\pi(Y)\pi^{-1}(k)v$. ($\mathrm{Ad}(k)Y = kYk^{-1}$.)*

Remark. Items (2) and (3) are natural compatibility relations arising between the Lie algebra and group actions.

Also, the third condition, for $k \in SO(2, \mathbb{R})$ follows from the first condition. The new requirement is about the action of $\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$.

Suppose that V is a continuous G-module. As our real goal is to understand G-modules, we begin with some definitions of the types of G-modules that we will be considering. By the facts of the previous section, under the action of K,

$$V = \overline{\bigoplus_{\gamma \in \hat{K}} V(\gamma)}.$$

Moreover, each isotypic component $V(\gamma)$ is itself a direct sum of irreducible finite dimensional K-modules.

Definition 4.2.2. *Let π be a continuous action of G on a Hilbert space V. V is called* (topologically) irreducible *if there is no nontrivial G-invariant closed subspace. $V = \overline{\oplus_{\gamma \in \hat{K}} V(\gamma)}$. We say that V is* admissible *if all $V(\gamma)$ are finite dimensional.*

Proposition 4.2.3. *If (π, V) is an irreducible unitary representation of G on a Hilbert space V then it's admissible.*

So, in order to understand unitary G-modules, we first try to understand admissible modules.

Definition 4.2.4. *Let (π, V) be a continuous action of G on a Hilbert*[13] *space V. A vector $v \in V$ is* smooth *if the map $g \to \pi(g)v$ is infinitely differentiable.*

Lemma 4.2.5. *For $f \in C_c^\infty(G)$ we can define an action $\pi(f)v := \int_G f(g)\pi(g)v dg$. (The integral being in the sense of Riemann.) Then $\pi(f)v$ is a smooth vector for any $f \in C_c^\infty$, $v \in V$. In particular, V^∞, the set of smooth vectors in V, is dense in V.*

Proof. Choose a sequence of (positive) functions $f_n \in C_c^\infty$ such that the support shrinks to the identity $\{e\}$ and $\int_G f_n(g)dg = 1$. So $\pi(f_n)v \to v$ as $n \to \infty$. Then

$$\begin{aligned} |\pi(f_n)v - v| &= \left| \int_G f_n(g)\pi(g)v dg - \int_G f_n(g) v dg \right| \\ &= |f_n(g)(\pi(g)v - v)dg| \\ &\le \max_{g \in \mathrm{supp}(f_n)} |\pi(g)v - v|. \end{aligned}$$

Since the action is continuous, the right side approaches zero as $n \to \infty$. ☐

Lemma 4.2.6. *Let (π, V) be a continuous representation of G, V a Hilbert space, V^∞ the set of smooth vectors of V. Then the following hold.*

(i) *For $X \in \mathfrak{g}$, can define*

$$\pi(X)v = \left.\frac{d}{dt}\right|_{t=0} (\pi(e^{tX})v) = \lim_{t \to 0} \frac{\pi(e^{tX})v - v}{t}$$

whenever $v \in V^\infty$.

(ii) *V^∞ is G-invariant.*

Definition 4.2.7. *A vector $v \in V$ is K-*finite *if $span\{\pi(k)v \mid k \in K\}$ is finite dimensional. V^f will denote the set of all K-finite vectors of V.*

Lemma 4.2.8. *If $v \in V^\infty \cap V^f$ then, for $X \in \mathfrak{g}$, $\pi(X)v \in V^f$.*

[13] One could actually consider V as a Frechet space, but for our purposes Hilbert space is sufficient.

Remark. In general, if v is K-finite $\pi(g)v$ need not be.

Proof. Consider

$$\begin{aligned}\pi(k)[\pi(X)v] &= \pi(k)\lim_{t\to 0}\frac{\pi(e^{tX})v-v}{t}\\ &= \lim_{t\to 0}\frac{\pi(k)\pi(e^{tX})v-\pi(k)v}{t}\\ &= \lim_{t\to 0}\frac{\pi(ke^{tX}k^{-1})\pi(k)v-\pi(k)v}{t}\\ &= \lim_{t\to 0}\frac{\pi(e^{t\operatorname{Ad}_k X})\pi(k)v-\pi(k)v}{t}\\ &= \pi(\operatorname{Ad}_k X)\pi(k)v\in\pi(\operatorname{Ad}_k X)span\{\pi(k)v\}.\end{aligned}$$

Since v is K-finite, $span\{\pi(k)v\}$ is finite dimensional, and so the action of $\operatorname{Ad}_k X$ on it must also give a finite dimensional space. □

The main corollary of this result is that $V^\infty\cap V^f$ is a $(\mathfrak{g},K)$-module.

The next result relates K-finiteness and smoothness.

Lemma 4.2.9. *Let V be a Hilbert space that is a continuous G-module. Let $\delta\in\widehat{K}$ and $V(\delta)$ the δ-isotypic component of δ in V. If $V(\delta)$ is finite dimensional then $V(\delta)\subseteq V^\infty$.*

Proof. Let $v\in V(\delta)$. We need to show that v is smooth. By Lemma 4.2.5 if $f\in C_c^\infty$ and $w\in V$, $\pi(f)w$ is always smooth. Also, $v=\lim_{n\to\infty}\pi(f_n)v$ where the support of f_n decreases to the identity and $\int_G f_n(x)dx=1$. Let $\xi=\dim(\delta)\operatorname{tr}(\delta^{-1})$, the elementary idempotent associated to δ. Since $v\in V(\delta)$, we have $v=\pi(\xi)v$. Also, $\pi(\xi*f_n*\xi)v\in V(\delta)$. Since $V(\delta)$ is finite dimensional, there exists $n_1,\dots,n_l$ such that $\pi(\xi*f_{n_i}*\xi)v$ is a basis of $W=span\{\pi(\xi*f_n*\xi)v\}$. Moreover, $v\in W$. So

$$v=\sum_{i=1}^{l}c_i\pi(\xi*f_{n_i}*\xi)v=\pi(f)v$$

where $f=\sum_{i=1}^{l}c_i\xi*f_{n_i}*\xi\in C_c^\infty$. By our remark above, this completes the proof. □

Suppose (π,V) is a continuous G-module with V a Hilbert space. Then Lemmas 4.2.8 and 4.2.9 imply that $V^f=V^\infty\cap V^f$ is an admissible $(\mathfrak{g},K)$-module. The next two propositions say that when π is unitary and irreducible this association is faithful.

Theorem 4.2.10. *If (π,V) is an admissible continuous G-module then it is topologically irreducible if and only if V^f is an irreducible admissible $(\mathfrak{g},K)$-module.*

Proof. Suppose V^f is an irreducible admissible $(\mathfrak{g},K)$-module. Assuming that V is not irreducible as a G-module, there exists $0\subsetneq W\subsetneq V$ fixed by G. Clearly $W^f\subseteq V^f$ and W^f is dense in W. Since V^f is dense in V this implies $W^f\neq V^f$, and W^f is a nontrivial $(\mathfrak{g},K)$-module, a contradiction.

Now suppose that V is topologically irreducible. Let $0\subsetneq W\subsetneq V^f$ be a $(\mathfrak{g},K)$-submodule. We claim that (1) $\overline{W}$ is G-invariant; (2) $\overline{W}\subsetneq V$. This would give a contradiction and complete the proof.

We prove (2) first. $W \subsetneq V^f$ implies that there exists $\delta \in \widehat{K}$ such that $W(\delta) \subsetneq V(\delta)$. (Note that both of these are finite dimensional because V is admissible.) Choose $v \in V(\delta) \setminus W(\delta)$. Then $v \notin \overline{W}$ because, if it were, there would exist $w_n \in W$ such that $w_n \to v$ hence $\pi(\xi_\delta)w_n \to \pi(\xi_\delta)v = v$, and this would imply that $v \in W(\delta)$ since $\pi(\xi_\delta)w_n \in W(\delta)$, and $W(\delta)$ is closed (since it's finite dimensional.)

To prove (1), we need to show that for $v \in \overline{W}$, $\pi(g)v \in \overline{W}$ for all $g \in G$. Choose $v_n \in W$ such that $v_n \to v$. Hence $\pi(g)v_n \to \pi(g)v$. It will suffice to show that

$$(*) \qquad \text{if } u \in W, \pi(g)u \in \overline{W}.$$

To do this, we prove that if $u \in W$ and $u' \in U^\perp$ then $\pi(g)u \perp u'$.

Also, note that (*) is true for $g \in K$, so it suffices to check it for elements of the type $\left(\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right)$ $(x \in \mathbb{R}^+)$ and $\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$. In both cases $g = e^X$ for some $X \in \mathfrak{g}$. It will suffice to show that $\pi(g)W \subset \overline{W}$ because given any $v \in \overline{W}$ there exist $w_n \in W$ with $w_n \to v$, and if $\pi(g)w_n \in \overline{W}$ then $\pi(g)v = \lim_{n\to\infty} \pi(g)w_n \in \overline{W}$.

We claim that if $v \in V^f$ then, when X is sufficiently small,

$$\pi(e^X)v = \sum_{n=0}^{\infty} \frac{\pi(X)^n v}{n!}.$$

Assuming this, for $w \in W$, $\pi(e^X)w = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\pi(X)^k w}{k!}$ when X is small. Since the term inside of the limit is in W the result follows. So Proposition 4.2.14 and Exercise 4.2.11 □

Exercise 4.2.11. Show further that W is invariant for all G. In other words, even if $g = e^X$ for X 'too big,' show that $\pi(g)W \subset \overline{W}$.

Definition 4.2.12. *A vector $v \in V$ is* smooth *if $g \mapsto \pi(v)g$ is a smooth function. Similarly, $v \in V$ is* analytic *if $g \mapsto \pi(v)g$ is an analytic function.*

Example 4.2.13. The function $f(t) = e^{-t^{-2}}$ satisfies $f^{(n)}(0) = 0$, but it is not constant. In other words, $f(t)$ does not equal its Taylor series in any neighborhood of 0.

When Harish Chandra first began his study of representations of Lie groups, this difference between smooth and analytic vectors occupied a lot of his work. For us, the following propositions make things easy, and allow one to complete the proof above.

Proposition 4.2.14. *When (π, V) is an irreducible continuous admissible G-module, every vector in V^f is analytic.*

We remark that irreducibility is crucial in the Proposition 4.2.14.

Proposition 4.2.15. *If V is analytic, then there exists a neighborhood $U \subset \mathfrak{g}$ of zero such that when $X \in U$, $\pi(e^X)v = \sum_{n=0}^{\infty} \frac{\pi(X)^n v}{n!}$.*

We skip the proofs. See Harish-Chandra[4] for details. (The paper is pretty readable.)

We conclude that if (π, V) is an irreducible admissible G-module then V^f is an admissible irreducible $(\mathfrak{g}, K)$-module. Moreover, if $G = \mathrm{GL}_2(\mathbb{R})$ then we have the following.

Proposition 4.2.16. *Given (π_1, V_1) and (π_2, V_2) irreducible $\mathrm{GL}_2(\mathbb{R})$-modules as above. Then $V_1 \simeq V_2$ if and only if $V_1^f \simeq V_2^f$ as $(\mathfrak{g}, K)$-modules.*

This proposition is not true for a general Lie group G. The proof is left to the reader.

To recap, we have shown that the classification of irreducible admissible G-modules can be accomplished by, first, classifying all irreducible admissible $(\mathfrak{g}, K)$-modules, and second, determining which give rise to a unitary G-module.

4.3. **An approach to studying $(\mathfrak{gl}_2, O(2,\mathbb{R}))$-modules.** In the previous section, one could have taken (except in Proposition 4.2.16) G to be any real Lie group, $\mathfrak{g}$ its Lie algebra and K a maximal compact subgroup of G. In this section we specialize to the specific situation

$$G = \mathrm{GL}_2(\mathbb{R}), \mathfrak{g} = \mathfrak{g}l_2, K = O(2,\mathbb{R}) = SO(2,\mathbb{R}) \times \{e, \epsilon = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)\}.$$

Recall that a $(\mathfrak{g}, K)$-module V is a $\mathfrak{g}$-module with an action of ϵ such that

- $\pi(\epsilon)\pi(X)\pi(\epsilon) = \pi(\mathrm{Ad}_\epsilon(X))$, and
- the action of $\mathfrak{s}o(2,\mathbb{R})$ on V gives a decomposition of V as a direct sum of spaces κ_n, $n \in \mathbb{Z}$ where κ_n is the action of $\mathfrak{s}o(2,\mathbb{R})$ induced from the irreducible representations of $SO(2,\mathbb{R})$.

(The irreducible representations of $SO(2,\mathbb{R})$ are given by $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{in\theta}$.)

The first item is a compatibility relation between the adjoint action of the group on its Lie algebra and the action on V. The second item is necessary because $\mathfrak{s}o(2,\mathbb{R}) \simeq \mathbb{R}$ so its representations are given by e^{ix} for arbitrary $x \in \mathbb{R}$, but we want only those that agree with a representation of $SO(2,\mathbb{R})$ after exponentiating.

Let $\mathcal{U}\mathfrak{g}_\mathbb{C}$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$.

Definition 4.3.1. *An* admissible *$\mathcal{U}\mathfrak{g}_\mathbb{C}$-module is a $\mathcal{U}\mathfrak{g}_\mathbb{C}$-module such that the action of $\mathfrak{s}o(2,\mathbb{R})$ gives direct sum of κ_n's each occuring finitely many times.*

Hence, we have the following correspondence.

$$\left\{ \begin{array}{c} \text{Admissible} \\ \mathcal{U}\mathfrak{g}_\mathbb{C}\text{-modules such that} \\ \pi(\epsilon)\pi(X)\pi(\epsilon) = \pi(\mathrm{Ad}_\epsilon X) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Admissible} \\ (\mathfrak{g}, K)\text{-modules} \end{array} \right\}$$

We call an element of the left category a $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$*-module.*

The next lemmas discuss restricting a $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$-module to a $\mathcal{U}\mathfrak{g}_\mathbb{C}$-module and inducing an action of ϵ on a $\mathcal{U}\mathfrak{g}_\mathbb{C}$-module to make it a $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$-module.

Lemma 4.3.2. *Let (σ, V) be an irreducible $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$-module. Then exactly one of the following holds:*

(i) *As a $\mathcal{U}\mathfrak{g}_\mathbb{C}$-module V is irreducible and $\sigma|_{\mathcal{U}\mathfrak{g}_\mathbb{C}} \simeq \sigma|_{\mathcal{U}\mathfrak{g}_\mathbb{C}} \circ \mathrm{Ad}_\epsilon$.*
(ii) *As a $\mathcal{U}\mathfrak{g}_\mathbb{C}$-module $V \simeq \sigma|_{\mathcal{U}\mathfrak{g}_\mathbb{C}} \oplus \sigma|_{\mathcal{U}\mathfrak{g}_\mathbb{C}} \circ \mathrm{Ad}_\epsilon$ and $\sigma_{\mathcal{U}\mathfrak{g}_\mathbb{C}} \not\simeq \sigma|_{\mathcal{U}\mathfrak{g}_\mathbb{C}} \circ \mathrm{Ad}_\epsilon$.*

Lemma 4.3.3. *Let (π_1, V_1) be an irreducible $\mathcal{U}\mathfrak{g}_\mathbb{C}$-module. Then*

(i) *If $\pi_1 \simeq \pi_1 \circ \mathrm{Ad}_\epsilon$, one can define an action of ϵ so that V_1 is an irreducible $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$-module.*
(ii) *If $\pi_1 \not\simeq \pi_1 \circ \mathrm{Ad}_\epsilon$ then it induces an irreducible $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$-module $V = V_1 \oplus V_1$ such that $\pi|_{\mathcal{U}\mathfrak{g}_\mathbb{C}} = \pi_1 \oplus \pi_1 \circ \mathrm{Ad}_\epsilon$ and $\pi(\epsilon)(v_1 \oplus v_2) = v_2 \oplus v_1$.*

Remark. If (σ, W) is an irreducible admissible $\mathcal{U}\mathfrak{g}_\mathbb{C}$-module then σ and $\sigma \circ \mathrm{Ad}_\epsilon$ induce the same representation. So in the induction map is either one-to-one or two-to-one depending on whether σ is equivalent to $\sigma \circ \mathrm{Ad}_\epsilon$ or not.

Similarly, if (π, V) is a $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$-module then $\pi|_{\mathcal{U}\mathfrak{g}_\mathbb{C}}$ is either irreducible or isomorphic to $\pi_1 \oplus \pi_2$. So the restriction map is either one-to-one or one-to-two.

Remark. This property of restriction and induction is essentially the same as what occurs in the theory of finite groups. A good exercise would be to work out the relation in the case of H an index two subgroup of a finite group G. (This will perhaps be one of the topics of the appendix.)

4.4. **Composition series of induced representations.** This section is very similar to the discussion of induced representations of $\mathrm{GL}_2(F)$ when F is a nonarchimedean field. In particular we will study

$$B(\mu_1,\mu_2) := \left\{ f : GL_2(\mathbb{R}) \to \mathbb{C} \;\middle|\; \begin{array}{c} f\left(\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g) \\ f \text{ is } K\text{-finite on the right} \end{array} \right\}$$

(Note: These conditions force $B(\mu_1,\mu_2)$ to consist of continuous functions.)

Under the right regular action ρ of G, $B(\mu_1,\mu_2)$ is admissible. In this section we determine the composition series of $B(\mu_1,\mu_2)$. The following theorem of Harish-Chandra (not proven) tells us that we don't need to look further for irreducible $(\mathfrak{g},K)$-modules.[14]

Theorem 4.4.1. *Any admissible irreducible $\mathcal{U}\mathfrak{g}_{\mathbb{C}}$-module is a subquotient of $B(\mu_1,\mu_2)$ for some choice of μ_1 and μ_2.*

We will do the following:

(1) determine all $B(\mu_1,\mu_2)$ $\mathcal{U}\mathfrak{g}_{\mathbb{C}}$-submodules and subquotients, (i.e. find the composition series such that each successive quotient is irreducible)
(2) glue the components together to make $(\mathfrak{g},K)$-modules

First, $B(\mu_1,\mu_2)$ has a basis consisting of functions $\{\varphi_n \mid n \in \mathbb{Z}\}$ where

$$\varphi_n\left(\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)\right) = e^{in\theta}.$$

Let $\mathfrak{g} = \mathfrak{g}l_2(\mathbb{R})$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and consider the elements

$$\begin{aligned} J &= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, X_+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_- \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ U &= \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, V_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, V_- \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}. \end{aligned} \tag{4.4.1}$$

Also, let $\Omega = X_+X_- + X_-X_+ + \frac{H^2}{2}$, the Casismir element. Notice that $\{X_+, X_-, H, J\}$ is a basis for $\mathfrak{g}$ (and of $\mathfrak{g}_{\mathbb{C}}$), and that $\{V_+, V_-, U, J\}$ is a basis for $\mathfrak{g}_{\mathbb{C}}$. The center of $\mathcal{U}\mathfrak{g}_{\mathbb{C}}$ is $Z(\mathcal{U}\mathfrak{g}_{\mathbb{C}}) = \mathbb{C}[J,\Omega]$.

Let $[\cdot,\cdot]$ denote the Lie bracket on $\mathfrak{g}$. A direct computation reveals that

$$[U, V_\pm] = \pm 2iV_\pm, \quad [H, X_\pm] = \pm 2X_\pm, \quad [V_+, V_-] = -4iU, \quad [X_+, X_-] = H.$$

Remark. Whenever $\mathfrak{g}$ acts on a finite dimensional space V, V decomposes as a direct sum of H-weight spaces (eigenspaces.) $X \in \mathfrak{g}$ acts on $\mathfrak{g}$ via $\mathrm{ad}_X = [X,\cdot]$. The above identities reflect the decomposition of $\mathfrak{g} = \mathbb{R}X_- \bigoplus (\mathbb{R}H \oplus \mathbb{R}J) \bigoplus \mathbb{R}X_+$ where $\mathbb{R}X_-$ is the weight -2 subspace, $\mathbb{R}H \oplus \mathbb{R}J$ the weight 0 subspace and $\mathbb{R}X_+$ the weight 2 subspace.

In this decomposition, $\mathbb{R}H$ is called the *Cartan subalgebra*. Choosing U instead of H (now in the comlexified algebra $\mathfrak{g}_{\mathbb{C}}$) the above identities give a similar decomposition of $\mathfrak{g}_{\mathbb{C}}$. This second choice is better for our purposes because $SO_2(\mathbb{R})$ is

[14] This is in contrast to the nonarchimedean theory where there are representations not appearing as a subquotient/submodule of $B(\mu_1,\mu_2)$, i.e. the supercuspidals.

obtained by exponentiating tU. Since our modules will be $K = O_2(\mathbb{R})$ finite, it is this action that we are interested in.

The following lemma describes the action of all of these elements on $B(\mu_1, \mu_2)$.

Lemma 4.4.2. *Let $\mu_1, \mu_2 : \mathbb{R}^\times \to \mathbb{C}^\times$ be quasicharacters. (Then $\mu_i(t) = |t|^{s_i}\left(\frac{t}{|t|}\right)^{m_i}$ with $m_i \in \{0, 1\}$.) Let $s = s_1 - s_2$ and $m = |m_1 - m_2|$. Then the action of $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$ on $\varphi_n \in B(\mu_1, \mu_2)$ is determined by*

$$\rho(U)\varphi_n = in\varphi_n, \quad \rho(\epsilon)\varphi_n = (-)^{m_1}\varphi_{-n}, \quad \rho(V_\pm)\varphi_n = (s + 1 \pm n)\varphi_{n\pm2},$$

$$\rho(J)\varphi_n = (s_1 + s_2)\varphi_n, \quad \rho(\Omega)\varphi_n = \frac{s^2 - 1}{2}\varphi_n.$$

Proof. We calculate $\rho(U)\varphi_n$. As it must be a scalar multiple of φ_n, and

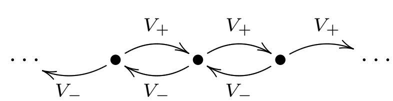
$$\begin{aligned}\rho(U)\varphi_n(e) &= \left.\frac{d}{dt}\right|_{t=0} (\rho(e^{tU})\varphi_n)(e) = \left.\frac{d}{dt}\right|_{t=0} \varphi_n(e^{tU}) \\ &= \left.\frac{d}{dt}\right|_{t=0} \varphi_n(\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}) = \left.\frac{d}{dt}\right|_{t=0} e^{int} = in,\end{aligned}$$

it follows that $\rho(U)\varphi_n = in\varphi_n$. The other cases are similar. We leave them as exercises. $\square$

Remark. $\varphi_n \in B(\mu_1, \mu_2)$ if and only if $n \equiv m \pmod 2$.

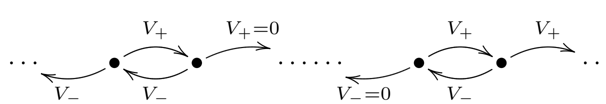
This lemma allows us to describe the irreducible submodules (and subquotients) of $B(\mu_1, \mu_2)$. First, note that $\varphi_n \in B(\mu_1, \mu_2)$ if and only if $n \equiv m \pmod 2$. While U and $Z(\mathcal{U}\mathfrak{g}_\mathbb{C})$ fix each $\mathbb{C}\varphi_n$, the elements V_+ and V_- raise and lower, respectively, the *weight* by two. The only question is whether or not the action of $V_\pm$ has any kernel.

Case I: When $s \notin \mathbb{Z}$ or $s \in \mathbb{Z}$ and $s \equiv m \pmod 2$, the action of V_+ and V_- has no kernel. In this case, $B(\mu_1, \mu_2)$ is irreducible. The following diagram illustrates the situation. Each dot represents $\mathbb{C}\varphi_n$ for some n and the arrows show how the operators $V_\pm$ permute these subspaces. (Note that $Z(\mathcal{U}\mathfrak{g}_\mathbb{C})$ and U act by scalar and hence preserve each dot.)



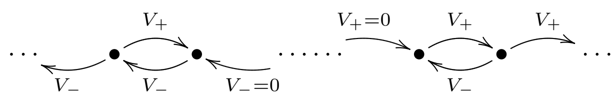
Now expand this action to $\epsilon = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$. We note that $\rho(\mu_1, \mu_2)\,\mathrm{Ad}_\epsilon \simeq \rho(\mu_1, \mu_2)$, so there is a natural action of ϵ on this module.

Case IIa: Suppose $s \in \mathbb{Z}$ and $s \not\equiv m \pmod 2$ and $s \geq 0$. By Lemma 4.4.2, $V_\pm\varphi_{\pm(s+1)} = 0$. So $W_+ = \bigoplus_{n \geq s+1} \mathbb{C}\varphi_n$ and $W_- = \bigoplus_{n \geq -s-1} \mathbb{C}\varphi_n$ are $\mathcal{U}\mathfrak{g}_\mathbb{C}$ invariant subspaces and $B(\mu_1, \mu_2)/W_+ \oplus W_-$ is a finite dimensional irreducible $\mathcal{U}\mathfrak{g}_\mathbb{C}$-module.



The irreducible subquotients are $B(\mu_1, \mu_2)/(B_+ \oplus B_-)$, B_+ and B_-. Notice that B_+ and B_- are ϵ-conjugates of each other. Combining them we get a $(\mathcal{U}\mathfrak{g}_\mathbb{C}, \epsilon)$ submodule. Also $B(\mu_1, \mu_2)/(B_+ \oplus B_-)$ is ϵ conjugate of itself. Recall that $\rho(\mu_1, \mu_2)(\epsilon)\varphi_n = (-1)^{m_1}\varphi_{-n}$.

Case IIb: Suppose $s \in \mathbb{Z}$ and $s \not\equiv m \pmod 2$ and $s < 0$. Now $V_{\mp}\varphi_{\pm(s+1)} = 0$, so $B_f = \bigoplus_{s+1}^{-s-1} \mathbb{C}\varphi_n$ is an irreducible subspace of $B(\mu_1, \mu_2)$ with quotient the sum of two irreducible subspaces.



Here we have the submodules

$$B_+(\mu_1, \mu_2) = \bigoplus_{n \geq s+1} \varphi_n \quad \text{and} \quad B_-(\mu_1, \mu_2) = \bigoplus_{n \leq -s-1} \varphi_n.$$

The intersection $B_f(\mu_1, \mu_2) = B_+ \cap B_-$ provides a third submodule. B_f is irreducible. The irreducible subquotients are B_f and B_+/B_f and B_-/B_f are as well. Additionally, B_+/B_f and B_-/B_f are ϵ-conjugates of each other and B_f is ϵ conjugate of itself.

Putting this together, we have the following description of all irreducible admissible $(\mathcal{U}\mathfrak{g}_{\mathbb{C}}, \epsilon)$-modules (i.e. $(\mathfrak{g}, K)$-modules.)

(1) Irreducible $B(\mu_1, \mu_2)$. We denote these by $\pi(\mu_1, \mu_2)$ and call them *principal series.*
(2) $s \in \mathbb{Z}$ such that $s \equiv m \pmod 2$ we get two types.
   (a) If $s \geq 0$, let $\pi(\mu_1, \mu_2)$ be the submodule $B/(B_+ \oplus B_-)$, and let $\sigma(\mu_1, \mu_2)$ be the $\mathcal{U}\mathfrak{g}_{\mathbb{C}}$-module induced from the $\mathcal{U}\mathfrak{g}_{\mathbb{C}}$-module B_+ (or B_- since they induce the same thing. Can think of this as $B_+ \oplus B_-$.)
   (b) If $s < 0$, let $\pi(\mu_1, \mu_2)$ be the module B_f, and let $\sigma(\mu_1, \mu_2)$ be the $(\mathcal{U}\mathfrak{g}_{\mathbb{C}}, \epsilon)$-module induced from the $\mathcal{U}\mathfrak{g}_{\mathbb{C}}$-module B_+/B_f (or B_-/B_f. They are conjugates so they induce the same module.)

Remark. Note the similarity to the nonarchimedean theory where the special representations are the infinite dimensional irreducible representations occurring in the composition series of the induced representations.

Lemma 4.4.3. *We have the following relations between $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1, \mu_2)$.*

(i) $\pi(\mu_1, \mu_2) \neq \sigma(\mu_1', \mu_2')$
(ii) $\pi(\mu_1, \mu_2) = \pi(\mu_1', \mu_2')$ *if and only if* $(\mu_1, \mu_2) = (\mu_2', \mu_1')$ *or* (μ_1', μ_2').
(iii) $\sigma(\mu_1, \mu_2) = \sigma(\mu_1', \mu_2')$ *if and only if* $(\mu_1, \mu_2) = (\mu_1', \mu_2')$, $(\eta\mu_1', \eta\mu_2')$, $(\eta\mu_2', \eta\mu_1')$, *or* (μ_2', μ_1'). *(*$\nu = \mathrm{sgn}.$*)*

Proof. For (i), we note that if π is finite dimensional they are not equal since σ is infinite dimensional. If π is infinite dimensional then π contains all φ_n with $n \equiv m \pmod 2$ but σ only contains part of the spectrum, hence they can not be equivalent representations.

For (ii) and (iii), we consider the action of a different module. Recall that by Lemma 4.4.2,

$$\rho(J)\varphi_n = (s_1 + s_2)\varphi_n, \qquad \rho(\Omega)\varphi_n = \frac{s^2 - 1}{2}\varphi_n,$$

with $s = s_1 - s_2$. If $\pi(\mu_1, \mu_2) = \pi(\mu_1', \mu_2')$ or $\sigma(\mu_1, \mu_2) = \sigma(\mu_1', \mu_2')$ then Ω and J should act by the same scalar on both sides.

So we get $s_1 + s_2 = s_1' + s_2'$ and $(s_1 - s_2)^2 = (s_1' - s_2')^2$. This implies that $(s_1, s_2) = \pm(s_1', s_2')$ or $\pm(s_2', s_1')$. Notice since $\mu_i = |t|^{s_i}\left(\frac{t}{|t|}\right)^{m_i}$ and m_i either ± 1.

So for fixed s_i there are finitely many choices. We leave it as an exercise to check the finitely many cases.

To complete the proof, one must actually define the intertwining maps between the spaces. This is straightforward. Once you have *a* map, it is unique. □

4.5. **Hecke algebra for** $\mathrm{GL}_2(\mathbb{R})$**.** As in the nonarchimedean theory, we have a notion of a Hecke algebra $\mathcal{H}$. Let $\mathcal{H}_1$ be the set of $f \in C_c^\infty(G)$ such that f is K-finite on the left and right, and let $\mathcal{H}_2$ be the $\mathbb{C}$-span of the matrix coefficients[15] of the finite dimensional irreducible complex representations of $K = O(2,\mathbb{R})$.

We set $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. For $f \in \mathcal{H}_1$ and $\xi \in \mathcal{H}_2$ we have

$$f * \xi = \int_K f(gu^{-1})\xi(u)du, \quad \xi * f = \int_K \xi(u) f(u^{-1}g)du.$$

Definition 4.5.1. *A representation of $\mathcal{H}$ (on a complex topological space) is called* admissible *if*

- *for all $v \in V$, $v = \sum_{i=1}^{l} \pi(f_i)v_i$ for some $v_i \in V$, and $f_i \in \mathcal{H}$,*
- *for all elementary idempotents ξ, $\pi(\xi)V$ is finite dimensional, and*
- *for all ξ and $v \in \pi(\xi)V$, the map $\xi\mathcal{H}_1\xi \to \pi(\xi)V$ given by $f \mapsto \pi(f)V$ is continuous. Note that $\xi\mathcal{H}_1\xi$ consists of smooth continuous functions of K-type ξ on both sides.*

For $f_n, f \in C_c^\infty(G)$ we write $f_n \to f$ if for every $\Omega \subset G(\mathbb{R})$ compact f_n converges uniformly to f on Ω and all derivatives $\frac{\partial^k f_n}{\partial x_1^{i_1}\cdots\partial x_4^{i_4}}$ converge to $\frac{\partial^k f}{\partial x_1^{i_1}\cdots\partial x_4^{i_4}}$ uniformly on Ω. So if $f_n \to f$ then $\pi(f_n)v \to \pi(f)v$.

Admissible $\mathcal{H}$-modules are closely related to $(\mathfrak{g}, K)$-modules, i.e. if V is an admissible $\mathcal{H}$-module, one can associate an admissible $(\mathfrak{g}, K)$-module as follows.[16] If $v = \pi(f)u = \int f(g)\pi(g)udg$ then we would like to have

$$\pi(h)v := \int f(g)\pi(h)\pi(g)udg = \int f(h^{-1}g)\pi(g)du = \pi(\lambda_h f)u.$$

The problem here is that for general $h \in G$, $\lambda_h f$ may not be in the Hecke algebra anymore. So we really only get this if we already have an action of G. However, this does work for Lie algebra.

Remark. In the nonarchimedean case $\mathcal{H}$ consists of locally constant functions so translates are still locally constant, and the above G-action goes through.

In the Lie algebra we have the following heuristic deduction

$$\pi(e^{tX})v = \pi(\lambda_{e^{tX}} f)v = \int_G f(e^{-tX}g)\pi(g)vdg.$$

Taking derivatives we have

$$\pi(X)v = \left.\frac{d}{dt}\right|_{t=0} \pi(e^{tX})v = \int_G \left.\frac{d}{dt}\right|_{t=0} f(e^{-tX}g)\pi(g)vdg.$$

We have the following actions:

[15] Matrix coefficients are functions of the form $\langle \delta(g)w_1, w_2\rangle$ where $\delta : K \to \mathrm{GL}(W)$ is a finite dimnsional (unitary) representation.

[16] We cannot define a G-action just a $(\mathfrak{g}, K)$-action.

- For $v \in V$ write $v = \sum_{i=1}^{l} \pi(f_i)v_i$, and define an action of $X \in \mathfrak{g}$ via
$$\pi(X)v := \sum_{i=1}^{l} \pi(Xf_i)v_i, \quad \text{with} \quad Xf = \frac{d}{dt}\bigg|_{t=0} f(e^{-tX}g).$$
(As in the p-adic case, this action doesn't depend on a choice of f_i and v_i.)
- Similarly, define the action of $k \in K$ to be
$$\pi(k)v = \sum \pi(\lambda_k f_i)v_i.$$
Again, this is well defined.

Theorem 4.5.2. *Let V be an admissible $\mathcal{H}$-module. By the above, one can consider it as a $(\mathfrak{g}, K)$-module. Then*

(i) *V is irreducible as an $\mathcal{H}$-module if and only if it is irreducible as a $(\mathfrak{g}, K)$-module.*
(ii) *$V_1 \simeq V_2$ as $\mathcal{H}$-modules if and only if $V_1 \simeq V_2$ as $(\mathfrak{g}, K)$-modules.*

By our above classification of $(\mathfrak{g}, K)$-modules, we also have.

Proposition 4.5.3. *Irreducible $\mathcal{H}$-modules are are $\pi(\mu_1, \mu_2)$, $\sigma(\mu_1, \mu_2)$.*

The proof of Theorem 4.5.2 requires differential operators. Jacquet-Langlands[5] has a sketch.

4.6. **Existence and uniqueness of Whittaker models.** The theory of Whittaker models of $G = \mathrm{GL}_2(\mathbb{R})$ is nearly parallel to the p-adic theory, but the techniques used are very different—the main technique being the use of differential equations.

Theorem 4.6.1. *Let π be an infinite dimensional admissible $\mathcal{H}$-module, and $\psi : \mathbb{R} \to \mathbb{C}^\times$ a nontrivial additive character. Then there exists a space $W(\pi, \psi)$, unique up to isomorphism, consisting of functions $W : G \to \mathbb{C}$ with the following properties.*

(1) $W(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g) = \psi(x)W(g)$.
(2) *Given W there exists $N > 0$ such that $W(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)) = O(|t|^N)$. (moderate growth)*
(3) *W is continuous, and the action of $\mathcal{H}$ on $W(\pi, \psi)$, given by*
$$\rho(f)W(g) = \int_G W(gh)f(h)dh$$
when $f \in \mathcal{H}_1$, and
$$\rho(f)W(g) = \int_K W(gk)f(k)dk$$
when $f \in \mathcal{H}_2$, is equivalent to π.

Proof of Uniqueness. Let $\pi = \pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$. Let $W(\pi, \psi)$ be a space satisfying the conditions of the theorem, and let κ_n denote the K-type sending $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{in\theta}$. Let W_n be the function in $W(\pi, \psi)$ of K-type κ_n. Set
$$f_n(t) = W_n \begin{pmatrix} t/|t|^{1/2} & \\ & 1/|t|^{1/2} \end{pmatrix}.$$

Since W_n is of type κ_n, the Iwasawa decomposition $G = PK$ implies that W_n is determined by f_n. (The function f_n is something like the Kirillov model of the p-adic case.) For $X \in \mathfrak{g}$, set

$$\rho(X)f_n = \rho(X)W\begin{pmatrix} t/|t|^{1/2} & \\ & 1/|t|^{1/2} \end{pmatrix}.$$

We want to analyze this action of $\mathfrak{g}$ on $\{f_n\}$. We denote the elements of $\mathfrak{g}$ as in (4.4.1). Using the fact that W_n has K-type κ_n, we calculate directly that

$$\begin{aligned}
\rho(U)f_n(t) =& \rho(U)W_n\begin{pmatrix} t/|t|^{1/2} & \\ & 1/|t|^{1/2} \end{pmatrix} \\
=& \frac{d}{ds}\bigg|_{s=0} W_n\left(\begin{pmatrix} t/|t|^{1/2} & \\ & 1/|t|^{1/2} \end{pmatrix}\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}\right) \\
=& \frac{d}{ds}\bigg|_{s=0} W_n\begin{pmatrix} t/|t|^{1/2} & \\ & 1/|t|^{1/2} \end{pmatrix} e^{ins} \\
=& inf_n(t),
\end{aligned}$$

and

$$\begin{aligned}
\rho(X_+)f_n(t) &= \frac{d}{ds}\bigg|_{s=0} W_n\left(\begin{pmatrix} t/|t|^{1/2} & \\ & 1/|t|^{1/2} \end{pmatrix}\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}\right) \\
&= \frac{d}{ds}\bigg|_{s=0} W_n\left(\begin{pmatrix} 1 & ts \\ & 1 \end{pmatrix}\begin{pmatrix} t/|t|^{1/2} & \\ & 1/|t|^{1/2} \end{pmatrix}\right) \\
&= \frac{d}{ds}\bigg|_{s=0} \psi(ts)f_n(t) = 2\pi iut f_n(t).
\end{aligned}$$

Note that $\psi(x) = e^{2\pi iux}$ for some $u \in \mathbb{R}^\times$. Similarly, we obtain

$$\rho(H)f_n = 2t\frac{df_n}{dt}, \qquad \rho(V_\pm)f_n = 2t\frac{df_n}{dt} \mp (4\pi ut \mp n)f_n.$$

Suppose $\pi = \sigma(\mu_1, \mu_2)$. So $\mu_i = |t|^{s_i}\left(\frac{t}{|t|}\right)^{m_i}$ with $s = s_1 - s_2$ and $m = |m_1 - m_2|$, and $s \in \mathbb{Z}$ with $s \not\equiv m \pmod 2$. If $s \geq 0$, $\rho(V_-)f_{s+1} = 0$. Combining this with the above, this gives the first order ODE

$$2t\frac{df_n}{dt} + (4\pi ut + n)f_n = 0. \tag{4.6.1}$$

($n = s+1$.) This is a separable ODE and easily solvable. We leave it as an exercise to verify that the solutions are of the form $C|t|^{-n/2}e^{-2\pi ut}$. Actually, since the domain is $\mathbb{R} \setminus \{0\}$ is disconnected,

$$f_n(t) = \begin{cases} Ct^{-n/2}e^{-2\pi ut} & \text{if } t > 0 \\ D|t|^{-n/2}e^{-2\pi ut} & \text{if } t < 0. \end{cases}$$

However, the growth condition forces one of C or D to be zero, hence there is a unique solution $f_n(t)$ (up to constant.) To see this, note that if $ut < 0$ then the growth condition is satisfied if and only if $C = 0$ (but D can be anything.) If $ut > 0$ any C satisfies the growth condition, but D must be zero. Putting this together gives uniqueness of W_n up to scalar. Thus $W(\pi,\psi)=\rho(\mathcal{U}\mathfrak{g}_{\mathbb{C}})W_n$ is unique. In the case that $s<0$, $\pi(V_-)\varphi_{-s+1}=0$. One can argue similarly to the above.

In the case that $\pi=\pi(\mu_1,\mu_2)$ is infinite dimensional, $s\notin\mathbb{Z}$ and $\pi=B(\mu_1,\mu_2)$. The relation coming from the action of Ω, recall that Lemma 4.4.2 says that $\rho(\Omega)f_n=\lambda f_n$, gives a second order differential equation. Using this and the growth condition, one argues uniqueness as above. $\square$

Remark. This same method can be used to show if $\pi(\mu_1,\mu_2)$ is finite dimensional then there does not exist $W(\pi,\psi)\neq 0$. The idea is that if $\pi(V_\pm)f_{\pm(s+1)}=0$, then only solution to the resulting ODEs is $f_n=0$.

We sketch the proof of existence when $\pi=\pi(\mu_1,\mu_2)$ infinite dimensional. In this case $\pi\simeq B(\mu_1,\mu_2)$. We use the same construction of the Whittaker functions as in the p-adic theory. Let $\Phi\in S(\mathbb{R}^2)^K=\{f\in S(\mathbb{R}^2)\mid f \text{ is } K\text{-finite}\}$[17]. Define

$$f_\Phi(g):=\mu_1(\det g)\,|\det g|^{1/2}\int_{\mathbb{R}^\times}\mu_1\mu_2(t)\Phi[(0,t)g]d^\times t$$

and

$$W_\Phi(g)=\mu_1(\det g)\,|\det g|^{1/2}\int_{\mathbb{R}^\times}\mu_1\mu_2^{-1}(t)[\rho(t)\Phi]^\sim(t,t^{-1})d^\times t.$$

Since Φ is K-finite, so are f_Φ and W_Φ, and $B(\mu_1,\mu_2)$ is the span of the functions f_Φ. Set $W(\mu_1,\mu_2,\psi)$ to be the span of $\{W_\Phi\mid\Phi\in S(\mathbb{R}^2)^K\}$. Now define $B(\mu_1,\mu_2)\to W(\mu_1,\mu_2,\psi)$ by $f_\Phi\mapsto W_\Phi$. This map has the same integral relations as in the p-adic case, see Section 3.14.4, and the following is true.

Theorem 4.6.2. *When* $\mathrm{Re}(s)>-1$ $f_\Phi\mapsto W_\Phi$ *is well defined, bijective and commutes with the action of* $(\mathcal{U}\mathfrak{g}_{\mathbb{C}},\epsilon)$.

Compare this to Corollary 3.14.10. Theorem 4.6.2, together with a simple check of the growth condition, implies that $B(\mu_1,\mu_2)$ irreducible has a Whittaker model. When $\pi=\sigma(\mu_1,\mu_2)$ we need a different (but similar) construction.

4.7. **Explicit Whittaker functions and the local functional equation.** From the discussion above (Theorem 4.6.2), we have the following diagram.

$$\begin{tikzcd}
B(\mu_1,\mu_2) \arrow[rr, "\mathrm{Re}\, s>-1"] & & W(\mu_1,\mu_2;\psi) \\
& S(\mathbb{R}^2)^K \arrow[ul, "\mathrm{Re}\, s>-1"] \arrow[ur] &
\end{tikzcd}$$

In this section we find $\Phi_n\in S(\mathbb{R}^2)^K$ such that $f_{\Phi_n}=\varphi_n\in B(\mu_1,\mu_2)$ where $\varphi_n\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)=e^{in\theta}$. The advantage of doing this is that we get W_{Φ_n}, and so we have explicit Whittaker functions. We then use these to assert the local functional equation.

Set

$$\Phi_n(x,y)=e^{-\pi(x^2+y^2)}(x+iy\,\mathrm{sgn}(n))^{|n|}.$$

In polar coordinates, $\Phi_n(r,\theta)=e^{-\pi r^2}r^{|n|}e^{in\theta}$. So in the r-direction the function is rapidly decreasing.

[17] An example of a K-finite function is $e^{-(x^2+y^2)}$.

We will show that $f_{\Phi_n} = \lambda\varphi_n$. Although it seems that $S(\mathbb{R}^2)$ is a huge space we will only need these simple functions to get the Whittaker functions. The point is that the kernel of $S(\mathbb{R}^2) \to W(\mu_1, \mu_2, \psi)$ is large.

First, note that f_{Φ_n} and φ_n have the same K-type. This follows from the following identity.

$$\Phi_n[(x,y)\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)] = \Phi_n[(x,y)]e^{in\theta}.$$

Therefore, to compare f_{Φ_n} and φ_n one only needs to calculate $f_{\Phi_n}(e)$.

$$\begin{aligned}
f_{\Phi_n}(e) &= \int_{\mathbb{R}^\times} \mu_1\mu_2(t)\,|t|\,\Phi_n((0,t)e)d^\times t \\
&= \int_{\mathbb{R}^\times} |t|^s \operatorname{sgn}(t)^m\,|t|\,e^{-\pi t^2}(it\operatorname{sgn}(n))^{|n|}d^\times t \\
&= (i\operatorname{sgn}(n))^{|n|}\int_{\mathbb{R}^\times} \operatorname{sgn}(t)^{m+|n|}\,|t|^{|n|+s+1}\,e^{-\pi t^2}d^\times t \\
&= 2(i\operatorname{sgn}(n))^{|n|}\int_0^\infty |t|^{|n|+s+1}\,e^{-\pi t^2}d^\times t \\
&= (i\operatorname{sgn}(n))^{|n|}\int_0^\infty e^{-t}(t/\pi)^{(s+|n|+1)/2}d^\times t \\
&= (i\operatorname{sgn}(n))^{|n|}\pi^{-\frac{s+|n|+1}{2}}\Gamma\left(\frac{s+|n|+1}{2}\right).
\end{aligned}$$

We have used the fact that $m \equiv n \pmod 2$ and the change of variables $t \to \frac{1}{\sqrt{\pi}}\sqrt{t}$. Note that when $\operatorname{Re}(s) > -1$ the argument of Γ is positive and thus $f_{\Phi_n}(e) \neq 0$. We conclude that $f_{\Phi_n} = \lambda\varphi_n$ for some $\lambda \neq 0$.

Now we consider

$$W_{\Phi_n}\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) = \mu_1(a)\,|a|^{1/2}\int_{\mathbb{R}^\times} \mu_1\mu_2^{-1}(t)\Phi^\sim(at, t^{-1})d^\times t. \tag{4.7.1}$$

If $\Phi = e^{-\pi(x^2+y^2)}P(x,y)$ and P is a polynomial and $\psi(x) = e^{2\pi iux}$ then $\Phi^\sim = e^{-\pi(x^2+u^2y^2)}Q(x,y)$ with Q a polynomial. Note that the Fourier transform with respect to $\psi(x) = e^{2\pi ix}$ of $e^{-\pi t^2}$ is $e^{-\pi s^2}$. This implies that (4.7.1) is equal to

$$W_{\Phi_n}\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) = \mu_1(a)\,|a|^{1/2}\int_{\mathbb{R}^\times} \mu_1\mu_2^{-1}(t)e^{-\pi(a^2t^2+u^2t^{-2})}Q(at, t^{-1})d^\times t. \tag{4.7.2}$$

Substituting $\mu_1\mu_2^{-1} = |t|^s\operatorname{sgn}(t)^m$, we want to verify the growth condition. We do not need worry about $\mu_1(a)\,|a|^{1/2}$ since this is already a power of a. We split the integral into a part near 0 and one near ∞.

$$\begin{aligned}
&\left|\int_0^\delta |t|^s\operatorname{sgn}(t)^m e^{-\pi(a^2t^2+u^2t^{-2})}Q(at,t^{-1})d^\times t\right| \\
&\qquad\qquad \leq \int_0^\delta |t|^s\,e^{-\pi(a^2t^2+a^2t^{-2})}\,|a|^{\deg Q}\,c_\delta d^\times t \\
&\qquad\qquad\qquad \leq c_\delta\,|a|^{\deg Q}\int_0^\delta |t|^s\,e^{-2\pi|au|}d^\times t \leq c'_\delta\,|a|^{\deg Q}\,e^{-2\pi|u||a|}.
\end{aligned}$$

We have used that $|a| \gg 0$ in the first line and Cauchy-Schwarz ($\alpha^2+\beta^2 \geq |\alpha\beta|$.) The final integral decays exponentially, so everything is good.

Now near ∞ we have

$$\left|\int_\delta^\infty |t|^s \operatorname{sgn}(t)^m e^{-\pi(a^2t^2+u^2t^{-2})} Q(at,t^{-1}) d^\times t\right| \leq \int_\delta^\infty c_\delta e^{-\pi(a^2t^2)} |a|^{\deg Q} |t|^{s+\deg Q} d^\times t = o(|a|^N)$$

for any $N>0$. So $W_{\Phi_n} = O(|a|^N)$ for any $N>0$.

Remark. For $\pi = \sigma(\mu_1,\mu_2)$ we have

$$W(\pi,\psi) = \left\{ W_\Phi \in W(\mu_1,\mu_2;\psi) \,\middle|\, \int_{-\infty}^\infty x^i \frac{\partial^j}{\partial y^j}\Phi^\sim(x,0)dx = 0 \text{ for all } i,j \right\}.$$

This is comparable to the p-adic case. See Lemma 3.14.12.

Note given $\varphi \in \pi$ with $\varphi = \sum c_i\varphi_i$ correspondingly we have W a Whittaker function with $W = \sum c_i W_{\Phi_i}$.

Theorem 4.7.1. *Let π be an infinite dimensional irreducible admissible representation of $\mathcal{H}$. For $W \in W(\pi,\psi)$ define*

$$\Psi(g,s,W) := \int_{\mathbb{R}^\times} W\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g\right) |a|^{s-1/2} d^\times a$$

and

$$\widetilde{\Psi}(g,s,W) := \int_{\mathbb{R}^\times} W\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\right) \omega^{-1}(a) |a|^{s-1} d^\times a$$

where ω is the central quasicharacter. Then there exists an Euler factor $L(s,\pi)$ and $L(s,\pi^\sim)$ such that

1. *$\Psi(g,s,W)$, $\widetilde{\Psi}(g,s,W)$ converge absolutely when* $\mathrm{Re}(s) \gg 0$*,*
2. *$\Phi(g,s,W) = \frac{\Psi(g,s,W)}{L(s,\pi)}$, $\widetilde{\Phi}(g,s,W) = \frac{\widetilde{\Psi}(g,s,W)}{L(s,\widetilde{\pi})}$ are holomorphic, and*
3. *there exists an exponential function $\epsilon(s,\pi,\psi)$ such that*

$$\widetilde{\Phi}(\omega g, 1-s, W) = \epsilon(s,\pi,\psi)\Phi(g,s,W).$$

If $W \in W(\pi,\psi)$ we have $W = \sum W_{\Phi_i}$ with each $\Phi_i = \Phi_{i,1}(x)\Phi_{i,2}(y)$. So to prove the functional equation we only need to check it for those W_{Φ_i}.

Remark. $\Phi(r,\theta) = \sum \Phi_i(r,\theta) = \sum f_i(r)e^{in\theta}$ so finite K-type alone is not what allows us to consider split cases.

Proof in case $\pi = \pi(\mu_1,\mu_2)$. As in the nonarchimedean case, it suffices to prove everything for $g=e$. When $\pi = \pi(\mu_1,\mu_2)$, $W(\pi,\psi) = W(\mu_1,\mu_2,\psi)$ and we consider the pure tensors $\Phi(x,y) = \Phi_1(x)\Phi_2(y) \in S(\mathbb{R}^2)^K$ then

$$\Psi(e,s,w) = Z(s,\mu,\Phi_1)Z(s,\mu_2,\Phi_2)$$

which are the GL_1 zeta integrals. Additionally,

$$\Psi(\omega,1-s,W) = Z(1-s,\mu_1^{-1},\Phi_1^\sim)Z(1-s,\mu_2^{-1},\Phi_2^\sim).$$

The meromorphic part of $Z(s,\mu,\Phi_1)$ is $L(s,\mu_1)$, and the calculations are the same as the p-adic theory.

Set $L(s,\pi) = L(s,\mu_1)L(s,\mu_2)$, $\epsilon(s,\pi,\psi) = \epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi)$. All statements for π follow from the corresponding statements for GL_1 theory for $\mathbb{R}$. (See below for the statements.) For $\pi = \sigma(\mu_1,\mu_2)$ one can argue in a simialr way. $\square$

In the GL_1 theory every quasicharacter $\mu : \mathbb{R}^\times \to \mathbb{C}^\times$ is of the form $x \mapsto |x|^r \operatorname{sgn}(x)^\delta$ for some $r \in \mathbb{C}$ and $\delta \in \{0,1\}$. Let $\psi : \mathbb{R} \to \mathbb{C}^\times$ be the character[18] $\psi(x) = e^{2\pi i u x}$. If $\Phi \in S(\mathbb{R})$, define

$$Z(s,\mu,\Phi) = \int_{\mathbb{R}^\times} \mu(x)\,|x|^s\,\Phi(x) d^\times x,$$

$$L(s,\mu) = \pi^{-(s+r+\delta)/2}\Gamma(\frac{s+r+\delta}{2}) =: \Gamma_{\mathbb{R}}(s+r+\delta)$$

and

$$\epsilon(s,\mu,\psi) = [i\operatorname{sgn}(u)]^\delta\,|u|^{s+r-1/2}\,. \tag{4.7.3}$$

Then the following are true.

- The $Z(s,\mu,\Phi)$ converges absolutely for $\operatorname{Re}(s) > s_0$ for certain s_0 depending on μ.
- $\frac{Z(s,\mu,\Phi)}{L(s,\mu)}$ is holomorphic.
- $\frac{Z(1-s,\mu^{-1},\Phi^\sim)}{L(1-s,\mu^{-1})} = \epsilon(s,\mu,\psi)\frac{Z(s,\mu,\Phi)}{L(s,\mu)}$.

4.7.1. *A brief description of the* $GL_1(\mathbb{C})$ *theory and the* L*-factor for* $\sigma(\mu_1,\mu_2)$. There is an analogous theory for quasicharacters $\omega : \mathbb{C}^\times \to \mathbb{C}^\times$. These are given by $z \mapsto |z|_{\mathbb{C}}^r\, z^m \overline{z}^n$ with $m,n \geq 0$ and on of them is 0. So, for $\Phi \in S(\mathbb{C})$,

$$Z(\omega,s,\Phi) = \int_{\mathbb{C}^\times} \omega(z)\,|z|^s\,\Phi(z) d^\times z,$$

and

$$L(s,\omega) = 2(2\pi)^{-(s+r+m+n)}\Gamma(s+r+m+n) =: \Gamma_{\mathbb{C}}(s+r+m+n).$$

Then $Z(\omega,s,\Phi)$ converges absolutely when $\operatorname{Re}(s) \gg 0$ and $\frac{Z(\omega,s,\Phi)}{L(s,\omega)}$ is holomorphic. There is a similar local functional equation. The ϵ-factor is a bit more complicated. All of this is done in [7].

Alternatively, $\mathbb{C}^\times \simeq \mathbb{R}^{>0} \times S^1$ via $z \mapsto (|z|_{\mathbb{C}}\,, \frac{z}{|z|}$, every character of $\mathbb{C}^\times$ is the product of a character on $\mathbb{R}^{>0}$ and a character on S^1. It is well known that such characters are of the form $x \mapsto x^t$ (for $t \in \mathbb{C}$) and $e^{i\theta} \mapsto e^{ik\theta}$ (for $k \in \mathbb{Z}$) respectively.

A simple calculation (left as an exercise) reveals that the character ω, which depends on m,n,r, above is equal to the character $z \mapsto |z|_{\mathbb{C}}^t\,(\frac{z}{|z|})^k$ if

$$k = m-n \qquad \text{and} \qquad t = \frac{m+n}{2} = \frac{|k|}{2}.$$

In the case that $\pi = \sigma(\mu_1,\mu_2)$, then π is a discrete series representation of weight k and

$$L(s,\pi) = L(s,\omega) = \Gamma_{\mathbb{C}}(s+t+\frac{k}{2}).$$

Note that in this case

$$\mu_1\mu_2^{-1} = x^p\operatorname{sgn}(x) \quad \text{and} \quad \mu_1\mu_2 = |x|^{2r}\operatorname{sgn}(x)^{p+1}.$$

Although not done here, it shouldn't be hard to see the relation between this r and p and the t and s above.

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[18]My original notes from the class didn't include any definition for ψ, but the u appearing in (4.7.3) is never defined, so this seemed a reasonable guess.

4.8. $\mathrm{GL}_2(\mathbb{C})$**.** Here the theory is similar. In short, every admissible representation is of the form $\pi = \pi(\omega_1, \omega_2)$ where $\omega_i : \mathbb{C}^\times \to \mathbb{C}^\times$ are characters. The local factors are (I think)

$$L(s, \pi) = L(s, \omega_1)L(s, \omega_2)$$

See [5] for details.

5. Global theory: automorphic forms and representations

In this final section we consider automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ where $\mathbb{A}_F$ is the ring of adeles associated to a number field F. Due to time limitations, we do not include proofs, but we will try to touch on the big ideas. We let $G = \mathrm{GL}_2$, $K_v = \mathrm{GL}_2(F_v)$ if v is finite, $K_v = O(2, \mathbb{R})$ if v is real and $K_v = U(2)$ if v is complex. Then define $K_\infty = \prod_{v|\infty} K_v$.

We remark that most of what we will say is true for G any reductive group over a number field, and K_v maximal compact subgroups of $G(F_v)$. However, we do not work in this generality as the definitions are slightly more complicated. When a result is particular to GL_2, we will try to make that clear.

5.1. **Restricted products.** We have $G(\mathbb{A}_F) = \prod_v^* G(F_v)$. The $*$ denotes the restricted product with respect to $\{K_v\}$. This means that $g = (g_v) \in G(\mathbb{A}_F)$ if and only if $g_v \in K_v$ for all but finitely many v. Also, let $G(\mathbb{A}_{F,f})$ the subgroup of $G(\mathbb{A}_F)$ consisting of elements (g_v) with $g_v = 1$ for all $v \mid \infty$. (Recall that these notions were introduced in Section 2.1.)

For each place v of F, let V_v is a complex vector space over $\mathbb{C}$. Fixing a vector $e_v \in V_v$ for almost all v allows us to define the restricted tensor product

$$V = \bigotimes_v^* V$$

which is generated by the set of all tuples $(w_v) \in \prod V_v$ such that $w_v e_v$ for almost every v, modulo the usual tensor relations.

We also define a global Hecke algebra in this fashion. Let

$$\mathcal{H} = \bigotimes_v^* \mathcal{H}_v$$

with respect to $e_v = 1_{K_v}$ (the characteristic function of K_v) for almost all v.

Definition 5.1.1. *A pure tensor $\xi = \otimes \xi_v \in \mathcal{H}$ is called an* elementary idempotent *if each ξ_v is an elementary idempotent in $\mathcal{H}_v$.*

Definition 5.1.2. *A representation π of $\mathcal{H}$ is a vector space V is a called* admissible *if*

(1) *every $w \in V$ is of the form $\sum_{i=1}^l \pi(f_i) w_i$ with $w_i \in V$, $f_i \in [\bigotimes_{v|\infty}^* \mathcal{H}_{1,v}] \otimes [\bigotimes_{v\nmid\infty}^* \mathcal{H}_v]$,*

(2) *for every elementary idempotent $\xi \in \mathcal{H}$ the range of $\pi(\xi)$ is finite dimensional, and*

(3) *if v_0 is an archimedean place and $\xi_{v_0} \in \mathcal{H}_{v_0}$ is an elementary idempotent for each $v \neq v_0$, choose an elementary idempotent ξ_v set $\xi = \xi_{v_0} \otimes (\otimes_{v \neq v_0} \xi_v)$. Then for any $w \in V$ the map $\xi_{v_0}\mathcal{H}_{v_0}\xi_{v_0} \to \pi(\xi)v$ by $f_{v_0} \mapsto \pi[f_{v_0} \otimes (\otimes_{v \neq v_0})\xi_v)]w$ is continuous.*

Remark. The first two conditions are as usual and the third is to control the archimedean place. See Definitions 2.2.7 and 4.5.1.

We now have the tools needed to build a global representation from local ones. To do this, for each v, let (π_v, V_v) be an admissible representation of $\mathcal{H}_v$. Assume that $\dim V_v^{k_v} \leq 1$ and that for almost all v equality holds. Choose $e_v \in V_v^{K_v}$ for almost all v and let V be the restricted tensor product with respect to these vectors. Define the action π of $\mathcal{H}$ on V to be

$$\pi((f_v)_v)(\otimes w_v) = \pi(f_v)w_v.$$

Note that e_v is a K_v fixed vector, so this action only depends on the finitely many places for which $f_v \neq \xi_v$ and $w_v \neq e_v$. So this makes V an admissible $\mathcal{H}$-module.

Definition 5.1.3. *Let (π, V) be an admissible representation of $\mathcal{H}$. It is* irreducible *if there is no nontrivial $\mathcal{H}$-invariant subspace.*

Remark. If $\pi = \otimes \pi_v$ then π is irreducible if and only if π_v is irreducible for every v.

Proposition 5.1.4. *If (π, V) is an irreducible admissible representation of $\mathcal{H}$, it is factorizable. In other words, there exists irreducible admissible representations (π_v, V_v) of $\mathcal{H}_v$ with π_v unramified for almost all v such that $V \simeq \bigotimes_v^* V_v$ and $\pi \simeq \bigotimes_v^* \pi_v$. The local representations π_v are unique up to isomorphism.*

In practice, one is often presented with a global expression, for example a period integral, and one would like to express this as a product of local objects. This passage from global to local, although expected, is often difficult to achieve.

5.2. **The space of automorphic forms.** A function $\varphi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ is an *automorphic form* if the following are satisfied.

- φ is right invariant by K_f an open compact subgroup of $G(\mathbb{A}_F)$.
- For any $g_f \in G(\mathbb{A}_{F,f})$ the map $G(F_\infty) \to \mathbb{C}$ given by $g_\infty \mapsto \varphi(g_\infty \cdot g_f)$ is smooth. Also, φ is K_∞ finite.
- φ is Z-finite.
- φ is of moderate growth.

We explain the terms K_∞*-finite*, Z*-finite* and *moderate growth* below.

Denote $\mathcal{A}$ the set of automorphic forms.

Definition 5.2.1. *A function $\varphi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ is* K_∞-finite *if $span\{K_\infty \varphi\}$ is finite dimensional. Compare with .*

Definition 5.2.2. *Let Z denote the center of $\mathcal{U}\mathfrak{g}_\mathbb{C}$, which is the tensor product of the universal enveloping algebras at each local place. Then $\varphi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ is* Z-finite *if φ is annihilated by an ideal of finite codimension in Z.*

We note that, in particular, all eigenvectors are Z-finite.

Consider $i : G \hookrightarrow \mathrm{SL}_4$ by $g \mapsto \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix}$ write $i(g) = (g_{m,n})$. We use this to define a norm on $G(\mathbb{A}_F)$: for each v set $|g_v|_v = \max_{m,n}\{|i(g_v)_{m,n}|_v\}$. Then define $|g| = \prod_v |g_v|_v$.

Definition 5.2.3. *$\varphi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ is of* moderate growth *(or* slowly increasing*) if*

$$|\varphi(g)| = O(|g|^N)$$

for some $N \geq 0$ as $|g| \to \infty$. In other words, φ is polynomially bounded as $|g| \to \infty$.

Remark. The norm $|\cdot| : G(\mathbb{A}_F) \to \mathbb{C}$ satisfies the following.

- There exists $c > 0$ such that $|g| > c$ for all $g \in G(\mathbb{A})$.
- There exists $c > 0$ such that $|g_1 g_2| < c|g_1||g_2|$.
- There exists $c, r > 0$ such that $|g^{-1}| < c|g|^r$.
- For compact sets $\Omega, \Omega' \subset G(\mathbb{A}_F)$ there exists c_1, c_2 such that $c_1|g| \leq |g_1 g g_2| \leq c_2|g|$ for $g_1 \in \Omega$ and $g_2 \in \Omega'$.

Theorem 5.2.4. *Fix an ideal I of Z with finite codimension, a K_∞-type ξ, and a compact open subgroup $K_f \subset G(\mathbb{A}_{F,f})$. Let V to be the set of $\varphi \in \mathcal{A}$ such that φ is right invariant by K_f, $\rho(\xi)\varphi = \varphi$ and $\rho(I)\varphi = 0$. Then $\dim V < \infty$.*

Corollary 5.2.5. *The action of $\mathcal{H}$ on $\mathcal{A}$ is admissible.*

Definition 5.2.6. *An irreducible representation of $\mathcal{H}$ is* automorphic *if it is a subquotient of $\mathcal{A}$, meaning that there exists $U \subset V \subset \mathcal{A}$ such that $\pi \simeq V/U$.*

Definition 5.2.7. *A continuous function $\varphi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ is* cuspidal *if $\int_{F\backslash\mathbb{A}_F} \varphi[\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g] = 0$ for all g. The condition for all g implies our usual notion of vanishing at all cusps.*

An automorphic form φ is a cusp form *if it is cuspidal. Set $\mathcal{A}_0$ to be the set of all cusp forms.*

Theorem 5.2.8. *$\mathcal{A}_0$ is a semisimple $\mathcal{H}$-module and each irreducible admissible representation π of $\mathcal{H}$ occurs with finite multiplicity.*

Remark. For our special case of $G = \mathrm{GL}_2$, the multiplicity is actually one.

5.3. **Spectral decomposition and multiplicity one.** One obtains a measure m on $G(\mathbb{A}_F)$ by specifying a Haar measure m_v on $\mathrm{GL}_2(F_v)$ for almost all v. We require $m_v(K_v) = 1$, and with this choice of measure the following is true.

Proposition 5.3.1. *If Z^+ is the identity component of the center of $G(\mathbb{A}_{F,\infty})$ then $m\,(G(F)Z^+\backslash G(\mathbb{A}_F)) < \infty$.*

This result is similar to the fact that $\mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R})$ has finite volume.

Suppose that $F = \mathbb{Q}$. In this case[19] we write $\mathbb{A}_\mathbb{Q} = \mathbb{A}$ and $\mathbb{A}_{\mathbb{Q},f} = \mathbb{A}_f$. Then there exists a finite set C such that

$$G(\mathbb{A}) = \bigsqcup_{c \in C} G(\mathbb{Q}) c G(\mathbb{R}) K_f$$

[19]If we allow other groups besides $G = \mathrm{GL}_2$, then this really isn't a restriction at all, and the argument here goes through.

for any compact open subgroup $K_f \subset G(\mathbb{A}_f)$. This implies that

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_f = \bigsqcup_{c\in C} \Gamma_c\backslash G(\mathbb{R})$$

where $\Gamma_c = G(\mathbb{Q}) \cap cK_f c^{-1}$ which is a congruence subgroup. In our case of $G = \mathrm{GL}_2$, this looks like finitely many copies of the upper half plane modulo a congruence subgroup which has finite volume.

The main corollary to Proposition 5.3.1 is that there exists a spectral decomposition of $L^2\left(G(F)Z^+\backslash G(\mathbb{A}_F)\right)$. The main references for this theory are Mœglin and Waldspurger[6] (which is a reworking of Langlands' work on Eisenstein series) and Godement's papers in [1].

Let $N = \{\left(\begin{smallmatrix} 1 & * \\ & 1\end{smallmatrix}\right)\}$ and $B = \{\left(\begin{smallmatrix} * & * \\ & *\end{smallmatrix}\right)\}$ as usual[20], and Z^+ the connected component of $Z(\mathbb{A})$. $G(\mathbb{A})$ acts (via the right regular representation) on $L^2(Z^+G(\mathbb{Q})\backslash G(\mathbb{A}))$. We restrict our discussion to the *cuspidal spectrum*:

$$L^2_0(G) := \left\{ \varphi \in L^2(Z^+G(\mathbb{Q})\backslash G(\mathbb{A})) \,\middle|\, \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)dn = 0 \text{ for all } g \in G \right\}.$$

The portion of $L^2(G)$ 'orthogonal' to $L^2_0(G)$ is called the *continuous spectrum.*

Remark. When the space is compact there is only a discrete spectrum. That is there is no continuous part.

Theorem 5.3.2. *$L^2_0(G)$ is the Hilbert space direct sum of irreducible unitary subrepresentations of $G(\mathbb{A})$ each of which occurs with finite multiplicity.*

The general proof follows from the theory of compact self adjoint operators and it works for all reductive groups. (See Godement's papers in [1].)

Example 5.3.3. For $G = S^1$, the Laplacian is $\Delta = \frac{\partial^2}{\partial\theta^2}$ and the eigenfunctions are $e^{in\theta}$. This is just classical Fourier analysis.

The action of the center of the universal enveloping algebra is critical. In our case it is freely generated by $\left(\begin{smallmatrix} 1 & \\ & 1\end{smallmatrix}\right)$ and the Kasimir element Ω. When $Z^+G(\mathbb{Q})\backslash G(\mathbb{A})/K$ consists of one component, and so equals $\Gamma\backslash\mathbb{H}$, the action of Ω as an operation on functions is -2Δ where

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

This is the hyperbolic Laplacian acting on the upper half space $\mathbb{H}$. Using this action, we decompose $L^2_0(G)$ into unitary representations. The papers by Godement in the PSPUM series from the Boulder Conference[1] discuss the classical situation.

Let $\chi : \mathbb{Q}^\times\mathbb{R}_+\backslash A^\times \to S^1$ be a character. Define

$$L^2_\chi(G) := \{\varphi \in L^2(G(\mathbb{Q})Z^+\backslash G(\mathbb{A})) : \varphi(zg) = \chi(z)\varphi(g) \text{ for } z \in Z(\mathbb{A})\},$$

and $L^2_{\chi,0}(G) = L^2_\chi(G) \cap L^2_0(G)$, the χ-part of the cuspidal spectrum. To deduce multiplicity one we use the Whittaker model and Fourier expansion. Let $\pi = \otimes_p\pi_p$ be an irreducible component in $L^2_0(G)$. Then define π^∞ to be the set of all $\varphi \in \pi$ such that φ is invariant by open subgroup K_f, $\varphi(g_\infty g_f)$ is a smooth function of $G(\mathbb{A}_\infty)$ for every choice of $g_f \in G(\mathbb{A}_f)$, and φ is K_∞-finite.

So $\pi^\infty = \otimes_p\pi_p^\infty$. ($\pi_p^\infty$ is just the set of smooth vectors in π_p as defined in the sections on local representations.) For $\varphi \in \pi^\infty$ we can do Fourier decomposition

[20] For a more general group, $B = NA$ is a Borel subgroup of G.

when restricting it to the nilpotent radical. By this we mean choose a nontrivial character $\psi : Q\backslash A \to \mathbb{C}$. (All character are then of the form $\psi_\xi(x) = \psi(\xi x)$.) Then for a fixed g, $\varphi[\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g]$ is a function on $\mathbb{Q}\backslash\mathbb{A}$, and Fourier expansion tells us that

$$\varphi(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g) = \sum_{\xi \in Q} \psi(\xi x) W_{\phi,\xi}(g), \tag{5.3.1}$$

where $W_{\phi,\xi}(g) = \int_{n \in Q\backslash A} \varphi(\left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right) g)\psi^{-1}(\xi n)dn$. $\phi \mapsto W_{\phi,\xi}(1)$ is a global Whittaker functional on π^∞. We derive this in Section 5.6.2.

Remark. Multiplicity one follows from the uniqueness (up to scalar) of the Whittaker model. This follows from the fact that the local Whittaker models are unique. See Proposition 5.6.3

Remark. Equation (5.3.1) implies that at least one of the Whittaker functional is nonzero. Therefore, all π_p^∞ are infinite dimensional. (See Theorem 3.13.4.)

Theorem 5.3.4 (Multiplicity One)**.** *Each irreducible component of $L_0^2(G)$ occurs only once.*

This theorem holds for $G = \mathrm{GL}_2$, but not in complete generality.

Proof. Suppose $\pi_1 \simeq \pi_2$ are irreducible components in $L_0^2(G)$, and let L be an intertwining operator between them. Then $\varphi \mapsto W_{\varphi,\xi}(1)$ and $\varphi \mapsto W_{L\varphi,\xi}(1)$ are two ψ_ξ Whittaker functionals. By uniqueness one is a scalar of the other. $W_{L\varphi,\xi}(1) = \lambda_\xi W_{\varphi,\xi}(1)$ for all $\varphi \in \pi^\infty$. But all λ_ξ are equal, say to λ. Using the Fourier inversion formula we see $L\varphi = \lambda\varphi$. So $\pi_1 = \pi_2$. $\square$

Theorem 5.3.5 (Strong Multiplicity One)**.** *For two irreducible components $\pi_1, \pi_2 \in L_0^2(G)$, $\pi_{1,p} \simeq \pi_{2,p}$ for almost all p implies $\pi_1 \simeq \pi_2$.*

The proof involves twisted L-functions.

Set $\mathcal{A}_0 = \mathcal{A} \cap L_0^2(G)$ and $\mathcal{A}_0(\psi) = \mathcal{A} \cap L_{\psi,0}^2(G)$. For an irreducible component π of $L_0^2(G)$ set $\mathcal{A}(\pi) = \mathcal{A} \cap \pi$.

Theorem 5.3.6. *Suppose π_p is an infinite dimensional irreducible admissible representation of $G(\mathbb{Q}_p)$. For $l \in \mathbb{Z}_{\geq 0}$ set*

$$K_0(p^l) = \{g_p \in G(\mathbb{Z}_p) \mid g_p \equiv \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \pmod{p^l}\}.$$

Then there exists a smallest l such that $\dim \pi_p^{K_0(p^l)} \neq 0$. Moreover, for this smallest l, $\dim \pi_p^{K_0(p^l)} = 1$.

$c(\pi_p) = p^l$ is called the *conductor* of π_p. If π is an irreducible component of $L_0^2(G)$, $c(\pi) = \prod c(\pi_p)$ is called the *conductor* of π. Since $l = 0$ whenever π_p is unramified, $c(\pi)$ is a well-defined integer.

5.4. **Archimedean parameters.** Since $L_0^2(G)$ is unitary, if π is an irreducible component of $L_0^2(G)$ then π_∞ is unitarizable.

Theorem 5.4.1. *If an infinite dimensional irreducible admissible represenation π of $\mathcal{H}_\mathbb{R}$ is unitarizable then it is either $\sigma(\mu_1, \mu_2)$ or $\pi(\mu_1, \mu_2)$ with $\mu_1 = \mu, \mu_2 = \overline{\mu}^{-1}$ and $\mu(t) = |t|^{s/2}\operatorname{sgn}(t)^m$ with $s \in (-1, 1)$ or $\mu_1 = |t|^{ia_1}\operatorname{sgn}(t)^{m_i}$ and $\mu_2 = |t|^{ia_2}\operatorname{sgn}(t)^{m_2}$.*

The requirement on s is so that we can make the form positive definite.

Sketch of Proof. Clearly $\sigma(\mu_1, \mu_2)$ are unitarizable. Suppose $\pi(\mu_1, \mu_2)$ is unitarizable then $\pi(\overline{\mu_1}, \overline{\mu_2}) \simeq \pi(\mu_1^{-1}, \mu_2^{-1})$ which implies that $\overline{\mu_i} = \mu_i^{-1}$ or $\overline{\mu_1} = \mu_2^{-1}$ and $\overline{\mu_2} = \mu_1^{-1}$. In the first case the norm part of μ_1 and μ_2 are imaginary. In the second case $\mu_1 = \mu$ and $\mu_2 = \overline{\mu}^{-1}$ with μ as above. To see the condition for s construct the Hermitian pairing. □

If $\pi_\infty \simeq \sigma(\mu_1, \mu_2)$ then π is called a *discrete series* representation and it corresponds to a space of holomorphic modular forms. The case $\pi \simeq \pi(\mu_1, \mu_2)$ corresponds to Maass forms. In either case, if $|t|^{ia}$ are purely imaginary these are called *continuous series*. In the other case we call the representations *complimentary series*.

5.5. **Classical forms on $\mathbb{H}$.** In this section we discuss the relation between classical modular and Maass forms and automorphic representations.

5.5.1. *Holomorphic cusp forms.* Define the action of $g \in \mathrm{GL}_2(\mathbb{R})^+$ on $z \in \mathbb{H}$ in the usual way: $z \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$, and define the automorphy factor $j(g, z) = (cz + d)(\det g)^{-1/2}$. We assume that $k \in \mathbb{Z}_{\geq 0}$. Define

$$f_{[g]_k}(z) = f(g \cdot z) j(g, z)^{-k}.$$

For Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ set $S_k(\Gamma)$ to be the set of all $f : \mathbb{H} \to \mathbb{C}$ such that

- $f \mid_{[\gamma]_k} = f$ for all $\gamma \in \Gamma$
- f is holomorphic on $\mathbb{H}$
- at every cusp of Γ f vanishes.

Some examples of congruence subgroups are

$$\begin{aligned} \Gamma(N) :=& \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \pmod N\}, \\ \Gamma_0(N) :=& \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \pmod N\}, \\ \Gamma_1(N) :=& \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \pmod N\}. \end{aligned}$$

We will focus on $\Gamma = \Gamma_0(N)$. Set

$$K_f(N) = \prod_{p \nmid \infty} K_0(p^{v_p(N)})$$

with $K_0(p^l) = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) \mid g \equiv \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \pmod{p^l}\}$ as before. Also, set $K_\infty = SO(2, \mathbb{R})$. Then if $K = K_f K_\infty$,

$$G(\mathbb{Q})Z^+ \backslash G(\mathbb{A})/K \simeq \Gamma_0(N)\backslash\mathbb{H}.$$

Now let χ be a character of $(\mathbb{Z}/N\mathbb{Z})^\times$ with $\chi(-1) = (-1)^k$. Set $S_k(N, \chi)$ to be the set of all $f \in S_k(\Gamma_1(N))$ such that $f_{[\gamma]_k}(z) = \chi(d) f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Note that $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$. Then $S_k(\Gamma_1(N)) = \otimes_\chi S_k(N, \chi)$.

5.5.2. *Maass forms.* By shifting weights we only need to worry about weight[21] 0 or 1. Let $W_s(\Gamma)$ be the set of all $f : \mathbb{H} \to \mathbb{C}$ with

- f is bounded and smooth
- $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$

[21]This seems a little funny because the definition seems to only involve weight 0 forms. Although, from the representation theory below it will be clear that it suffices to consider weight 0 and weight 1 only.

- $\Delta f = \frac{1-s^2}{4} f$

The reason for writing the eigenvalue as $\frac{1-s^2}{4}$ is that if $\mu_1\mu_2^{-1} = |t|^s \operatorname{sgn}(t)^m$ then Ω acts according to $(s^2-1)/2$ and $\Delta = -\frac{1}{2}\Omega$. So Δ acts by $(1-s^2)/4$.

5.5.3. *Mapping $S_k(N,\chi)$ to $\mathcal{A}_0$.* Given $f \in S_k(N,\chi)$ we associate a function ϕ_f on $G(\mathbb{A})$. Write $g = \gamma g_\infty k_0$ with[22] $\gamma \in G(\mathbb{Q})$, $g_\infty \in G_\infty$, $k_0 \in K_f(N)$. Set

$$\phi_f(g) = f(g_\infty \cdot i) j(g_\infty, i)^{-k} \widetilde{\chi}(k_0).$$

Here $\widetilde{\chi} : K_f(N) \to S^1$ is a character of the level group with $\widetilde{\chi}_p = 1$ for $p \nmid N$. When $p \mid N$, $\widetilde{\chi}_p \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi([a])$ where $[\cdot] : \mathbb{Z}_p^\times \to (\mathbb{Z}_p^\times / N\mathbb{Z}_p^\times)^\times \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$. This map is well defined because $G(\mathbb{Q})^+ \cap G_\infty K_f(N) = \Gamma_0(N)$. What is the image of this map? They are special types of automorphic forms.

Proposition 5.5.1. *Consider the set of all $\phi \in \mathcal{A}_0$ such that ϕ is right translation invariant by $K_f(N)$, of K_∞ type k, having central character χ and Ω eigenvalue $\frac{(k-1)^2-1}{2}$. Then $f \mapsto \phi_f$ is an isomorphism from $S_k(N,\chi)$ to this space.*

Injectivity is easy. Surjectivity is not too difficult. The k type tells you the lowest point in the discrete series. The conditions on the central character and the action of Ω are equivalent to f having character χ and being killed by the order one differential operator $d/d\bar{z}$. This last equation is exactly the condition for f to be holomorphic.

5.6. **Hecke theory and the converse theorem.** The final section of these notes will be to apply the theory we have developed, first to discuss the theory of L-functions attached to a global representation, and second to apply this theory to prove Weil's converse theorem. We will see how the representation theoretic viewpoint makes the converse theorem actually quite easy to prove. We first state the theorems:

Theorem 5.6.1. *Let $\pi = \otimes_v \pi_v$ be a irreducible admissible $\mathcal{H}$-module and assume π occurs in $\mathcal{A}$. Define $L(s,\pi) = \prod_v L(s,\pi_v)$. Then*

(i) *$L(s,\pi)$ and $L(s,\widetilde{\pi})$ converge in a right half plane and can be meromorphically continued to $\mathbb{C}$.*
(ii) *If $\pi \subset \mathcal{A}_0$, the continuations of $L(s,\pi)$ and $L(s,\widetilde{\pi})$ are holomorphic on all of $\mathbb{C}$. If $\pi \not\subset \mathcal{A}_0$, they have finitely many poles. In either case they are bounded in any finite width vertical strip (not containing the poles).*
(iii) *$L(s,\pi) = \epsilon(s,\pi) L(1-s,\widetilde{\pi})$, with $\epsilon(s,\pi) = \prod \epsilon(s,\pi_v,\psi_v)$ where $\psi : \mathbb{A}/F \to \mathbb{C}$ is a character.*

Remark. We will see that $\epsilon(s,\pi)$ does not depend on the choice of ψ even though each local ϵ-factor does.

Theorem 5.6.2 (Converse Theorem)**.** *Let $\pi = \otimes_v \pi_v$ be an irreducible admissible $\mathcal{H}$-module. Then π occurs in $\mathcal{A}$ if and only if for all quasicharacters $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}$ $L(s,\pi\otimes\chi)$ is holomorphic and satisfies*

$$L(s,\pi\otimes\chi) = \epsilon(s,\pi\otimes\chi) L(1-s,\widetilde{\pi}\otimes\chi').$$

[Include a discussion of the classical formulation of the converse theorem here.]

[22] For any choice of N, $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K_f(N)$.

Remark. One advantage of the number theoretic method is that it is it generalizes Weil's original proof to any number field without making the proof any more difficult.

5.6.1. *Global Whittaker model.* Fix $\psi : \mathbb{A}_F/F \to \mathbb{C}$ a character. Let $\pi = \otimes\pi_v$ be an irreducible admissible $\mathcal{H}$-module and all π_v infinite dimensional.

Proposition 5.6.3. *There exists a unique space $W(\pi,\psi)$ of continuous functions on $G_{\mathbb{A}}$ with the following properties:*

- $W\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g\right) = \psi(x)W(g)$
- *$W(\pi,\psi)$ is right invariant $\mathcal{H}$ and this action is $\simeq \pi$.*
- *For each $v \mid \infty$ there exists $N \in \mathbb{R}$ such that $W\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) = O(|a|^N)$ as $a \to \infty$ in $F_v^\times$.*

Proof. We begin with existence. For each v there exists local Whittaker model $W(\pi_v,\psi_v)$. For almost all v, π_v is spherical. Therefore, for such v, let W_v° be the spherical vector of $W(\pi_v,\psi_v)$ such that $W_v^\circ(e) = 1$. Consider $\otimes_{vW_v^\circ} W(\pi_v,\psi_v)$ and define

$$W(\pi,\psi) := \{W \in \prod_v W_v(g_v) : W_v(e_v) = 1 \quad \otimes W_v \in \otimes_{vW_v^\circ} W(\pi_v,\psi_v)\}.$$

Uniqueness follows from local uniqueness. We leave this as an exercise or one can look at the book. $\square$

5.6.2. *Fourier expansion.* Suppose π is an irreducible component in $\mathcal{A}_0$, $\psi : \mathbb{A}_F/F \to \mathbb{C}$, $\varphi \in \pi$,

$$W_\varphi^\psi(g) = \int_{F\backslash\mathbb{A}_F} \varphi[\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g]\psi^{-1}(x)dx.$$

Then the span of $\{W_\varphi^\psi \mid \varphi \in \pi\}$ is a Whittaker model of π. If $\xi \in F$ then ψ_ξ-Whittaker model is determined by the ψ-model:

$$\begin{aligned}
W_\varphi^{\psi_\xi}(g) &= \int_{F\backslash\mathbb{A}_F} \varphi[\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g]\psi_\xi^{-1}(x)dx \\
&= \int_{F\backslash\mathbb{A}_F} \varphi[\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g]\psi^{-1}(\xi x)dx \\
&= \int_{F\backslash\mathbb{A}_F} \varphi[\begin{pmatrix} 1 & \xi^{-1}x \\ & 1 \end{pmatrix} g]\psi^{-1}(x)dx \\
&= \int_{F\backslash\mathbb{A}_F} \varphi[\begin{pmatrix} \xi^{-1} & \\ & 1 \end{pmatrix} \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right) g]\psi^{-1}(x)dx \\
&= \int_{F\backslash\mathbb{A}_F} \varphi[\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right) g]\psi^{-1}(x)dx \\
&= W_\varphi^\psi\left(\left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right) g\right).
\end{aligned}$$

Since every character is of the form ψ_ξ, as a consequence

$$\varphi\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g\right) = \sum_{\xi\in F} W_\varphi^{\psi_\xi}(g)\psi_\xi(x) = \sum_{\xi\in F} W_\varphi^\psi\left(\left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right) g\right)\psi(\xi x).$$

We can recover ϕ from W_ϕ^ψ by setting $x = 0$:

$$\varphi(g) = \sum_{\xi\in F} W_\varphi^\psi(\left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right) g).$$

This is crucial for the converse theorem as we will see.

By uniqueness, the Whittaker model defined from the Fourier coefficients must coincide with the Whittaker model obtained from local the Whittaker models. So, one can modify the local identification $\pi_v \simeq W(\pi_v, \psi_v)$ such that **ADD DIAGRAM** (One has $e_v \mapsto W_v$ for almost all v. Then the remaining choices can be made so that the diagram commutes.

5.6.3. *Functional equation for global* Ψ*-integral.*

Proposition 5.6.4. *Let* π *be an irreducible component of* $\mathcal{A}_0$ *with central character* η*. Then there exists* $s_0 \in \mathbb{R}_{>0}$ *such that for all* $W \in W(\pi, \psi)$

$$\Psi(g,s,W) = \int_{\mathbb{A}_F^\times} W\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g\right) |a|^{s-1/2}\, d^\times a,$$

$$\widetilde{\Psi}(g,s,W) = \int_{\mathbb{A}_F^\times} W\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g\right) |a|^{s-1}\, \eta^{-1}(a) d^\times a$$

converge absolutely if $\mathrm{Re}(s) > s_0$*. Furthermore,* Ψ *and* $\widetilde{\Psi}$ *have holomorphic continuation to* $\mathbb{C}$ *and satisfy*

$$\Psi(wg, 1-s, W) = \Psi(g,s,W).$$

Proof. When $\mathrm{Re}(s) \gg 0$

$$\begin{aligned}\Psi(g,s,W) &= \int_{\mathbb{A}_F^\times} W\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g\right) |a|^{s-1/2}\, d^\times a \\ &= \int_{F^\times\backslash\mathbb{A}_F^\times} \sum_{\xi\in F^\times} W\left(\left(\begin{smallmatrix} \xi & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g\right) |a|^{s-1/2}\, d^\times a \\ &= \int_{F^\times\backslash\mathbb{A}_F^\times} \varphi[\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g]\, |a|^{s-1/2}\, d^\times a\end{aligned}$$

Similarly,

$$\widetilde{\Psi}(g,s,W) = \int_{F\backslash\mathbb{A}_F^\times} \varphi\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g\right) \eta^{-1}(a)\, |a|^{s-1/2}\, d^\times a.$$

It follows that

$$\begin{aligned}\widetilde{\Psi}(wg,1-s,W) &= \int_{F^\times\backslash\mathbb{A}_F^\times} \varphi\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) wg\right) \eta^{-1}(a)\, |a|^{1/2-s}\, d^\times a \\ &= \int \varphi\left(w\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) wg\right) \eta^{-1}(a)\, |a|^{1/2-s}\, d^\times a \\ &= \int \eta(a)\varphi\left(\left(\begin{smallmatrix} a^{-1} & \\ & 1\end{smallmatrix}\right) g\right) \eta^{-1}(a)\, |a|^{1/2-s}\, d^\times a \\ &= \int \varphi\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right) g\right) |a|^{s-1/2}\, d^\times a \\ &= \Psi(g,s,W).\end{aligned}$$

In the second equality we have used that φ is $G_{\mathbb{Q}}$-invariant on the left. Recall that $w = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$. Convergence property and being holomorphic follows from the fact that φ is cuspidal. $\square$

Proof of Theorem 5.6.1. We prove Theorem 5.6.1 in the case when $\pi \subset \mathcal{A}_0$. For example take $F = \mathbb{Q}$. Then $L(s,\pi) = \prod_v L(s,\pi_v)$ and $L(s,\widetilde{\pi}) = \prod_v L(s,\widetilde{\pi}_v)$. For almost all p, π_p is spherical, so it equals $\pi(\mu_{1,p},\mu_{2,p})$. Hence

$$L(s,\pi_p) = \frac{1}{1-\mu_{1,p}(p)p^{-s}} \cdot \frac{1}{1-\mu_{2,p}(p)p^{-s}}$$

When η is unitary π is unitary which implies that π_p is unitary. Hence $\mu_{j,p}$ are either purely imaginary or $\mu_1 = \overline{\mu_2}^{-1} = \mu$ with $\mu(x) = |x|^t$ and $\mathrm{Re}(t) \in (-1/2, 1/2)$ or maybe $(-1,1)$. So $|\mu_{j,p}(p)| < p$. This implies that $L(s,\widetilde{\pi}_v)$ and $L(s,\pi_v)$ converge on the right half plane. If η is not unitary, then twist: $|\mu_{j,p}(p)| < p^N$ for fixed N. Similar conclusion.

For for $W = \prod W_v$ we have the following:

$$\widetilde{\Psi}(wg, 1-s, W) = \Psi(g,s,W)$$

but since $\widetilde{\Psi}$ is an integral over $\mathbb{A}_F^\times$ and not $F^\times\backslash\mathbb{A}_F^\times$ we have

$$\begin{aligned}\widetilde{\Psi}(wg,1-s,W) &= \prod_v [L(1-s,\widetilde{\pi}_v)\Phi_v(wg,1-s,W)] \\ &= \prod_v [L(s,\pi_v)\Phi_v(g,s,W)] \\ &= L(1-s,\widetilde{\pi}) \prod_v \epsilon(s,\pi_v,\psi_v)\Phi_v(g,s,\pi_v).\end{aligned}$$

Cancel the $\Phi_v(g,s,\pi_v)$ to get for $\epsilon(s,\pi) = \prod_v \epsilon(s,\pi_v,\psi_v)$ and

$$L(s,\pi) = \epsilon(s,\pi)L(1-s,\widetilde{\pi}).$$

For almost all v, $\epsilon(s,\pi_v,\psi_v) = 1$, hence $\epsilon(s,\pi)$ is an exponential function. $\square$

The advantage of this method is that we can handle all levels simultaneously.

Proof of Converse Theorem. Given $\pi = \otimes\pi_v$ satisfying the listed conditions, we want to show that π occurs in $\mathcal{A}_0$. We use the Whittaker models to define a map

$$\pi \simeq \otimes W(\pi_v,\psi_v) \to \mathcal{A}_0.$$

For $W = \prod W_v$ with $\otimes W_v \in \otimes_v^\circ W(\pi_v,\psi_v)$ set

$$\phi_W(g) = \sum_{\xi\in F^\times} W\left(\begin{pmatrix}\xi & \\ & 1\end{pmatrix} g\right).$$

Then ϕ_W is left invariant by $B(F^\times)$. Need to check that ϕ_W is left invariant by $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. This would imply (Bruhat decoposition) ϕ_W is a function on $G(F)\backslash G(\mathbb{A})$ and the conditions of ϕ_W being an automorphic form follow from the conditions for Whittaker functions naturally. This implies that $\pi \hookrightarrow \mathcal{A}_0$ via $W \mapsto \phi_W$. So π occurs in $\mathcal{A}_0$.

To show $\phi_W(wg) = \phi_W(g)$ one can argue $\phi_W\left(w\begin{pmatrix}x & \\ & 1\end{pmatrix}g\right) = \phi_W\left(w\begin{pmatrix}x & \\ & 1\end{pmatrix}g\right) = \phi_W\left(\begin{pmatrix}x & \\ & 1\end{pmatrix}g\right)$ as functions on $F^\times\backslash\mathbb{A}_F^\times$. To do this one needs only to show that for all characters $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ the Mellin transform

$$\int \Phi_W\left(w\begin{pmatrix}x & \\ & 1\end{pmatrix}g\right)\chi(x)\,|x|^{s-1/2}\,d^\times x = \int \phi_W\left(\begin{pmatrix}x & \\ & 1\end{pmatrix}g\right)\chi(x)\,|x|^{s-1/2}\,d^\times x$$

But the first of these integrals equals

$$\widetilde{\Psi}(wg,1-s,\chi^{-1}\otimes\widetilde{\pi}) = \Psi(g,s,\chi\otimes\pi)$$

and the second is equal to $\Psi(g, s, W)$. Thus it is true by assumption that the L-factors satisfy the functional equation. □

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