INSTRUCTIONAL CONFERENCE ON REPRESENTATION THEORY AND ARITHMETIC

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Contents

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Goal of Conference

These are notes for an instructional conference on "Representation Theory and Arithmetic" held at Northwestern University May 5-9, 2008. The premise of the conference was to explain the representation theoretic ideas that often arise in the number theory. It was expected that participants were familiar with modular forms but maybe not their generalizations to the language of representation theory. The conference goal was to explain this aspect to the participants.

1. Matt Emerton: Classical Modular Forms to Automorphic Forms

We begin with the point of view taken in Serre's Course In Arithmetic. Let

$$\mathcal{L} = \{\text{discrete rank 2 lattices in } \mathbb{C}\}.$$

Definition 1. *A* modular form of weight k (and level 1) *is a function* $f : \mathcal{L} \to \mathbb{C}$ *satisfying*

- f *is holomorphic,*
- $f(\alpha L) = \alpha^{-k} f(L)$ *for all* $\alpha \in \mathbb{C}^\times$ *and* $L \in \mathcal{L}$*,*
- f *satisfies a certain growth condition.*

The reason that higher levels need also be considered is related to the fact that $\mathrm{SL}_2(\mathbb{Z})$ has a large commenserate group in $\mathrm{SL}_2(\mathbb{R})$. Two subgroups of $\mathrm{SL}_2(\mathbb{R})$ are commenserable if their intersection has finite index in each other. From this fact, the notions of adelic groups, Hecke operators and so forth result.

We define the *lattices with level N structure* to be

$$\mathcal{L}(N) := \{(L, j) \mid L \in \mathcal{L}, j : (\mathbb{Z}/N\mathbb{Z} \to L/NL\}.$$

Then the forgetful functor gives a surjection

$$\mathcal{L}(N) \longrightarrow \mathcal{L}.$$

$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts in the natural way on $\mathcal{L}(N)$, and the above map can be considered as a covering space with $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ being the group of deck transformations.

If $\alpha \in \mathbb{C}^\times$ then $\alpha : L \to \alpha L$ is an isomorphism. This induces an isomorphism $\alpha_N : L/NL \to \alpha L/N\alpha L$. With this we can now define modular forms of higher levels.

Definition 2. *A* modular form of weight k and level N *is a function* $f : \mathcal{L} \to \mathbb{C}$ *satisfying*

- f *is holomorphic,*
- $f(\alpha L, \alpha_N \circ j) = \alpha^{-k} f(L, j)$ *for all* $\alpha \in \mathbb{C}^\times$ *and* $(L, j) \in \mathcal{L}(N)$*,*
- f *satisfies a certain growth condition.*

Let B be the *based lattices*, in other words B is the space of real bases for $\mathbb{C}$.

$$\begin{aligned} B &= Iso_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}) \\ &= \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1, z_2 \text{ are } \mathbb{R} \text{ linearly independant}\} \end{aligned}$$

These distinct models are related by the map $\iota \mapsto (\iota(1,0), \iota(0,1))$. Note that the second characterization of B implies that B is open subset of $\mathbb{C}^2$ and hence has a complex structure.

$\mathrm{GL}(\mathbb{C}) \supset \mathbb{C}^\times$ acts on B on the left, and change of basis gives an action of $\mathrm{GL}_2(\mathbb{R})$ on the right. This gives a third characterization of B:

$$\begin{array}{ccc} B & \longrightarrow & \mathbb{C} \setminus \mathbb{R} \times \mathbb{C}^\times \\ (z_1, z_2) & \mapsto & \left(\frac{z_1}{z_2}, z_2\right) =: (\tau, z). \end{array}$$

In this model, the actions given above (at least of $\mathbb{C}^\times$ and $\mathrm{GL}_2(\mathbb{R})$) are easily checked to be:

$$(\tau, z) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\frac{a\tau + c}{b\tau + d}, (b\tau + d)z\right),$$

$$\alpha(\tau, z) = (\tau, \alpha).$$

(b and c are switched in the above formula compared to the standard formula because we have a right action.)

Fixing $i \in B$, the actions of $\mathrm{GL}(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{R})$ are simply (?) transitive, meaning that $\mathrm{GL}(\mathbb{C})i \simeq B \simeq i\mathrm{GL}_2(\mathbb{R})$. Therefore, $\mathrm{GL}(\mathbb{C}) \simeq \mathrm{GL}_2(\mathbb{R})$. (The only difference being that $\mathbb{C}$ doesn't have a canonical choice of $\mathbb{R}$ basis.) Then

$$\begin{aligned} (B \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))/\mathrm{GL}_2(\mathbb{Z}) &\to \mathcal{L} \\ (i, g) &\mapsto (i(\mathbb{Z}^2), i_N \circ g^{-1}) \end{aligned}$$

is an isomorphism. Note that $\mathrm{GL}_2(\mathbb{Z})$ acts diagonally on $B \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ in the obvious way and $i_N : (\mathbb{Z}/N\mathbb{Z})^2 \to i(\mathbb{Z}^2)/i(N\mathbb{Z}^2)$ is the map induced from $i : \mathbb{Z}^2 \to i(\mathbb{Z}^2)$ as we have seen before.

In terms of our other description $\mathbb{C} \setminus \mathbb{R} \times \mathbb{C}^\times$ a modular form f of level N corresponds to a function

$$\widetilde{f} : \mathbb{C} \setminus \mathbb{R} \times \mathbb{C}^\times \to \mathbb{C}$$

given by

$$\widetilde{f}\left(\frac{a\tau + c}{b\tau + d}, g \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = (b\tau + d)^k f(\tau, g), \tag{1}$$

that satisfies an appropriate growth condition. It is not hard to see that (1) implies that $\widehat{f}$ is determined by its restriction to $\mathbb{H} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

We give an explanation of how the classical notion of level appears in $\widehat{f}$. Suppose that $\widehat{f}(\tau, \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) g) = \widehat{f}(\tau, g)$ for all $\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Then we can assume that g has determinate 1 by multiplying by such an element. Then $\widehat{f}\mid_{\mathbb{H}}: \mathbb{H} \to \mathbb{C}$ is a function satisfying $\widehat{f}(\frac{a\tau+c}{b\tau+d} = (b\tau + d)^k \widehat{f}(\tau)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Similarly, invariance by $\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$ would give such a condition for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$.

1.1. **The Growth Condition.** Remark: for each $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ the action of $\left(\begin{smallmatrix} 1 & N \\ & 1 \end{smallmatrix}\right)$ gives a Fourier series for $f(\tau) = \widehat{f}(\tau, g)$. Then the growth condition says that for all g these Fourier series have no nonzero negative terms.

1.2. **Passage to Representation Theory.** Fix $i_0 = (i, 1)$. Then this gives an isomorphism between $\mathrm{GL}(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{R})$. The restriction to $\mathbb{C}^\times$ is given by $a+bi \mapsto \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. So

$$f : (\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))/\mathrm{GL}_2(\mathbb{Z}) \to \mathbb{C}$$

satisfies

$$f(\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) g_1, g_2) = (a+bi)^k f(g_1, g_2).$$

Moreover, it can be checked that the growth condition as given above implies a polynomial bound on $|f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) g_1, g_2)$ in terms of a, b, c, d.

What does "holomorphic" mean in these coordinates?

The basic idea is that the Cauchy-Riemann equations can be expressed in terms of differential operators, and the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$ acts by differential operators as well, so there one just needs to find the right relation between these two ideas.

By the choice of i_0 there is an isomorphism

$$\mathrm{GL}_2(\mathbb{R}) \longrightarrow i_0 \mathrm{GL}_2(\mathbb{R}) = B.$$

Since these are both smooth manifolds, one can consider the corresponding linear map on the tangent spaces to the identity:

$$T_1 \mathrm{GL}_2(\mathbb{R}) \longrightarrow T_{i_0} B \tag{2}$$

The left side of (2) is $\mathfrak{g}l_2 = M_2(\mathbb{C})$. The right side of (2) contains $(T_{i_0}B)^{hol} = \{\partial z_1, \partial z_2\}$. The question is now to determine what the subspace of $\mathfrak{g}l_2$ is that corresponds to $(T_{i_0}B)^{hol}$ is. This is equivalent to describing $(T_{i_0}B)^{hol}$ in terms of certain 1-parameter subgroups.

Write $z_j = x_j + iy_j$. Then

$$(i, 1) \left(\begin{smallmatrix} a & 0 \\ c & 1 \end{smallmatrix}\right) = (c + ai, 1).$$

So, fixing $a = 1$ and varying c, one finds that $\partial x_1 |_{i_0} = \left(\begin{smallmatrix} 0 & \\ 1 & 0 \end{smallmatrix}\right)$, and by fixing $c = 0$ and varying a, $\partial y_i |_{i_0} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. In a similar fashion we take

$$(i, 1) \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) = (i, a + bi)$$

to determine that $\partial x_2 |_{i_0} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\partial y_2 |_{i_0} = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$. Putting this together, we have

$$\partial z_1 |_{i_0} = \frac{1}{2}(\partial x_1 |_{i_0} - i\partial y_1 |_{i_0}) = -\frac{1}{4}(Z + H + Y_+)$$

and

$$\partial z_2 |_{i_0} = \frac{1}{2}(\partial x_2 |_{i_0} - i\partial y_2 |_{i_0}) = \frac{1}{2}(Z + H)$$

where

$$Z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, H = \begin{pmatrix} 0 & -1 \\ 1 & \end{pmatrix}, Y_\pm = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}.$$

This tells us that $T_{i_0}B^{hol}$ corresponds the subspace of $\mathfrak{g}l_2$ generated by $\langle Z + H, Y_+\rangle$, and $T_{i_0}B^{antihol}$ corresponds to that generated by $\langle Z - H, Y_-\rangle$.

More generally, if h is any element of $\mathrm{GL}_2(\mathbb{R})$ then, since the right action of $\mathrm{GL}_2(\mathbb{R})$ is holomorphic,

$$T_{i_0 h}B = \langle h^{-1}(Z + H)h, h^{-1}Y_- h\rangle.$$

Therefore, $f : \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}$ is holomorphic if and only if

$$(Z - H)f = 0 \quad \text{and} \quad V_- f = 0.$$

Actually, the first of these conditions follows from the fact that for $\mathbb{C}^\times \mathrm{GL}(\mathbb{C})$ the subspace of $\mathfrak{gl}_2$ corresponding to $\mathbb{C}^\times$ is $\langle Z, H\rangle$. The condition that f be of weight k is equivalent to $Zf = Hf = kf$.

Such a function f is an element of $C^{\infty,mod}(\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z}), \mathbb{C})$ (smooth functions of moderate growth) which is a $\mathfrak{gl}_2$-module. One can then consider the $\mathfrak{gl}_2$-module generated by $v_k \mapsto f$: $D_k^+ := \bigoplus_{n/geq0} \mathbb{C}v_{k+2n}$ where $v_{k+2n} = Y_+^n v_k$. One can verify that $Hv_i = iv_i$, $Zv_i = kv_i$ and Y_- takes $\mathbb{C}v_i$ to $\mathbb{C}v_{i-2}$. Under this correspondence, the space of modular forms of weight k and level N is

$$M_k(N) = \mathrm{Hom}_{\mathfrak{gl}_2}(D_k^+, C^{\infty,mod}(\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}), \mathbb{C})$$

Finally, one can then consider the projective limit of these spaces which equals $\mathrm{GL}_2(\mathbb{R})\times\mathrm{GL}_2(\widehat{\mathbb{Z}}/\mathrm{GL}_2(\mathbb{Z})$, and by strong approximation this is equal to $\mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathbb{Q})$. So the (even larger) space from which the modular forms can be obtained is smooth functions on this final adelic space of moderate growth.

2. David Nadler: Real Lie Groups

2.1. Basic Notions.

Definition 3. *A* Lie Group *G is a group in the category of (smooth) manifolds, which means that G is a manifold with maps*

$$m : G \times G \to G, i : G \to G, \dots$$

that are morphisms in the category of smooth manifolds, i.e. differentiable maps.

Examples:

- Finite groups. (The ability to really use the differentiable structure is lost for these groups, but they still show up in the study of Lie groups, for example, as the group of connected components and as quotient groups.)
- $\mathbb{R}^n$ with addition.
- $T^n = R^n/\mathbb{Z}^n$.
- Small dimensional spheres: $S^0 \simeq \mathbb{Z}/2\mathbb{Z}$, $S^1 \simeq SO_2(\mathbb{R})(\simeq U(1) \simeq T^1)$, $S^3 = SU(2) =$ unit quaternions.

Notice that these examples are all compact groups.

A *matrix representation* of a group is a map $\rho : G \to \mathrm{GL}(V)$ where V is a finite dimensional vector space over $\mathbb{C}$. Morally, all of the Lie groups that we care about have some such ρ that is injective. So, choosing a basis of V, we can think of Lie groups as being subgroups of $\mathrm{GL}_n(\mathbb{C})$ for some n.

2.2. Examples.

- $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$,
- (volume preserving) $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$,
- (quadratic form preserving) $SO_n(\mathbb{C})$, $SO(p,q)$,
- (symplectic form preserving) $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{Sp}(p,q)$.

(He originally planned to have a section on "Rational Forms." It's likely that he intended to talk more about these groups in that section, but he ran out of time before getting to it.)

2.3. **Classification.** There are two fundamental methods that are used to study Lie groups–the first, exploiting the manifold structure, and the second the group structure.

2.3.1. *Linearize.* This is the notion of the Lie algebra. It is formed by considering the conjugation map
$$G \times G \to G \quad (g, h) \mapsto g^{-1}hg,$$
and differentiating twice. First, differentiating with respect to h gives
$$Ad : G \times T_1G \to T_1G,$$
and differentiating again with respect to g gives
$$ad : T_1G \times T_1G \to T_1G.$$

We denote $\mathfrak{g} = T_1G$ and $[\cdot,\cdot] = ad$. These are the familiar Lie algebra and Lie bracket. Concretely, if $G \hookrightarrow \mathrm{GL}(V)$ then $\mathfrak{g} \hookrightarrow \mathfrak{g}l(V) = End(V)$ and $[\cdot,\cdot]$ is the usual bracket. For example, $SO_n(\mathbb{R})$ has Lie algebra $\mathfrak{so}_n(\mathbb{R})$ which, under the identification $SO_n(\mathbb{R}) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$, consists of trace zero skew-symmetric matrices in $\mathfrak{g}l_n(\mathbb{R}) = M_n(\mathbb{R})$.

Just as differentiation provides a map $G \to \mathfrak{g}$, there is an exponential map $\exp : \mathfrak{g} \to G$. Each vector $X \in \mathfrak{g}$ determines a one parameter subgroup $\gamma : \mathbb{R} \to G$ such that $\gamma(0) = 1$ and $\gamma'(0) = X$. Then $\exp(X) = \gamma(1)$.

"Picture" in two examples:

(1) $G = SU(2)$. He drew a picture of S^2 with the origin at a pole, and said appealled to our intuition to believe that one-parameter subgroup corresp-donding to a tangent vector at the identity would give a great circle through the opposite pole eventually returning to the identity again. Geometrically, $SU(2) = S^3$, and this same idea applies there making the exponential map surjective.

(2) $G = \mathrm{SL}_2(\mathbb{C})$. Since $SU(2) \subset \mathrm{SL}_2(\mathbb{C})$ his picture started with a (two) sphere to which he added cones at each of the poles which represent nilpotent elements such as $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right) \notin SU(2)$. Elements not in the sphere itself nor on the cones include things such as $\begin{pmatrix} 2 & \\ & \frac{1}{2} \end{pmatrix}$.

Exercise 4. *For the second example above, determine the exponential map. (Remark: in a small neighborhood of the identity, the exponential map is a bijection.) Guess:* $\begin{pmatrix} 2 & \\ & \frac{1}{2} \end{pmatrix}$ *is not in the image of* exp, *and, more generally, the image is only that which is pictured in the cones or on the sphere.*

A final reason that linearization is a useful tool is that
$$\left\{ \begin{array}{c} \text{Connected and simple} \\ \text{connected Lie groups} \end{array} \right\} \leftrightarrow \{\text{Lie algebras}\}$$

Definition 5. *A Lie algebra* $\mathfrak{g}$ *is* simple *if it is not abelian and it has no nontrivial ideals.* $\mathfrak{g}$ *is* semisimple *if it is a direct sum of simples.* $\mathfrak{g}$ *is* reductive *if* $\mathfrak{g}$ *is a direct sum of semisimple and a center. We use the same adjectives for a Lie group* G *if its Lie algebra satisfies the particular property.*

The affine symmetries of $\mathbb{R}^k$ is a Lie group $\mathrm{GL}_k(\mathbb{R}) \rtimes \mathbb{R}^k$ which is a matrix group with elements $\begin{pmatrix} K & v \\ 0 & 1 \end{pmatrix}$ where $K \in \mathrm{GL}_k(\mathbb{R})$, $v \in \mathbb{R}^k$ (and 0 is the $1 \times k$ zero vector.) We don't like groups of this type.

Examples of each: $\mathrm{GL}_n(\cdot)$ is reductive, $SO(4)$ is semisimple, $\mathrm{SL}_2(\cdot)$ is simple. (I don't know how the affine symmetries group fits into this picture, or why he says we don't like it, but my guess is that it is not a reductive group.)

Exercise 6. *Prove the following*

- *$SO(4)$ is semisimple but not simple.*
- *Any Lie algebra $\mathfrak{g}$ fits into an exact sequence*
$$0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{b} \to 0$$
where $\mathfrak{a}$ is solvable and $\mathfrak{b}$ is semisimple.
- *If G is compact then it is reductive.*

2.3.2. *Group Theory.* Since abelian groups are easier than nonabelian ones, one might attempt first to study the abelian subgroups of a Lie group G. Two 1-dimensional examples of abelian Lie groups are $\mathbb{R}^\times$ (which is cut out by $xy = 1$, and S^1 which is cut out by $x^2 + y^2 = 1$. Note that over $\mathbb{R}$ these equations give different topological groups, but over $\mathbb{C}$ they coincide. This has to do with the notion of splitting described below.

Remark: $\mathbb{R}$ does not embed in $\mathrm{GL}(V)$ in any way such that it is closed under transpose.

Assume that G is a reductive Lie group. A *maximal torus* $T \subset G$ is an abelian subgroup that is diagonalizable and maximal among such. If $G = \mathrm{SL}_2(\mathbb{R})$ then

$$T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^\times \right\} \simeq \mathbb{R}^\times, and T = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \simeq S^1$$

are maximal tori. The second is *non split.*

Definition 7. *A* maximal split torus *of a Lie group G is a maximal torus $T \subset G$ with the property that $(\mathbb{R}^\times)^k \hookrightarrow T$ with k maximal.*

Some exampls of reductive groups: $S^1 \simeq SO_2(\mathbb{R})$, $SU(2)$, $\mathrm{SL}_2(\mathbb{R})$.

Exercise 8. *Do the following:*

- *Classify (finite dimensional) representations of $SO_2(\mathbb{R})$.*
- *Classify (finite dimensional) representations of $SU(2)$. (Hint(?): spherical harmonics)*
- *Show that all maximal tori of* $\mathrm{SL}_2(\mathbb{R})$ *are conjugate to one of the two seen above.*

Some pictures:

(1) $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C})$ which acts on $\mathbb{P}^1(\mathbb{C})$. He drew the Riemann sphere with 0 and ∞ at poles and an equator separating the upper and lower half planes $\mathbb{H}$ and $\mathbb{H}_-$. $\mathrm{SL}_2(\mathbb{R})$ preserves each half plane.
(2) $\mathrm{SL}_2(\mathbb{R})$ acts on its Lie algebra $\mathfrak{sl}_2(\mathbb{R}) \simeq \mathbb{R}^3$. He then drew some $\mathrm{SL}_2(\mathbb{R})$ orbits in $\mathbb{R}^3$. There were three types: nilpotent (cone), elliptic (two compenents each a parabaloid) and hyperbolic (a hyperboloid.)

2.4. **Useful Decompositions.** Assume that G is the real points of a some polynomial equations defined over $\mathbb{R}$ in $\mathrm{GL}_n(\mathbb{C})$ for some n. Also, assume that G is reductive. Then there exist decompositions called *Cartan*, *Iwasawa* and *Bruhat.* In the case $G = \mathrm{SL}_2(\mathbb{R})$ these decompositions are:

G	Iwasawa	Bruhat	Cartan
$\mathrm{SL}_2(\mathbb{R})$	KA^+N	$B\widetilde{W}B = NN_G(A)N$	KA^+K

where $K = SO_2(\mathbb{R})$, $A^+ = \{\left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right) \mid a > 0\}$, $N = \{\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right) \mid x \in \mathbb{R}\}$.

3. Jacob Lurie: Lie Theory and Algebraic Groups

Definition 9. *A* Lie algebra *over* $\mathbb{C}$ *is a* $\mathbb{C}$*-vector space* V *with a map* $[\cdot,\cdot] : V \otimes V \to V$ *satisfying*

1. $[x, y] = -[y, x]$,
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

If A is any associative $\mathbb{C}$-algebra then $[x, y] = xy - yx$ makes A a Lie algebra. If $\mathfrak{g}$ is a Lie algebra, $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/([x, y] - xy + yx)$, the universal envoloping algebra, is an associative algebra. These notions are adjoint:

$$\mathrm{Hom}_{assoc-alg}(\mathcal{U}(\mathfrak{g}), A) = \mathrm{Hom}_{Lie-alg}(\mathfrak{g}, A).$$

If M is a manifold then V the set of tangent vector fields on M, acts on smooth functions $f : M \to \mathbb{C}$ and is a Lie algebra. $[v, u](f) = v(u(f)) - u(v(f))$. If G is a Lie group we consider the set of left invariant vector fields. These are closed under the bracket operation, and so form a Lie subalgebra whose dimension is $\dim G$. It is isomorphic to the tangent space at the identity. This is the Lie algebra of G. Under this mapping from Lie groups to Lie algebras we have the following picture:

$$\left\{\begin{array}{c}\text{connected and simply}\\ \text{connected Lie groups}\end{array}\right\} \subset \{\text{Lie Groups}\} \to \left\{\begin{array}{c}\text{Finite dimensional}\\ \text{Lie algebras}\end{array}\right\},$$

and, moreover, the left and right sides are equivalent. In other words, the Lie algebra of G is unique up to the connected component of the universal covering space $\widetilde{G}^0$.

Let G be a Lie group with $\mathfrak{g}$ its Lie algebra. Then, as mentioned before, there is a map $\exp : \mathfrak{g} \to G$. In the case $G = \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$, $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$. The following explains why the category of connected simply connected Lie groups and the category of Lie algebras are equivalent.

Theorem 10 (Campbell Hausdorff formula)**.** *If* $A, B \in \mathfrak{g}$ *then*

$$\exp(tA)\exp(tB) = \exp(t(A + B) + \frac{t^2}{2}[A, B] + \cdot)$$

converges and reconstructs the group structure of G *in a neighborhood of 1.*

Sketch of Proof. Note that $\exp(tA)\exp(tB) \in 1 + t\mathcal{U}(\mathfrak{g})[[t]]$.

$\mathcal{U}(\mathfrak{g})$ is a Hopf algebra. $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ via $x \mapsto 1 \otimes x + x \otimes 1$ $(x \in \mathfrak{g})$. Such an element of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is called *primitive*, and $\mathfrak{g} = Prim(\mathcal{U}(\mathfrak{g}))$, the set of primitive elements.

$x \in 1 + t\mathcal{U}(\mathfrak{g})[[t]]$ if and only if $\exp(x)$ is *group like*, meaning $x \mapsto x \otimes x$. Group like elements are closed under multiplication. □

Another "proof" that $\mathfrak{g}$ *determines* G. G acts on itself by conjugation, and so it acts on $\mathfrak{g}$. This makes $\mathfrak{g}$ a representation of G (the adjoint.)

$$\phi : G \to \mathrm{Aut}_{Lie-alg}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$$

Assertion: $\mathfrak{g}$ semisimple implies that ϕ is a local isomorphism. In other words, we recover G as something defined purely in terms of the Lie algebra (up to connect component and simply connectedness.) Actually

$$0 \to Z(G) \to G \to \mathrm{Aut}(\mathfrak{g})$$

is exact. This implies that $\mathrm{Aut}(\mathfrak{g}) \simeq G^0/Z(G^0)$. □

Definition 11. *A connected semisimple group G is called* adjoint *if $Z(G)$ is the identity.*

Corollary 12. *If G is adjoint then G has the structure of a linear algebraic group. (G is the $\mathbb{R}$-points of a linear algebraic group of over $\mathbb{R}$.)*

In general, G fits into an exact sequence: $\widetilde{G} \to G \to G/Z$ where $\widetilde{G}$ is simply connected (the universal covering space) and G/Z is adjoint. For example, if $G = SO_{2n}(\mathbb{C})$ then $\widetilde{G} = Spin_{2n}(\mathbb{C})$ and $G/Z = PSO_2(\mathbb{C})$.

3.1. **Classification.** From now on we assume that all groups and Lie algebras are over $\mathbb{C}$.

If G is semisimple then $Z(G)$ finite implies that $G \to G/Z(G)$ is a finite cover of an algebraic group, and so G itself must be algebraic.

In the diagram

$$\begin{array}{ccc} \{\text{linear algebraic groups}\} & \longrightarrow & \{\text{Lie groups}\} \\ \cup| & & \cup| \\ \{\text{semisimple lin. alg. groups}\} & \longrightarrow & \{semisimple Lie groups\} \end{array}$$

the bottom arrow is an isomorphism.

Recall that a *representation* of G on V is a holomorphic moap $G \times V \to V$ satisfying the usual identities. Then it is a fact that any such map is automatically algebraic. So we have three ways to describe the same thing (in the case that G is simply connected):

(1) Complex analytic representations of G.
(2) Algebraic representations of G.
(3) Maps $\mathfrak{g} \to \mathfrak{g}l_n(\mathbb{C})$ such that there is a map from G to $\mathrm{GL}_n(\mathbb{C})$ making the following commute:

$$\begin{array}{ccc} \widetilde{G} & \longrightarrow & \widetilde{\mathrm{GL}_n}(\mathbb{C}) \\ \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \end{array}$$

The goal is to understand representations of $\mathrm{GL}_2(\mathbb{C})$. The Lie algebra $\mathfrak{s}l_2(\mathbb{C})$ consists of traceless matrices in $\mathfrak{g}l_2(\mathbb{C})$. It has basis consisting of

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Lie brackets are easily checked to be

$$[H, E] = 2E, [H, F], -2F, [E, F] = H.$$

Suppose that $\mathfrak{s}l_2(\mathbb{C})$ acts irreducibly on a finite dimensional vector space V. For the action of H, choose a λ-eigenvector $v \in V$ such that the real part of λ is maximal. Since

$$2Ev = [H, E]v = H(Ev) - E(Hv),$$

Ev has H eigenvalue $\lambda + 2$, but by the assumption that λ was maximal, this says that $Ev = 0$. Similarly, one sees that $F^k v$ has H-eigenvalue $\lambda - 2k$.

Claim: $EF^k v = (k\lambda - k(k-1))F^{k-1}v$. We prove this by induction. $k = 0$ is the calculation above. Suppose the statement is true for k. Then

$$E(F^{k+1})v = HF^k v + FEF^k v = (\lambda - 2k)F^k v + (k\lambda - k(k-1))FF^{k-1}v.$$

Simplifying this expression proves the claim.

So V is spanned by $v, Fv, F^2v, \ldots$. We conclude that $F^kv = 0$ for k large. Choose n such that $F^nv \neq 0$ but $F^{n+1}v = 0$. Then

$$0 = E(F^{n+1}v) = (\lambda(n+1) - n(n+1))F^nv.$$

Since $F^nv \neq 0$ this implies that $\lambda = n$, and so $V = \langle v, Fv, \ldots, F^nv$. We have proved

Propostion 13. *Irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ (up to isomorphism) are in one-to-one correspondence with the nonnegative integers.*

Examples: $n = 0$ gives the trivial representation, $n = 1$ gives the tautological representation of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$ and $n = 2$ gives the adjoint representation on $\mathfrak{g}$.

Since a representation of (the adjoint group) $\mathrm{PGL}_2(\mathbb{C})$ is a representation of $\mathrm{SL}_2(\mathbb{C})$ on which -1 acts trivially, we see that the representations of $\mathrm{PGL}_2(\mathbb{C})$ are classified by nonnegative even integers.

4. Jacob Lurie: Representations of algebraic groups

Definition 14. *An* affine (or linear) algebraic group *is a group in the category of affine varieties. i.e. $G = Spec(A)$, and the maps $m : G \times G \to G$, $i : G \to G$ satisfy the appropriate properties.*

In terms of the ring A, this means that the $A \to A \otimes A$ is a comultiplication, and A is a commutative noncocommutative Hopf algebra.

Definition 15. *A* representation *of an algebraic group is*

1 *a vector space V*

2 *an automorphism $\eta : V \to V$ for every point η of G.*

2' *This is equivalent to saying that for every R point of G, $\eta : Spec(R) \to G$ if and only if $A \to R$, $\eta : V \otimes R \to V \otimes R$. Since the statement is functorial, it suffices to define this in the universal case $A = R$. $\eta : V \otimes A \to V \otimes A$ is determined by restriction $V \to V \otimes A$.*

3 *The diagram*

$$\begin{array}{ccc} V & \longrightarrow & V \otimes A \\ \downarrow & & \downarrow \\ V \otimes A & \longrightarrow & (V \otimes A) \otimes A \end{array}$$

is commutative.

In summary, a representation of G is a comodule over A.

Examples: $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times = \mathbb{G}_m = Spec(\mathbb{C}[t, t^{-1}]$. The comultiplication is $t \mapsto t \otimes t$. A representation is thus a vector space V with a map $V \to V[t, t^{-1}]$ $v \mapsto \sum \lambda_i(v)t^i$. Condition 3 implies that $v = \sum \lambda_i(v)$, so $\lambda_i \circ \lambda_j = \lambda_i \delta_{ij}$. This implies $V = \bigoplus_{i \in \mathbb{Z}} V_i$ where V_i is the image of λ_i. We conclude that representations of $\mathbb{G}_m$ are graded vector spaces.

Corollary 16. *Any nontrivial representation of $\mathbb{G}_m$ contains a 1-dimensional summand.*

Definition 17. *An algebraic group G is called* torus *if $G \simeq \mathbb{G}_m \times \cdot \times \mathbb{G}_m$. In this case $X^*(G)$ is the set of maps $G \to \mathbb{G}_m$.*

In general, if G is a torus then there is a one-to-one correspondence

$$\{\text{representations of } G\} \leftrightarrow \{\text{graded vector spaces by } X^*(G)\}. \tag{3}$$

Example: $(\mathbb{C},+) \simeq \mathbb{G}_a = Spec(\mathbb{C}[t])$. The comultiplication is $t \mapsto t\otimes 1 + 1\otimes t$. Representations of $\mathbb{G}_a$ correspond to vector spaces V with $V \to V[t]$, $v \mapsto \sum_{i\geq 0}\mu_i(v)t^i$. The conditions in this case are $\mu_0 = id$, $\mu_i \circ \mu_j = (i+j\ i)\mu_{i+j}$, and this implies that $\mu_n = \frac{\mu_1^n}{n!}$. Let $\mu_1 = d$, so $v \mapsto \sum \frac{d^i v t^i}{i!} = \exp(td)v$. Since this map is algebraic if follows that d is nilpotent. In summary,

$$\{\text{representations of } \mathbb{G}_a\} \leftrightarrow \{\text{Vector spaces with a nilpotent endomorphism}\}.$$

To summarize, if $\mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ is a representation, then the restriction to $\mathbb{R}$ is of the form $t \mapsto \exp(At)$ for some $A \in \mathfrak{gl}_n(\mathbb{C})$. In order to be algebraice, A must be nilpotent.

Corollary 18. *Any nonzero algebraic represntation V of $\mathbb{G}_a$ contains a nonzero fixed vector.*

Definition 19. *An algebraic group G is called* unipotent *if G has a filtration by normal subgroups*

$$0 \subset \cdot \subset G_{n-1} \subset G_n \subset G,$$

such that $G_{i+1}/G_i \simeq \mathbb{G}_a$.

For example, upper trianglar unipotent matrices in GL_n form a unipotent group.

Corollary 20. *If U is unipotent, and U acts on a nonzero vector space V then $V^U \neq 0$.*

Definition 21. *An algebraic group G is* solvable *if B is a semidirect product of a torus acting on a unipotent group.*

This definition is not the "right" definition. The real definition is that G has filtration

$$0 \subset \cdot \subset G_{n-1} \subset G_n \subset G,$$

such that $G_{i+1}/G_i \simeq \mathbb{G}_a$ or $\mathbb{G}_m$.

An example of a solvable group is B the group of upper triangular matrices in GL_n.

Definition 22. *Let G be an algebraic group. A* Borel subgroup *$B \subset G$ is a subgroup such that B is solvable and G/B is proper (compact.)*

If $G = \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$ $(V = \langle v_1, \dots, v_n\rangle)$ then B, as above, is the stabalizer of the flag

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

with $V_i = \langle v_1, \dots, v_i\rangle$. A *flag* of V is a nested sequence of vectors spaces as above. Then $\mathrm{GL}(V)$ acts transitively on the space of flags, so G/B is the collection of all flags in V. This is compact. In the case $n = 2$, $G/P = \mathbb{CP}^1 \simeq S^1$.

Theorem 23. *Every algebraic group admits a Borel subgroup.*

Idea of proof. There is an injection $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ for some n. One might try $B = (B_0 \cap G)$ where B_0 is the standard Borel of $\mathrm{GL}_n(\mathbb{C})$. One just needs to show that G/B is compact. In other words, want to show

$$G/B \to \mathrm{GL}_n(\mathbb{C})/B_0 = X$$

is closed. This won't necessarily be the case. But G acts on X, so choose $x \in X$ such that $Gx \subset X$ is closed. Choosing B_0 to be the stabalizer of x now works. $\square$

Definition 24. *So if G is an algebraic group and B a Borel subgroup $B = T \ltimes U$. T is called a* maximal torus *of G. $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r$ for some r. The number r is called the* rank *of G.*

Since G acts on $\mathfrak{g}$, $\mathfrak{g}$ is a representation of T. This gives a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha.$$

$\alpha \in X^*(T)$ is called a *root* (of G) if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. (If G is semisimple, $\mathfrak{g}_0 = Lie(T)$.)

Example: $G = \mathrm{SL}_2$. $B = T \ltimes U$.

$$T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right\} \simeq \mathbb{G}_a, \quad U = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\}.$$

So $X^*(T) \simeq \mathbb{Z}$. The Lie algebra of T is generated by H, and

$$\mathfrak{s}l_2(\mathbb{C}) = \mathbb{C}F \oplus \mathbb{C}H \oplus \mathbb{C}E = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus g_2.$$

So the roots of SL_2 can be identified with $\{-2, 2\} \subset \mathbb{Z} \simeq X^*(T)$.

Example: $G = \mathrm{GL}_n$. The Borel $B \simeq T \ltimes U$ with T and U as above. In particular $T \simeq (\mathbb{G}_m)^n$, so $X^*(T) \simeq \mathbb{Z}^n$ with basis, say, $\{e_1, \ldots, e_n\}$. Then

$$\mathfrak{g}l_n = Lie(T) \oplus \bigoplus_{\substack{\alpha = e_i - e_j \\ i \neq j}} \mathfrak{g}_\alpha,$$

and

$$Lie(B) = Lie(T) \oplus \bigoplus_{\substack{\alpha = e_i - e_j \\ i < j}} \mathfrak{g}_\alpha.$$

Picture of the weight lattice of PGL_3 with the roots of $Lie(B)$ and $Lie(T)$ circled. Apparently, this is the picture in Harris–Fulton.

Note the following features of GL_n:

(1) $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$ with $\mathfrak{g}_0 = Lie(T)$.
(2) If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, $\dim \mathfrak{g}_\alpha = 1$.
(3) If $\mathfrak{g}_\alpha \neq 0$ then $\mathfrak{g}_{-\alpha} \neq 0$.
(4) If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$ then

$$\mathfrak{s}l_2 \simeq \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha.$$

(5) If $\mathfrak{g}_\alpha \neq 0$ then exactly one of α and $-\alpha$ is a root of B.

Theorem 25. *The above features hold if G is semisimple or, more generally, reductive.*

(G is reductive if G is an extension of a semisimple group by a torus. This a property of G that can not be deduced from its Lie algebra.)

Definition 26. *A* weight *of G is a character of the maximal torus. If μ, λ are weights, we write $\mu \leq \lambda$ if $\lambda = \mu + \sum_{\substack{r_i \geq 0 \\ \alpha_i \text{ positive}}} r_i \alpha_i$. A root α is* positive *if $\mathfrak{g}_\alpha \subset Lie(B)$ for a fixed Borel B.*

Definition 27. *The* Weyl group $W(G)$ *of a reductive algebraic group* G *is* $N(T)/T$*. (N referring to normalizer.) This is always a finite group. A weight* λ *is* dominant *if for all* $w \in W(G)$ $w\lambda \leq \lambda$*.*

Exercise 28. *Let* $G = \mathrm{GL}_n$*. Show that* $N(T) = \{\text{permutation matrices}\} \ltimes T$*, and that therefore* $W(\mathrm{GL}_n) = N(T)/T \simeq S_n$ *(the symmetric group on n letters.)*

Example: $G = \mathrm{SL}_2$, $W \simeq \mathbb{Z}/2\mathbb{Z} \simeq S_2$ has generator $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$. The weight lattice of G is $\mathbb{Z}$, and the Weyl group acts on it by -1. So λ is dominant if and only if $\lambda \geq 0$. Recall from above that these dominant weights classified the representations of $\mathfrak{sl}_2(\mathbb{C})$.

Definition 29. *Let* G *be a reductive algebraic group,* V *a representation of* G*.* B *a Borel and* $U \subset B$ *a unipotent subgroup. A* highest weight vector of weight λ *is a nonzero* $v \in V^U$ *such that* T *acts on* v *by* λ*.*

Lemma 30. *Let* G *and* V *be as in the definition. If* $V \neq 0$*, there exists a highest weight vector.*

Proof. Previously we saw that V^U is nonzero, and it is clear that $B/U \simeq T$ acts on V^U. So $V^U = \bigoplus_\lambda V^U_\lambda$. Any nonzero $v \in V^U_\lambda$ for a maximal weight λ is a highest weight vector. □

Lemma 31. *If* V *is irreducible, the highest weight vector is unique up to scaling.*

Proof. Suppose v is a highest weight vector of weight λ. Consider $V_0 \subset V$ generated by v under the action of $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{v}^- \oplus \mathcal{U}(\mathfrak{b})$. We claim that $V_0 = \bigoplus_{\mu \leq \lambda} (V_0)_\mu$ and $(V_0)_\lambda = \mathbb{C}v$. Since V is irreducible, $V_0 = V$. Thus any other highest weight vector not a multiple of v must be in V_μ for $\mu < \lambda$. But, arguing as above, we would also have $\lambda < \mu$. □

Lemma 32. *If* V *and* V' *are irreducible and both generated by a highest vector of weight* λ *then* $V \simeq V'$*.*

Proof. Consider $V \oplus V'$ with the projections ψ, ψ', and let $v \oplus v'$ be the highest weight vector of weight λ. Let $V_0 \subset V \oplus V'$ be the space generated by $v \oplus v'$. By irreducibility of V, V', V_0 must surject onto each via the projection maps. Then

$$\ker \psi \subset V_0 \subset V \oplus V' \to V'.$$

If $\ker \psi \neq 0$, $\ker \psi$ must surject onto V'. This would imply there exists $w \in \ker \psi$ such that $w \mapsto v'$, but this is not possible. □

In summary, if G is a reductive group then we have an injection

$$\{\text{Irreducible reps. of } G\}/\sim \hookrightarrow X^*(T).$$

The image is precisely the set of dominant weights.

5. Matt Emerton

In Emerton's previous talk he finished with

$$\varphi \in \mathrm{Hom}_{\mathfrak{g}}(D_k^+, C^{\infty,mod}(\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z}), \mathbb{C})).$$

D_k^+ was generated by v_k and $\varphi(v_k) \in M_k(N)$. ($C^\infty(X, Y)$ is the space of continuous functions from X to Y of moderate growth.) In this talk the goal was to discuss the general theory into which D_k^+ fits. (Note that D_k^+ is not finite dimensional, and so it is not explained by the theory of Lurie's talk.)

We will have

$$\mathrm{Hom}_{\mathfrak{g}}(V, C^{\infty,mod}(\Gamma\backslash G(\mathbb{R}), \mathbb{C}))$$

where V is some (to be seen) type of vector space, G a Lie group with Lie algebra $\mathfrak{g}$ and Γ an arithmetic subgroup. This theory is the theory of "infinite place." There are many results that one needs to be familiar with even though one (presumably a number theorist) will spend most of his life concentrating on the p-adic part, the archimedean theory can't be ignored altogether. For one, the infinite places controls "weight."

$\Gamma\backslash G(\mathbb{R})$ is a *Frechet space*: there are a bunch of seminorms. All of these give rise to a topology.

Let G be a reductive linear algebraic group over $\mathbb{R}$. (This means that $G \subset of\,\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathrm{GL}_n)$ and G is invariant under taking inverse conjugate transpose $\theta(g) = ({}^t g^*)^{-1}$.) Let $K = G^\theta$. Then

$$\begin{array}{rcl} G(\mathbb{R}) & \subset & \mathrm{GL}_n(\mathbb{C}) \\ K(\mathbb{R}) & = & G(\mathbb{R}) \cap \mathrm{GL}_n(C)^\theta \\ & = & G(\mathbb{R}) \cap U(n), \end{array}$$

and so $K(\mathbb{R})$ is compact. Actually, $K(\mathbb{R})$ is maximal compact and unique up to conjugacy. (However, $K(\mathbb{C})$ is *not* compact.) The reason that a compact group is so useful is that an irreducible action of a compact group is an algebraic representation as classified in Lurie's talk.

To describe why recall the following algebraic tautologies. When $A \to B$ is a unital map of rings, and M is an A-module and N is a B-module,

(1) $\mathrm{Hom}_B(B \otimes M, N) \simeq \mathrm{Hom}_{\mathbb{C}}(M, N)$, and
(2) $\mathrm{Hom}_B(N, \mathrm{Hom}_A(B, M)) \to^{\sim} \mathrm{Hom}_A(N, M)$.

The second tautology implies that

$$\mathrm{Hom}_{G(\mathbb{R})}(V, \text{“}\mathbb{C}^\infty\text{”}(G(\mathbb{R}), \mathbb{C})) \to \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = V^\vee.$$

(The inverse map is $v \mapsto (g \mapsto \varphi(gv))$.)

So, to understand representations of K we need only look in $C^\infty(K(\mathbb{R}), \mathbb{C})$. If $\widehat{K(\mathbb{R})}$ is the set of equivalence classes of irreducible representations of $K(\mathbb{R})$ then

$$\bigoplus_{\mu \in \widehat{K(\mathbb{R})}} \mu^\vee \oplus \mu \hookrightarrow C^\infty(K(\mathbb{R}), \mathbb{C})$$

Fourier theory implies that the image is dense. In fact the image is $\mathcal{O}(K) =$ algebraic functions on K, and a version of Weierstrass approximation implies that $\mathcal{O}(K)$ is dense.

Let V be a continuous representation of $G(\mathbb{R})$ on a complete locally convex topological vector space. If $\mu \in \widehat{K(\mathbb{R})}$, define $m_\mu = \mathrm{Hom}_{K(\mathbb{R})}(\mu, V)$. Let $V(\mu)$ be the image of the evaluation map $m_\mu \otimes \mu \hookrightarrow V$. Again, a version of Weierstrass approximation implies that

$$V = \widehat{\oplus} V(\mu) \supset \oplus V(\mu) =: V_{K-fin}.$$

$\widehat{\oplus}$ means the Hilbert space closure. For $v \in V_{K-fin}$, the span of Kv is finite dimensional.

Definition 33. *V is* admissible *if all $V(\mu)$ are finite dimensional.*

Theorem 34. *If V is admissible,*

(1) *all $v \in V_{K-fin}$ are* smooth, *i.e. the map $G(\mathbb{R}) \to V$ given by $g \mapsto gv$ is smooth,*[1]
(2) *$\mathfrak{g}$ preserves the space V_{K-fin}, (V_{K-fin} is a $(\mathfrak{g}, K)$*-module,
(3) *$W \mapsto W_{K-fin}$ and $U \mapsto \overline{U}$ are mutually inverse bijections between the lattice of closed subrepresentations and the lattice of $(\mathfrak{g}, K)$-submodules.*

Proof of 2. Suppose $v \in V_{K-fin}$. Let $W \subset V$ be finite dimensional $K(\mathbb{R})$-invariant subspace containing v. Let $X \in \mathfrak{g}$ and $k \in K$.

$$kX \cdot W = \mathrm{ad}_k(X)kW \subset \mathrm{ad}_k(X)W \subset \mathfrak{g}W.$$

$\mathfrak{g}W$ need not be finite dimensional, but the action of K on YW for any particular $Y \in \mathfrak{g}$ is finite dimensional. □

Example: Let's suppose that G is *quasi-split*, meaning that G contains a Borel $B = TU$ over $\mathbb{R}$. (G is *split* if it contains a maximal torus T over $\mathbb{R}$.) Then $G(\mathbb{R}) = K(\mathbb{R})B(\mathbb{R})$. ($K(\mathbb{R}) \cap B(\mathbb{R})$ need not be trivial.) Let $\chi : T(\mathbb{R}) \to \mathbb{C}^\times$ be a continuous character. Define

$$I(\chi) = \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} \chi := \{f : G(\mathbb{R}) \to \mathbb{C} \mid f(bg) = \chi(b)f(g) \text{ for all } b \in B(\mathbb{R}), g \in G(\mathbb{R})\}.$$

[2] $I(\chi)$ is an admissible $(\mathfrak{g}, K)$-module.

Why admissible? Since $\mathcal{O}(K) = \bigoplus_{\mu \in \widehat{K(\mathbb{R})}} \mu^\vee \otimes \mu$ and the restriction of $I(\chi)$ to $K(\mathbb{R})$ is

$$\{f : K(\mathbb{R}) \to \mathbb{C} \mid f(bk) = \chi(b)f(g) \text{ for all } b \in B(\mathbb{R}) \cap K(\mathbb{R}), k \in K(\mathbb{R})\},$$

it follows that

$$I(\chi)_{K-fin} = \bigoplus_\mu (\mu^\vee)^{B\cap K} \otimes \mu.$$

Since $\mu^\vee$ is finite dimensional, this implies that $I(\chi)_{K-fin}$ is admissible.

Example: $\mathrm{SL}_2(\mathbb{R}) = B(\mathbb{R})K(\mathbb{R})$ where

$$B(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ & a^{-1} \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times, b \in \mathbb{R}\}$$

and $K(\mathbb{R}) = SO_2(\mathbb{R})$. In this case, $B(\mathbb{R}) \cap K(\mathbb{R}) = \{\pm\}$. Also, a character of B is of the form $\chi : a \mapsto |a|^s \operatorname{sgn}(\epsilon)$ with $s \in \mathbb{C}$ and $\epsilon \in \{0, 1\}$. So

$$I(\chi)_{K-fin} = \bigoplus_{\substack{n \in \mathbb{Z} \\ n \equiv \epsilon \pmod 2}} (\cdot)^n.$$

In order to determine the action of $\mathfrak{g} = \mathfrak{sl}_2 = \langle Y_-, Y_+, H\rangle$, it suffices to know how Y_+Y_- acts since $\mathcal{U}(\mathfrak{sl}_2) \supset Z(\mathfrak{sl}_2) = \mathbb{C}[Y_+Y_- + H^2 - 2H]$.

Exercise 35. *Compute λ the $C = Y_+Y_- + H^2 - 2H$-eigenvalue on $(\cdot)^n \in I(\chi)$, and conclude that Y_+Y_- acts by $\lambda - n(n-2)$.*

To finish of his lecture, he drew a picture of the D_8 root system for Sp(4) and said that it can be found in the book of Fulton and Harris.

[1]finite dimensional representations are automatically smooth, but this need not be true for infinite dimensional representations.

[2]Emerton said something here about how regularity conditions, such as smooth, L^2 etc., on the type of functions in $I(\chi)$ are not necessary because $(\mathfrak{g}, K)$-modules force the functions to satisfy these properties already.

6. Florian Herzig: Representation theory of p-adic groups

Florian provided written notes of his talk (and much more) so I don't include any notes here.

7. David Nadler: Hecke algebras

7.1. Distributions: from groups to algebras. References: Gross "On the Satake Isomorphism" and Bernstein notes. (Should be able to find them on University of Chicago webpage on geometric Langlands.)

Let G be a finite group. Then $\mathbb{C}[G]$ can be thought of as a way of assigning a distribution δ_g to each element $g \in G$ with the property $\delta_g \delta_{g'} = \delta_{gg'}$.

What is a distribution? The dual space. (Hecke algebras are distribution algebras.)

$$\{G - \text{representations}\} \leftrightarrow \{\mathbb{C}[G] - \text{modules}\}$$

What are the $\mathbb{C}[G]$-modules? For one, $\mathbb{C}[G]$ is itself a $\mathbb{C}[G]$-module. This is sometimes too big to handle, but $\mathbb{C}[G/N]$ is a (left) $\mathbb{C}[G]$-module that can be easier to work with. $\mathbb{C}[G/N] = N$-invariant functions on G is called (?) a *residual right symmetry* and it is governed by a Hecke algebra.

If f is right N-invariant, $\delta_{g'} * f(g) = f((g')^{-1}g)$ is still right N-invariant, but $f(g) * \delta_{g'}$ is not necessarily right N-invariant. However $\mathbb{G}[N\backslash G/N]$ *is* right N-invariant. This is a Hecke algebra via convolution:

$$m : G \times G \to G \quad m_* \mathbb{C}[G] \otimes \mathbb{C}[G] \to \mathbb{C}[G]$$

If $N\backslash G \times_N G/N = \{n_1 g n, n^{-1} g' n_2\}$ then the group law on G gives

$$\begin{array}{ccccc} & & N\backslash G \times_N G/N & & \\ & \swarrow^{p_1} & \downarrow^{p_2} & \searrow^{m} & \\ N\backslash G/N & & N\backslash G/N & & N\backslash G/N \end{array}$$

This gives the multiplication $(\delta_1, \delta_2) \mapsto m_*(p_1^* \delta_1, p_2^* \delta_2)$.

Exercise 36. *Describe the following Hecke algebra:* $G = \mathrm{SL}_2(\mathbb{F}_q)$, $N = B(\mathbb{F}_q)$ *and* $\mathcal{H} = \mathbb{C}[N\backslash G/N]$. $\dim_{\mathbb{C}}(\mathcal{H}) = 2$ *(because the Weyl group is* S_2*.) How do you multiply the two functions?*

Another example to return to:

F	$\mathcal{O}$	ϖ	k
$\mathbb{F}_q((t))$	$\mathbb{F}_p[[t]]$	t	$\mathbb{F}_p$
$\mathbb{Q}_p$	$\mathbb{Z}_p$	p	$\mathbb{F}_p$
prod. of 2 formal disks	formal disk		

Cartan decomposition: $G = G(F)$, $K = G(\mathcal{O})$, $K\backslash G/K \simeq A^+$. In the case that $G = \mathrm{GL}_n$,

$$A^+ = \left\{ \left(\begin{array}{ccc} \varpi^{k_1} & & 0 \\ & \ddots & \\ 0 & & \varpi^{k_n} \end{array} \right) \middle| k_1 \leq \cdots \leq k_n, k_i \in \mathbb{Z} \right\}$$

The goal is to understand $\mathcal{H}_{sph} = \mathbb{C}_c[K\backslash G/K] \simeq \mathbb{C}_c[A^+]$. (The c denotes compactly supported distributions.) Note that, a priori, the isomorphism here is only as vector spaces–not as algebras.

Examples:

(1) $\mathrm{GL}_1(\mathbb{Q}_p)$. $A^+ = \{(p^k) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}$.
(2) $\mathrm{GL}_2(\mathbb{Q}_p)$. $A^+ = \{\begin{pmatrix} p^{k_1} & \\ & p^{k_2} \end{pmatrix} \mid k_i \in \mathbb{Z}, k_1 \leq k_2\}$.

Exercise 37. *In the case that $G = \mathrm{GL}_1$, $\mathcal{H} = \mathbb{C}_c[K\backslash G/K]$, show that $\mathcal{H}$ is isomorphic to $\mathbb{C}_c[A^+]$ as algebras.*

At this point he tried to draw a picture of G/K the set of rank 2 $\mathbb{Z}_p$-modules in $\mathbb{Q}_p^2$. ($G = \mathrm{PGL}_2(\mathbb{Q}_p)$, $K = \mathrm{PGL}_2(\mathbb{Z}_p)$.) $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is the center element and it is surrounded by a $\mathbb{P}_1(\mathbb{F}_p)$ of "closest" neighbors. Probably he was drawing the neighbors in the case that $p = 2$. In this case the lattices are $2\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$, $\mathbb{Z}_2(1,1) + 2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$. Each of these has $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a neighbor and two other (different for each.)

A K-orbit of G/K consists of all of the points of the above picture of a given distance from the origin. This is $K\backslash G/K = A^+$ and corresponds to the natural numbers.

Aside: this could be done for $F = \mathbb{C}((t))$, G as above. In this case G/K is a free group on $\mathbb{P}^1(\mathbb{C})/(xx^\perp = 1)$.

Theorem 38 (Satake Isomorphism)**.** *If G (over a local field) is split then $\mathcal{H}_{sph} = RepRing(G^\vee) \otimes_{\mathbb{Z}} \mathbb{C}$. $G^\vee$ is the Langlands dual group.)*

Example: $G = \mathrm{GL}_1$. We have seen $\mathcal{H} = \mathbb{C}_c[\mathbb{Z}]$. The irreducible representations of $\mathrm{GL}_1^\vee = \mathrm{GL}_1$ are parametrized by the integers.

Comment on why this is important: If V is a representation of G then $\mathcal{H}$ acts on the K invariants V^K.

How to prove Satake isomorphism: Both $\mathbb{C}_c[K\backslash G/K]$ and $\mathbb{C}_c[A]$ act on $\mathbb{C}_c[K\backslash G/K] \simeq \mathbb{C}_c[A]$ by Iwasawa decomposition. This gives a map

$$\mathbb{C}_c[K\backslash G/K] \longrightarrow \mathbb{C}_c[A] \text{“}=\text{”}\{\text{functions on a lattice}\}.$$

8. Frank Calegari: Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

Let $G = \mathrm{GL}_2(\mathbb{F}_q)$. Note that $\#G = (q^2-1)(q^2-q)$, and that it is partitioned by the conjugacy classes $\mathcal{C}$:

$g \in \mathcal{C}$	$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$	semisimple non-split
$\#\mathcal{C}$	$q-1$	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q^2-q}{2}$

The 1-dimensional representations correspond to the conjugacy classes of center elements. (The first column of the above table.)

The next "class" of representations come from the principal series representations. Let $M = \{\begin{pmatrix} * & \\ & * \end{pmatrix}\}$, $N = \{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\}$ and $B = MN$. Let χ_1 and χ_2 be characters of $\mathbb{F}_q$. Then we can define a character on B:

$$\chi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \chi_1(a)\chi_2(d).$$

Let $B(\chi) = B(\chi_1, \chi_2) = \mathrm{Ind}_B^G(\chi)$. Notice that $\#G/B = \#G/\#B = \#\mathbb{P}^1(\mathbb{F}_q) = q+1$.

The *Jacquet module* of a representation V is

$$J_N V = V/\langle nv - v \mid n \in N, v \in V\rangle.$$

This is the biggest quotient of V on which N acts trivially. It has an action by B.

Lemma 39. $\mathrm{Hom}_M(J_N V, \chi) = \mathrm{Hom}_G(V, B(\chi))$.

Proof. We have

$$\mathrm{Hom}_M(J_N M, \chi) = \mathrm{Hom}_B(J_N V, \chi) = \mathrm{Hom}_B(V, \chi) = \mathrm{Hom}_G(V, B(\chi)).$$

The final equality is Frobenius reciprocity. The others are easy. □

Corollary 40. *If V is irreducible and $J_N V \neq 0$ then $V \subset of B(\chi)$ for some χ.*

Lemma 41. *Let A, B be subgroups of a finite group H with χ, β characters on A, B respectively. Then*

$$\mathrm{Hom}_H(\mathrm{Ind}_A^H \alpha, \mathrm{Ind}_B^H \beta) = \bigoplus_{w \in A\backslash H/B} \mathrm{Hom}_{w^{-1}Aw \cap B}(\alpha^w, \beta).$$

"Proof". $\mathrm{Hom}_B(\mathrm{Ind}_A^H \alpha, \beta)$ consists of functions of coests Ah. When you restrict to B this gives functions on the cosets AhB which is

$$\bigoplus_{w \in A\backslash H/B} \mathrm{Hom}_{w^{-1}Aw \cap B}(\text{“}\alpha\text{”}^w, \beta).$$

□

We now apply this lemma to $A = B = B$ and $H = G$. The *Weyl group* is

$$B\backslash G/B = \{1, w\},$$

and

$$\dim \mathrm{Hom}(B(\chi), B(\chi')) = \begin{cases} 1 & \text{if } \chi = \chi' \text{ or } \chi^w = \chi', \\ 0 & \text{otherwise.} \end{cases}$$

8.1. **Whittaker models and the Weil representation: Cuspidal representations.** There is a Whittaker model.

There are $(q-1)(q-2)/2$ irreducible representations corresponding to split semisimple elements.

Theorem 42. *Let F be a field. Consider the group H presented as follows. The generators are:*

- $m(y)$, $y \in F^\times$,
- $n(z)$, $z \in F$,
- w,

and the relations are:

- $m(y)m(y') = m(yy')$,
- $n(z)n(z') = n(z+z')$,
- $m(y)n(z)m(y^{-1}) = n(y^2 z)$,
- $wm(y)w = m(-y^{-1})$,
- $wn(y)w = m(-y^{-1})n(-y)wm(-y^{-1})$.

The map that sends $m(y)$ to $\begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}$, $n(z)$ to $\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}$ and w to $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ is a surjective group homomorphism from H to $\mathrm{SL}_2(F)$ that is in fact an isomorphism.

"Proof". It is easy to check that the map is surjective and a homomorphism. (Check that elements of the given type generate $\mathrm{SL}_2(F)$ and that the relations hold.) To get that it is an isomorphism, consider the inverse map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} n(a/c)m(-c^{-1})wn(d/c) & \text{if } c \neq 0 \\ m(a)n(b/a) & \text{if } c = 0. \end{cases}$$

To check that this is a group homomorphism is messy but possible. □

Let E/k be a rank 2 algebra over $k = \mathbb{F}_q$, either $E = k \oplus k$ or E is an unramified field ℓ with $[\ell : k] = 2$. Choose $\psi : k \to \mathbb{C}^\times$ a nontrivial character. There is a conjugation map $x \mapsto \overline{x}$ on E given by the usual Galois conjugation if $E = \ell$ and $\overline{(x, y)} = (y, x)$ in the case $E = k \oplus k$. This gives trace and norm maps

$$tr : x \mapsto x + \bar{x}, \quad N : x \mapsto x\bar{x}.$$

The Weyl representation will be an action of G. Let W be the space of $\mathbb{C}$ valued functions on E. So $W \simeq \mathbb{C}^{q^2}$. There exists a Fourier transform on W:

$$\widehat{f}(x) = \epsilon \frac{1}{q} \sum_{y \in E} f(y)\psi(tr(\bar{x}y))$$

satisfying $\widehat{\widehat{f}}(x) = f(-x)$. ($\epsilon$ is 1 if E is split and -1 if not.) Define the action via

- $\left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right) f(x) = f(ax)$,
- $\left(\begin{smallmatrix} 1 & z \\ & 1 \end{smallmatrix}\right) f(x) = \psi(zN(x))f(x)$,
- $wf(x) = \widehat{f}(x)$.

To verify that this is a valid action one needs only check the five relations of Weil's theorem.

This gives a representation of $\mathrm{SL}_2(k)$. To extend it to all of G, let

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix} f(x) = \widetilde{\chi}(a)f(\alpha x)$$

where $\alpha \in F$ satisfies $N(\alpha) = a$ and $\widetilde{\chi}$ is chosen in a consistent way.

W (which has dimension q^2) is not irreducible, but it can be broken up in the following way. Choose $\chi : E^\times \mathbb{C}^\times$ that does not factor through the the norm N. Let W^0 be the set of functions $f \in W$ such that $f(1) = 0$. Clearly, $\dim W^0 = q^2 - 1$, and G acts on W^0. Let

$$W(\chi) = \{f \in W^0 \mid f(yx) = \chi(y)f(x) \text{ whenever } y \in \ker N\}.$$

If $E = \ell$, we get $q + 1$ nonsemisimple representations $W(\chi)$ of dimension $(q^2 - 1)(q - 1) = q + 1$.

(Does next part correspond to $E = k \otimes k$? I don't know.)

$$\dim\{V \subset W(\chi) \mid \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) v = \psi(ax)v\} = \begin{cases} 1 & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

If $v \in V$, $v' = \left(\begin{smallmatrix} a^{-1} & \\ & 1 \end{smallmatrix}\right) v$ lives in the same type of space with $a' = ad$. This implies $W(\chi)$ is irreducible of dimension $q - 1$, $J_N W(\chi) = 0$ and the number of such $W(\chi)$ up to isomorphism is $\frac{q^2-1-(q-1)}{2} = \frac{q^2-q}{2}$.

8.2. **Final thoughts.** What happens for $\mathbb{F}_q$ replaced by a p-adic field K? In this case there are three choices for E, namely $K \oplus K$, a field extension of degree 2, L or a nonsplit quaternion algebra D/K. In these cases one takes W to be the space of locally constant compactly supported functions on E. Again, there exists a Fourier transform on W, and one wants to break up W into indecomposable parts.

For a representation V of $E^\times$ there is a lift $\mathrm{JL}(\theta)$. These will give all representations (all supercuspidals?) as long as $p > 2$.

9. Frank Calegari: class field theory

Let $K/\mathbb{Q}$ be a number field, $I = I_K$ the group of fractional ideals, S a set of rational primes. $I^S \subset I$ will be the subgroup of generated by primes not in S. Set $K^S = \{a \in K^\times \mid v_p(a) = 0 \text{ if } p \notin S\}$. Note that $a \in K^S$ if and only if $(a) \in I^S$.

There is a natural map $\varphi : K^\times \to I$, and if $C = I/K^\times$ is the class group and $U = U_K = \mathcal{O}_K^\times$ is group of units then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & I & \longrightarrow & K^\times & \longrightarrow & C & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \Vert & & \\ 0 & \longrightarrow & U & \longrightarrow & I^S & \longrightarrow & K^S & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Definition 43. *A* modulus $\mathfrak{m} = \mathfrak{m}_\infty \mathfrak{m}_f$ *consists of an ideal* $\mathfrak{m}_f \subset \mathcal{O}_K$ *and a collection of infinite real places* $\mathfrak{m}_\infty$. *Let* $S(\mathfrak{m})$ *equal the places dividing* $\mathfrak{m}$. *Let*

$$K_{\mathfrak{m},1} = \left\{a \in K^\times \mid a_v > 0 \text{ if } v \mid \mathfrak{m}_\infty, v(a-1) \geq v(\mathfrak{m}_f) \text{ if } v \mid \mathfrak{m}_f\right\} \subset K^{S(\mathfrak{m})}.$$

The Ray class group of conductor $\mathfrak{m}$ *is* $C_\mathfrak{m} = I^S/\varphi(K_{\mathfrak{m},1})$.

The first observations are that $C_\mathfrak{m}$ are finite and that

$$I^S/\varphi(K_{\mathfrak{m},1}) \to I^S/\varphi(K^{S(\mathfrak{m})}) = C$$

is surjective.

Let $K_\mathfrak{m} = K^{S(\mathfrak{m})}$. Then

$$0 \to U/U_{\mathfrak{m},1} \to K_\mathfrak{m}/K_{\mathfrak{m},1} \to C_\mathfrak{m} \to C \to 0$$

is exact.

Lemma 44. $K_\mathfrak{m}/K_{\mathfrak{m},1} \simeq \prod_{v \mid \mathfrak{m}_\infty} \{\pm 1\} \times (\mathcal{O}/\mathfrak{m}_f\mathcal{O})^\times$. *In particular,* $C_\mathfrak{m}$ *is finite. More precisely,* $\#C_\mathfrak{m}/\#C$ *is equal to the index of the image of* $U/U_{\mathfrak{m},1}$ *in* $K_\mathfrak{m}/K_{\mathfrak{m},1}$.

Example 45. $K = \mathbb{Q}$, $\mathfrak{m} = \infty \cdot N$, $N > 2$.

$$0 \to \{\pm 1\} \to \{\pm 1\} \times (\mathbb{Z}/N\mathbb{Z})^\times \to 1 \to 0.$$

So $C_\mathfrak{m} \simeq (\mathbb{Z}/N\mathbb{Z})^\times$.

9.1. **Frobenius elements.** Let L/K be Galois, $\mathfrak{p}$ a primes of K and $\mathfrak{P}$ a prime in L above $\mathfrak{p}$. Consider the following diagram.

$$\begin{array}{ccccccc} L & \longrightarrow & L_\mathfrak{P} & \hookleftarrow & \mathcal{O}_{L_\mathfrak{P}} & \twoheadrightarrow & \ell \\ \uparrow & & & & & & \uparrow \\ K & \longrightarrow & K_\mathfrak{p} & \hookleftarrow & \mathcal{O}_{K_\mathfrak{p}} & \twoheadrightarrow & k \end{array}$$

This gives an injective map $i : Gal(L_\mathfrak{P}/K_\mathfrak{p}) \to Gal(\ell/k)$, and a surjective map $j : Gal(L_\mathfrak{P}/K_\mathfrak{p}) \to Gal(L/K)$. If $\mathfrak{p}$ is unramified, i is an isomorphism. This gives a canonical element in $Gal(L_{\mathfrak{P}}/K_{\mathfrak{p}})$ called Frobenius, and, via j, an element (also called Frobenius) in $Gal(L/K)$ denoted $(\mathfrak{P}, L/K)$. An elementary computation shows that

$$(\tau\mathfrak{P}, L/K) = \tau(\mathfrak{P}, L/K)\tau^{-1}.$$

So if L/K is abelian, the Frobenius element doesn't depend on the choice of prime lying over $\mathfrak{p}$. In this case we write $(\mathfrak{p}, L/K)$.

9.2. **Artin map.** Let L/K be an abelian extension that is ramified only at $v \in S$. The Artin map $\psi_{L/K} : I_K^S \to Gal(L/K)$ is given by $\psi_{L/K}(\mathfrak{p}) = (\mathfrak{p}, L/K)$. We say that $\psi : I_K^S \to Gal(L/K)$ *admits a modulus* $\mathfrak{m}$ if $\psi(\varphi(K_{\mathfrak{m},1})) = 0$ for $S = S(\mathfrak{m})$.

Theorem 46. *The Artin map admits a modulus.*

What does this mean? $(\mathfrak{p}, L/K)$ is trivial if $L_{\mathfrak{P}}/K_{\mathfrak{p}}$ is trivial. Since $n = efr$, in this case $e = f = 1$, so this implies that $\mathfrak{p}$ splits completely. In fact, $\mathfrak{p}$ splits completely implies that $(\mathfrak{p}, L/K)$ is trivial.

Theorem 46 implies that if L/K is abelian, any principal prime ideal in $\varphi(K_{\mathfrak{m},1})$ must split completely. Returning to Example 45, this says every prime ideal congruent to 1 modulo N splits completely.

Remark: Theorem 46 is not fully satisfactory because it doesn't imply the existence of any abelian extensions. It says that if ψ has modulus $\mathfrak{m}$ then it factors through $I_K^S/K_{\mathfrak{m},1} = C_{\mathfrak{m}}$. The following theorem gives existence of abelian extensions.

Theorem 47. *[Existence Theorem] There is a field $L_{\mathfrak{m}}/K$ such that $\psi_{L/K} : I_K^S \to Gal(L_{\mathfrak{m}}/K)$ has kernel exactly $\varphi(K_{\mathfrak{m},1})$. i.e. $C_{\mathfrak{m}} \simeq Gal(L_{\mathfrak{m}}/K)$.*

We call $L_{\mathfrak{m}}$ the *Ray class field* of conductor $\mathfrak{m}$. (Prime ideals of K that split completely are...)

If $\mathfrak{m} = 1$, $L_{\mathfrak{m}}$ is called the *Hilbert class field*.

Example: $\mathbb{Q}$, $U = \{\pm 1\}$, $\mathfrak{m} = \infty \cdot p^k$. The exact sequence is

$$0 \to \{\pm 1\} \to \{\pm 1\} \times (\mathbb{Z}/p^k\mathbb{Z})^\times \to C_{\mathfrak{m}} \to 1 \to 0.$$

So, the existence theorem says that there is a field L over $\mathbb{Q}$ such that $Gal(L/\mathbb{Q}) \simeq C_{\mathfrak{m}} = (\mathbb{Z}/N\mathbb{Z})^\times$. $L = \mathbb{Q}(\zeta_{p^k})$ indeed has this Galois group.

Example: $\mathbb{Q}$, $U = \{\pm 1\}$, $\mathfrak{m} = p^k$. The subgroup of order $\#C_{\infty \cdot p^k}/2$ corresponds to the maximal real subfield of $\mathbb{Q}(\zeta_{p^k})$. Note that there exist field extensions that are unramified at all finite primes, but don't correspond to Hilbert class fields.

Example: K an imaginary quadratic field. $\#U = 2, 4$ or 6. $\mathfrak{m} = p^k$. The exact sequence is

$$0 \to U \to (\mathcal{O}_K/\mathfrak{m})^\times \to C_{\mathfrak{m}} \to 1 \to 0.$$

As $k \to \infty$,

$$\mathcal{O}_{K,p} = \begin{cases} \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times & \text{if } p \text{ splits,} \\ \mathbb{Z}_{p^2}^\times & \text{if } p \text{ is inert.} \end{cases}$$

In either case, the K_{p^∞}/K is a $\mathbb{Z}_p^2$ plus something finite. Over $\mathbb{Q}$ the existence theorem gives a $\mathbb{Z}_p$ extension called the *cyclotomic extension*, and so there exists a so-called cyclotomic tower of fields. Over K we get two $\mathbb{Z}_p$ extensions. $Gal(K/\mathbb{Q})$ acts on $Gal(K_{p^\infty}/K)$. The second extension is called the *anticyclotomic $\mathbb{Z}_p$ extension.*

Example: K a totally real field over $\mathbb{Q}$. The following sequence is exact.

$$0 \to U/U_{\mathfrak{m},1} \to \prod_{v|\infty} \{\pm 1\} \times (\mathcal{O}_K/\mathfrak{m})^\times \to C_\mathfrak{m} \to 1 \to 0 \tag{4}$$

Dirichlet's unit theorem says that $U \simeq T \otimes \mathbb{Z}^{r_1+r_2-1}$ with T finite and $[K : \mathbb{Q}] = n = r_1 + 2r_2$, so in this case $U \simeq T \oplus \mathbb{Z}^{n-1}$, and $U \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p^{n-1}$. We take $\mathfrak{m} = p^k$ and consider what happens to (4) as $k \to \infty$. In the limit, the sequence is only known to be right exact. That is, modulo torsion,

$$\mathbb{Z}_p^{n-1} \to \mathbb{Z}_p^n \to C_{p^\infty} \to 0$$

is "exact." If the first map here were injective, this would imply that $C_{p^\infty} \simeq \mathbb{Z}_p$ (plus torsion,) so totally real fields have no new abelian extensions.

Conjecture 48 (Leopoldt)**.** *The map* is *injective, and so over totally real fields there are no new $\mathbb{Z}_p$ extensions.*

Remark: This discussion dispels the 'belief' that class field theory explains *everything* about abelian extensions.

9.3. Adelic language.

Definition 49. *The* adeles *of a global field are defined to be*

$$\mathbb{A}_K = \{x = (x_v) \in \prod_v K_v \mid x_v \in \mathcal{O}_v \textit{ for almost all } v,$$

and the ideles *are*

$$\mathbb{A}_K^\times = \{x = (x_v) \in \prod_v K_v f^\times \mid x_v \in \mathcal{O}_v^\times \textit{ for almost all } v.$$

(v runs through all valuations of K and K_v are the corresponding complete local fields, with valuation rings $\mathcal{O}_v$.)

There is a topology on $\mathbb{A}_K$. The following (open) sets form a cover.

$$\left\{ \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v \mid S \text{ is a finite set of primes} \right\}$$

It follows that A_K is locally compact.

Example: Assume that $S_\infty \subset S$. Then

$$U(S, \epsilon) = \left\{ (a_v) \,\middle|\, \begin{array}{ll} |a_v| = 1 & \text{if } v \notin S \\ |a_v - \epsilon| < 1 & \text{if } v \in S \end{array} \right\}$$

is an open set of $\mathbb{A}_K$.

There is a natural map

$$K^\times \hookrightarrow \mathbb{A}_K^\times \quad a \mapsto (a, a, \ldots).$$

The image is discrete. To see this, it suffices to show that if $a \in K^\times$ then there exists $U(S, \epsilon)$ such that $a \notin U(S, \epsilon)$. If $a \in U(S, \epsilon)$, then $\prod |a_v - 1| < \epsilon$. But ϵ can be chosen to be less than 1, and this would contradict the product formula unless $a = 1$.

Another natural map is

$$\mathbb{A}_K^\times \to I_K \quad (a_v) \mapsto \prod_{\mathfrak{p} \nmid \infty} \mathfrak{p}^{v_\mathfrak{p}(a_\mathfrak{p})}.$$

Note that $(a, a, \ldots)$ goes to (a) under this map. So, this induces a map $K^\times\backslash\mathbb{A}_K^\times$ to the ideal class group.

Definition 50. *The quotient* $K^\times\backslash\mathbb{A}_K^\times$ *is the* idele class group.

Recall that $\psi : I_K^S \to G$, G an abelian group (think $G = Gal(L/K)$), admits a modulus if $\psi(\varphi(K_{\mathfrak{m},1}))$ is trivial (?) Therefore, G must be a quotient of $C_{\mathfrak{m}}$.

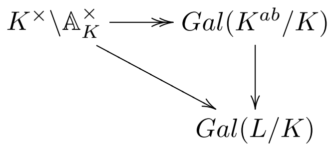
Theorem 51. *There exists a unique homomorphism* $\phi : K^\times\backslash\mathbb{A}_K^\times \to G$ *such that*

- ϕ *is continuous,*
- $\phi((a_v)) = \psi($*image of* (a_v) *in* $I_K^S)$*. (*$\mathbb{A}_K^\times$ *maps to* I_K^S *if* $a_v = 1$ *for all* $v \in S$*.)*

Moreover, any such map of $K^\times\backslash\mathbb{A}_K^\times$ *of finite order is of this type.*

This theorem "packages all ray class fields together in some nice way."

If L/K is an abelian extension, we have the following diagram of maps.

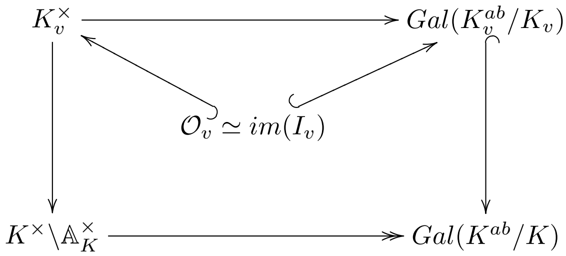


The kernel of $K^\times\backslash\mathbb{A}_K^\times \to Gal(L/K)$ has kernel exactly $N(L^\times\backslash\mathbb{A}_L^\times)$.

9.3.1. *Local class field theory.* There is an injection

$$K_v \hookrightarrow \mathbb{A}_K^\times \quad a_v \mapsto (1, \ldots, 1, a_v, 1, \ldots)$$

where a_v is inserted in the v-component. This then gives the following commutative diagram.



The "smallest" quotient of $K^\times\backslash\mathbb{A}_K^\times$:

$$K^\times\backslash\mathbb{A}_K^\times/\prod_{v\nmid\infty}\mathcal{O}_v^{\times?}\longrightarrow Cl(K)$$

There is an absolute value $|\cdot| \to \mathbb{R}^{\geq 0}$ on the adeles given by $|(a_v)| = \prod_{v\nmid\infty}|a_v|_v$. (Normally, the absolute value on the adeles is given by the product over *all* places. In this case, one has the product formula: $|(a)| = 1$ for $a \in K^\times$. I don't know if I copied down the wrong formula, if Calegari wrote the wrong thing on the board, or if he meant what I wrote above.)

9.3.2. *Hecke characters.* Suppose $\psi : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ has conductor $\mathfrak{m} = \mathfrak{m}_f$. If ψ is trivial on $\prod_{\substack{v\nmid\infty \\ v\nmid\mathfrak{m}}} \mathcal{O}_v^\times \times \prod_{v|\mathfrak{m}}(1+\mathfrak{m})$, it is called a *Hecke character.* All characters at the infinite places are of the form:

$K_v^\times$	χ
$\mathbb{R}^\times$	$x \mapsto x^\epsilon \lvert x\rvert^s,\ \epsilon \in \{\pm 1\}, s \in \mathbb{C}$
$\mathbb{C}^\times$	$z \mapsto z^A\bar{z}^B(z\bar{z})^s,\ A, B \in \mathbb{Z}, s \in \mathbb{C}$

ψ is an *algebraic Hecke character* if $\psi_{\mathbb{R}^\times}(x) = x^\epsilon |x|^n$ for some $n \in \mathbb{Z}$, and $\psi_{\mathbb{C}^\times}(z) = z^A\bar{z}^B$ for some $A, B \in \mathbb{Z}$.

Why do we care about algebraic Hecke characters? If $\psi : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}$ is an algebraic hecke character, there exists a compatible family of Galois representations $\rho_{\psi,v} : Gal(K^{ab}/K) \to E^\times$ unramified if $q \mid mN(v)$, $[E : K_v] < \infty$. Moreover, if we fix $i_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $i_v : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$,

$$\rho_{\psi,v}(Frob_q^{-1}) = i_p \circ i_v^{-1}(1, \ldots, 1, \pi_q, 1, \ldots).$$

Example: $\psi : Q^\times\backslash\mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}$, $\mathfrak{m} = 1$. We take the ordering of the adeles: $\mathbb{Q}_\infty \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \cdots$. Define

$$\psi(1, \prod \mathbb{Z}_p) = 1 \quad \text{and} \quad \psi(x, 1, \ldots) = |x|.$$

Then

$$\begin{aligned}\psi(1, \ldots, 1, q, 1, \ldots) &= \psi(\frac{1}{q}, \ldots, \frac{1}{q}, 1, \frac{1}{q}, \ldots) \\ &= \psi(\frac{1}{q}, 1, \ldots)\psi(1, \frac{1}{q}, \ldots, \frac{1}{q}, 1, \frac{1}{q}, \ldots) \\ &= \frac{1}{q} \cdot 1 = \frac{1}{q}.\end{aligned}$$

This construction gives the cyclotomic character.

What do these look like?

- Finite characters.
- ψ and ψ' characters, then $\psi\psi'$ is too.
- ψ a Hecke character for K. $\psi_L := \psi \circ N_{L/K}$.

Theorem 52. *Suppose that ψ is an algebraic Hecke character of L. Then*

$$\psi_L = \textit{finite} \times \psi_K \circ N_{L/K}.$$

$K \subset L$ is a CM extension of a totally real field $K^+/\mathbb{Q}$.

Comments: He finished the lecture with a Venn diagram: the left circle being 1-dimensional Galois representations unramified outside a finite set of primes S. The right circle being Hecke characters. Algebraic Hecke characters are the intersection.

10. Matt Emerton: Automorphic forms and representations

There are two contexts for the definition:

1. $C^{\infty,mod}(\Gamma\backslash G(\mathbb{R}), \mathbb{C})$. ($\Gamma$ need not be a congruence subgroup, but Hecke algebras don't come up if Γ is not congruence.)
2. Adelic framework. This requires a lot more set up to understand, but is better for number theory.

10.1. **Groups over $\mathbb{Q}$.** Let G be a reductive group defined over $\mathbb{Q}$. (One can take reductive to mean that, after base change to $\mathbb{C}$, G is reductive over $\mathbb{C}$ as defined before.) There is some closed embedding $G \hookrightarrow \mathrm{GL}_N$.

Examples? There are many. You have to give them names which are usually the same as for those defined over $\mathbb{R}$ and $\mathbb{C}$. e.g. $U(p,q)/\mathbb{Q} = ?$ In analogy to the $\mathbb{R}, \mathbb{C}$ case there is an imaginary quadratic extension E of $\mathbb{Q}$:

$$\begin{array}{ccc} \mathbb{C} & \rightsquigarrow & E \\ & & \\ \mathbb{R} & \rightsquigarrow & \mathbb{Q} \end{array}$$

With these groups there is a theory similar to the section in Serre's Course in Arithmetic on quadratic forms: you have to choose local data hoping that they can be glued together to form a unique global object. (Hasse principle)

(References: Boulder volumes, Casselman's notes on p-adic groups.)

When you have a group over $\mathbb{Q}$, denominators can be cleared so that the group is reduced, and the morphism and can be made to be defined over $\mathbb{Z}[\frac{1}{N}]$ for some N.

Example: $x^2 + y^2 = 1$. Multiplication is defined by

$$(x,y)\cdot(x',y') = (xx' - yy', xy' + x'y).$$

(This is complex multiplication on S^1.) Since over $\mathbb{F}_2$ get $(x+y+1)^2 = 0$, this group is not reduced over $\mathbb{Z}$, but it is (?) over $\mathbb{Z}[\frac{1}{2}]$.

Why are theorems proved for reductive groups and not just semisimple ones? Possible explanations are:

- GL_n is reductive but not semisimple.
- The process of inductions does not preserve semisimplicity, but it does preserve reductivity.

Consider

$$\begin{array}{ccc} SU(2) & \hookrightarrow & U(2) \\ & \searrow & \\ & & \mathbb{H}^\times \end{array}$$

The center of $U(2)$ is in $U(1)$, and the center of $\mathbb{H}^\times$ (units in the Hamiltonians, which is reductive but not semisimple) is contained in $\mathbb{R}^\times$. Over $\mathbb{Q}$, is there a connection with this fact?

Let $D^\times/\mathbb{Q}$ be a quaternion algebra. (D can be thought of as a function: $R \mapsto (D \otimes_{\mathbb{Q}} R)^\times$.) The center is $\mathbb{G}_m$ which is split. On the other hand, $U(2)$ has center $(E^\times)^{N=1}$ which is not split. Some groups that are split over $\mathbb{Q}$ will not be split over $\mathbb{Z}[\frac{1}{N}]$ for any N. This is a reason that people work with nonsplit groups.

(Reference: Gross's Algebraic Modular Forms does Satake isomorphism for non-split groups.)

Definition 53. *$G/\mathbb{Q}_p$ is* unramified *if G is quasi-split and there exists a maximal torus that splits over an extension of $\mathbb{Q}_p$.*

It is a fact that $G/\mathbb{Q}$ is unramified at all but finitely many primes. Assume all ramified primes are in N. So $G/\mathbb{Z}[\frac{1}{N}]$ is reduced and unramified at all closed points. For $p \nmid N$, $G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ is a *hyperspecial maximal compact subgroup.*

G affine means that there is a closed embedding $G \hookrightarrow$ affine r-space. The $G(\mathbb{A}) \hookrightarrow \mathbb{A}^r$ is topologically closed. Note: in the ideles, two adeles are close if they're close *and* their inverses are close—the topology of the ideles is finer than that induced by the adeles. This has to do with the fact that

$$\prod \mathcal{O}_v \cap \prod \mathcal{O}_v^\times \neq \prod \mathcal{O}_v^\times.$$

$\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$, $\mathbb{A}_f = \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. What are the open compact subgroups of $\mathbb{A}_f$? If $K_p \subset G(\mathbb{Q}_p)$ is a compact subgroup such that $p \mid M$ and $N \mid M$ then

$$\prod_{p|M} K_p \times \prod_{p\nmid M} G(\mathbb{Z}_p) \subset G(\mathbb{A}_f)$$

are the compact subgroups. They induce a topology. $G/\mathbb{Q}$ will agree with an integral model over $\mathbb{Z}[\frac{1}{N'}]$.

The adeles say something about arithmetic, so taking the adelic points of a variety over $\mathbb{Q}$ actually says something about the integers. This means that we don't need to explicitly work over $\mathbb{Z}[\frac{1}{N'}]$, but it is good, philosophically, to realize that this can be done.

10.2. **Authomorphic forms.** These are functions of $G(\mathbb{Q})\backslash G(\mathbb{A})$. What does this quotient look like? It is a fact that given $K_f \subset G(\mathbb{A}_f)$ a compact open subgroup, there exists a finite subset C such that

$$G(\mathbb{A}) = G(\mathbb{Q}) \cdot C \cdot G(\mathbb{R})K_f \tag{5}$$

Why? In GL_1:

$$\mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^\times K_f = \widehat{\mathbb{Z}}/\{\pm 1\}K_f,$$

which is finite. This has something to do with the fact that Ray class groups are finite.

Remark: It suffices to discuss all of this theory over $\mathbb{Q}$ instead of an arbitrary number field F. Suppose G is an algebraic group over F. Then $G' = \operatorname{Res}_F^{\mathbb{Q}} G$, which satisfies $\operatorname{Res}_F^{\mathbb{Q}}(R) = G(F \otimes_{\mathbb{Q}} R)$, has $G'(\mathbb{A}) = G(F \otimes_{\mathbb{Q}} \mathbb{A}) = G(\mathbb{A}_F)$. In other words, you don't have to say anything about F unless you want to.

The group SL_2 is generated by two copies of $\mathbb{G}_a$:

$$\langle \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \rangle$$

Since $\mathbb{Q}$ is dense in $\mathbb{A}_f$, it follows that $\left(\begin{smallmatrix} 1 & \mathbb{Q} \\ 0 & 1 \end{smallmatrix}\right)$ is dense in $\left(\begin{smallmatrix} 1 & \mathbb{A}_f \\ 0 & 1 \end{smallmatrix}\right)$. Thus $\mathrm{SL}_2(\mathbb{Q})$ is dense in $\mathrm{SL}_2(\mathbb{A}_f)$, and

$$\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\mathrm{SL}_2(\mathbb{R})K_f$$

for any open subgroup $K_f \subset \mathrm{SL}_2(\mathbb{A}_f)$. In other words, we can take $C = 1$ in (5). This basic argument works for any split group.

Rewriting (5), we have

$$G(\mathbb{A}) = \bigsqcup_{c \in C} G(\mathbb{Q})cG(\mathbb{R})K_f,$$

and so

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_f = \sqcup c \in CG(\mathbb{Q})\backslash G(\mathbb{Q})cG(\mathbb{R})K_f/K_f = \sqcup c \in C\Gamma_c\backslash G(\mathbb{R}). \tag{6}$$

$gc \in cG(\mathbb{R}) \cdot K_f$ if and only if $g \in G(\mathbb{R})cK_fc^{-1}$, and so $g \in G(\mathbb{Q}) \cap cK_fc^{-1}$ which is a congruence subgroup.

$$\begin{array}{ccc} G(\mathbb{R}) & \hookleftarrow & K_\infty^0 Z_\infty^0 \\ \downarrow & & \\ G(\mathbb{R})/K_\infty^0 Z_\infty^0 \text{ “ = ” } AN & & \end{array}$$

(A, N as in the decomposition KAN.)

$G(\mathbb{R})/K_\infty^0 Z_\infty^0$ is a topologically contractible (with components) *symmetric space* for $G(\mathbb{R})$. In the case $G = \mathrm{SL}_2$ this is the upper half plane $\mathbb{H}$. Another example is

$$\begin{array}{ccc} \mathrm{Sp}_{2g} & \hookleftarrow & U(g) \\ \downarrow & & \\ \text{Siegel } g\text{-space} & & \end{array}$$

Final comment: the real points are a big deal. The finite adeles control C and Γ_c, but the geometry comes from $G(\mathbb{R})$. As an example, think about how the grossencharakteres do involve the infinite place.

11. Frank Calegari: Problem session

$\mathrm{SL}_2(\mathbb{R})$ is topologically isomorphic to $\mathbb{R}^\times \times S^1$. $\mathrm{SL}_2(\mathbb{R})$ acts on $\tau \in \mathbb{H}$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau+b}{c\tau+d}$. Since $\begin{pmatrix} a^{1/2} & b \\ & a^{-1/2} \end{pmatrix} i = a + bi$, the action is transitive.

11.1. **Proof of Theorem 52.** Consider the tower of fields: $\mathbb{Q} \subset K^+ \subset K \subset L$. Consider a character $\psi : L^\times \backslash \mathbb{A}_L^\times \to \mathbb{C}^\times$ such that

$$\psi_\infty = \psi \mid_{\prod v|\infty L_v^\times}$$

is of the form $x \mapsto x^n$ at real places and $z \mapsto z^a \bar{z}^b$ at complex places. $\mathrm{Hom}(L, \mathbb{C}) \leftrightsquigarrow \sigma_i \in G = Gal(L/\mathbb{Q})$.

$$\begin{aligned} \psi(\underbrace{u, \dots, u}_{\infty}, u, u, \dots) &= \psi(\underbrace{u, \dots, u}_{\infty}, 1, 1, \dots)\psi(\underbrace{1, \dots, 1}_{\infty}, u, u, \dots) \\ &= \psi_\infty(u) \cdot 1 = \psi_\infty(u). \end{aligned}$$

If $\mathfrak{m}_f \neq 1$, there exists $k \in \mathbb{Z}$ such that $\psi_\infty(u^k) = 1$ for all $u \in U = \mathcal{O}_K^\times$. Then

$$|\psi_\infty(u)| = \prod |\sigma_i(u)|^{a_i}$$

where real $\sigma_i \leftrightarrow a_i$ and complex $\sigma_j, \sigma_{j+1} \leftrightarrow a_j, a_{j+1}$.

Let $\eta = \sum a_i[\sigma_i] \in \mathbb{Z}[Gal(L/\mathbb{Q})]$. So $|\psi_\infty(u)| = |\eta u|$ and $|\sigma\psi_\infty(u)| = |\sigma\eta u|$, and therefore $(\eta u)^k = 1$, and $\eta(u^{k'}) = 1$ for some k'. We conclude that η annihilats $U \otimes_{\mathbb{Z}} \mathbb{R}$.

Let c denote complex conjugation. Then

$$U \otimes \mathbb{C} = \prod_{\substack{\text{irreducible reps} \\ \text{of } G \text{ of dimension}>1}} V^{\dim(V|c=1)}.$$

Notice that if V is a representation then $ann(V) = ann(V \otimes V)$. If every representation occurs above then

$$0 \to \mathbb{C} \to \mathbb{C}[Gal(L/\mathbb{Q})] \to \underset{\sum[g]}{?} \to 0$$

If V doesn't occur, $\dim(V \mid c = 1) = 0$, and thus c acts by -1. G surjects onto G/H which acts on V, and $c \mapsto c$ is a nontrivial central element of G/H. By Galois theory it must act through the Galois group of K which is CM. So

$$0 \longrightarrow K \longrightarrow \mathbb{C}[G] \xrightarrow{N_{G/H}} \mathbb{C}[G/H] \longrightarrow 0$$

is exact. Therefore η factors through $N_{G/H}$. This implies that $\psi_L = \chi \cdot \psi_K \circ N_{L/K}$ where χ is a finite order character and ψ_K is a grossencharakter.

Corollary 54. *$\rho : Gal(K^{ab}/K) \to E^\times$, $[E : \mathbb{Q}_p] < \infty$. Then ρ is unramified outside pN.*

"HT" at $v \mid p$.

12. Matt Emerton: Automorphic forms

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra of a reductive Lie group $G/\mathbb{C}$. Let $\mathfrak{t} \subset \mathfrak{g}$ be Lie algebra of a maximal torus $T \subset G$. Set $r = \dim \mathfrak{t}$. The Weyl group $W = N(T)/T$ acts on T and so on $\mathfrak{t}$. If $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$, the Harish-Chandra isomorphism says that

$$Z(\mathfrak{g}) \simeq (\operatorname{Sym} \mathfrak{t})^W \simeq \mathbb{C}[x_1, \ldots, x_r].$$

Where does this come from? The idea is that the characteristic polynomial of an element $g \in \mathfrak{g}l_n$ is of the form $x^n - a_{n-1}x^{n-1} + \cdots a_0$, and the a_i are symmetric polynomials.

For a more general group, can still do this, but there could be some additional relations on the n parameters. In the end, there will be r independant pieces of information—each one can then be converted into an element of $Z(\mathfrak{g})$. To actually determine these: if the group is semisimple then $\mathfrak{g} \subset \mathfrak{sl}_n$, but the polynomial corresponds to elements.

Exercise 55. *Find the Casimir of* $\mathrm{SL}_2, \mathrm{Sp}_4$ *in this way.*

$$\begin{array}{ccccc} & & G & & \\ & & \uparrow & & \\ U & \hookrightarrow & P = MU & \twoheadrightarrow & M \end{array}$$

We can assume that $T \subset M \subset G$. so $W_M = N_{M(T)}/T \subset N_{G(T)}/T = W_G$. Thus $Z(\mathfrak{g})$, which equals the set of W_G invariants is a subset of $Z(\mathfrak{m})$ (the set of W_M invariants) which is a subset of $\operatorname{Sym} T$. If $a \in \operatorname{Sym} T$ then $\prod_{w \in W_G}(x - a^w)$ is W_G invariant.

12.1. **Definition of automorphic forms.** Let $G/\mathbb{Q}$ be a connected reductive linear algebraic group. Fix a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. Let

$$\mathcal{A} = \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \mathbb{C})$$

be the set of functions $f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying:

(1) $f\,|_{G(\mathbb{R})}$ is smooth.
(2) There exists $K_f \subset G(\mathbb{A}_f)$ open and compact such that K_f fixes f.
(3) f is K_∞-finite in the $G(\mathbb{R})$ restriction.
(4) $Z(\mathfrak{g}) \cdot f$ is finite dimensional.
(5) f is slowly increasing.

For GL_1, this says that

$$f : \mathbb{R}^\times_{>0} \times (\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}$$

should be a generalized eigenvector of $\frac{\partial_t}{t}$. If $\frac{1}{t}\partial_t f = cf$ then $f \mid_{\mathbb{R}^\times} (t) = \mathrm{sgn}(t)^a \, |t|^s$. We conclude that automorphic forms on GL_1 are linear combinations of idele class characters.

In a torus (as above) there is no interaction between the real place the finite places. This will not be true in general.

In the case of GL_2:

$$\begin{aligned} \mathrm{GL}_2(\mathbb{A}_f) &= \begin{pmatrix} \mathbb{A}_f^\times & \\ & 1 \end{pmatrix} \mathrm{SL}_2(\mathbb{A}_f) \\ &= \begin{pmatrix} \mathbb{Q}^\times & \\ & 1 \end{pmatrix} \mathrm{SL}_2(\mathbb{Q}) \begin{pmatrix} \widehat{\mathbb{Z}}^\times & \\ & 1 \end{pmatrix} \mathrm{SL}_2(\widehat{\mathbb{Z}}) \\ &= \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\widehat{\mathbb{Z}}). \end{aligned}$$

This implies that if K_f is a principal congruence subgroup of level N,

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K_f = \mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}). \tag{7}$$

Choosing N large enough, we can assume that f is a function on (7). Assuming that $Z(\mathfrak{g})$ acts one-dimensionally, one finds that the Casimir acts by hyperbolic Laplacian.

If G is a quaternion group split at ∞: $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$. In this case, one gets Shimura curves: $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ where Γ is a cocompact discrete group.

Suppose $K_\infty \cdot f = U$ is an irreducible representation of K_∞. (A priori, $K_\infty \cdot f$ is a direct sum of irreducibles, but, choosing f in one of these would generate such a U.) Then there is a K_∞-equivariant map $U \hookrightarrow \mathcal{A}$. This of this as $\widetilde{f} : \mathbb{G}(\mathbb{Q}\backslash G(\mathbb{A})/K_f \to U^*$ with a K_∞ equivariance condition. This gives a K_∞ bundle:

$$\begin{array}{c} \mathbb{G}(\mathbb{Q}\backslash G(\mathbb{A})/K_f \\ \big\downarrow \\ \mathbb{G}(\mathbb{Q}\backslash G(\mathbb{A})/K_\infty K_f \end{array}$$

The quotient $\mathbb{G}(\mathbb{Q}\backslash G(\mathbb{A})/K_\infty K_f$ looks like a modular curve or a generalization. To this there is

$$\begin{array}{c} \text{associated vector bundle} \\ \big\downarrow {\scriptstyle U^* \text{ fibers}} \\ \mathbb{G}(\mathbb{Q}\backslash G(\mathbb{A})/K_\infty K_f \end{array}$$

Then "$\widetilde{f}$" is a fiber.

If we fix U, $ann_{Z(\mathfrak{g})}(f)$ and K_f, we get a finite dimensional space. This is a generalization of the finiteness of spaces of classical modular forms. (The growth condition is important here in order to get finiteness.)

12.2. **Eisenstein series.** Let $G/\mathbb{Q}$ be as above. Let $P = MU$ be the Levi decomposition of a parabolic. Let $V \subset \mathcal{A}_M$, so $(\mathfrak{m}, M(\mathbb{A}_f))$ acts on V.

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V = \{f : G(\mathbb{A}) \to v \mid f(pg) = pf(g)\}.$$

There is a map $\mathcal{E}is : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V \to \mathbb{A}_G$.

Philosophically: Maybe

$$V^* \otimes \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V \hookrightarrow \mathbb{A}_G.$$

$$v^* \otimes f : G(\mathbb{A}) \to V^* \otimes V \to \mathbb{C}.$$

This would satisfy all but maybe $G(\mathbb{Q})$ invariance and slowly increasing. (Note that $(v^* \otimes f)(pg) = v^* \otimes pf(g) = p^{-1}v^* \otimes f(g)$.)

To take care of $G(\mathbb{Q})$ invariance, define

$$\mathcal{E}is(v^* \otimes f)(g) = \sum_{\gamma \in G(\mathbb{Q})} (v^* \otimes f)(\gamma g). \tag{8}$$

Does this converge?

If V is automorphic (i.e. $V \subset \mathcal{A}_M$) then there is a special functional $\ell : V \to \mathbb{C}$ called the *automorphic functional*: $\ell(f) = f(1)$.

$$\begin{array}{ccc} V & \hookrightarrow & \mathcal{A}_M \\ & \searrow^{\ell} & \downarrow \\ & & \mathbb{C} \end{array}$$

Then $M(\mathbb{Q})\ell = \ell$ which implies $P(\mathbb{Q})\ell = \ell$. So we choose $v^* = \ell$ in (8) to get

$$\mathcal{E}is(f)(g) = \sum_{\gamma \in G(\mathbb{Q})/P(\mathbb{Q})} (\ell \otimes f)(\gamma g), \tag{9}$$

which is $G(\mathbb{Q})$ invariant whenever the sum converges. In this case it is called an *Eisenstein series.*

Example: Recall the example from Emerton's first lecture:

$$B = (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}) \simeq (i, 1)\mathrm{GL}_2(\mathbb{R}).$$

Let $G = \mathrm{GL}_2$, $P = B_- = \{(\begin{smallmatrix} * & \\ * & * \end{smallmatrix})\}$, $M = T = \{(\begin{smallmatrix} * & \\ & * \end{smallmatrix})\}$. Consider V"="$|a|^{s_1}\,|d|^{s_2}$. Then

$$\begin{aligned} \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} |\cdot|^{s_1} \otimes |\cdot|^{s_2} &= \bigotimes_v \mathrm{Ind}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} |\cdot|_v^{s_1} \otimes |\cdot|_v^{s_2} \\ &= \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} |\cdot|_\infty^{s_1} \otimes |\cdot|_\infty^{s_2} \otimes \bigotimes_{v \text{ finite}} \cdots \\ &\ni f_\infty \otimes 1 \otimes 1 \otimes \cdots \end{aligned}$$

where f_∞ is an even function on $SO(2)$. Can take $f_\infty = (\cdot)^k$ for $k \in 2\mathbb{Z}$. For $\tau = x + iy$, the resulting Eisenstein series $\mathcal{E}is(\tau, 1)$ is

$$\sum_{\substack{b,d \geq 0 \\ (b,d)=1}} \frac{y^{-s_1}}{(b\tau + d)^k \, |b\tau + d|^{s_2 - s_1 - k}}.$$

Setting $s_1 = 0$ and $s_2 = k$ gives the classical Eisenstein series.

He didn't have time to discuss these, but he intended to have the following sections:

12.3. **Cusp forms.**

12.4. **Ramanujan's conjecture.**

12.5. **Hecke operators.**

12.6. L-**functions.**

13. Calegari: Automorphic forms and cohomology

Let f be a classical cusp form of weight $k = 2$. Then

$$\omega = f dz \longrightarrow \Gamma\backslash\mathbb{H}^* = X(\Gamma)$$

gives a class in

$$H^1_{dR}(X(\Gamma)) \to H^1(X(\Gamma), \mathbb{C}).$$

The right hand side is easy to compute. One can compute Hecke operators, and facts about the original modular form. Some questions: Is there a way to find multiplicities of various spaces of modular forms? Can one do this in more generality? (Maass forms are not of cohomological type.) Which automorphic forms contribute to the cohomology of these symmetric spaces? How do you compute these?

Distinction: f a classical modular form of weight k, level Γ. If $k \geq 2$, Eichler-Shimura implies that $H^1_\Gamma(\Gamma, \mathrm{Sym}^{k-2}\mathbb{R}^2)$. This is what is meant by "cohomological type." For $k \geq 1$ there is an algebraic thing that gives some type of cohomology. This is not what we are looking at here.

Cohomology of symmetric spaces or arithmetic quotients:

$$\Gamma\backslash G(\mathbb{R}) \quad \text{or} \quad \Gamma\backslash G(\mathbb{R})/K.$$

Fix $K \subset G = G(\mathbb{R})$ a maximal compact. The symmetric spaces G/K are diffeomorphic to $\mathbb{R}^n$ for some n. If $X = \Gamma\backslash G/K$ then X is an Eilenberg-MacLane $K(\Gamma, 1)$. Hence, the group cohomology $H^1(\Gamma, E)$ is isomorphic to $H^1(X, \overline{E})$. If X is not compact, there are some extra difficulties.

Assume for now that G is semisimple and $X = \Gamma\backslash G/K$ is compact. Let $C^\infty(X)$ be the smooth functions on X, which correspond to 0-forms.

1-forms: Tangent space to G at 1 is $\mathfrak{g}$. A 1-form on G is a section of the n-th exterior power. (When $n = 1$ this is the cotangent bundle.) The cotangent bundle gives (?) by $\mathfrak{g}$. So, the 1-forms are:

$$\mathrm{Hom}(\mathfrak{g}, C^\infty(G)).$$

What about n-forms on G/H? Let $\pi : G \to G/H$ be the projection map. Then can pullback n-forms of G/H via $\pi^* \in \mathrm{Hom}(\bigwedge^n \mathfrak{g}, C^\infty(G))$ to get 1-forms on G/H.

(1) $\omega \in \mathrm{Hom}_H(\bigwedge^0 \mathfrak{g}, C^\infty(G))$.
(2) $\omega(X \wedge \cdots) = 0$ if $X \in \mathfrak{h}$.

n-forms on G/H:

$$\mathrm{Hom}_H(\bigwedge^n \mathfrak{g}/\mathfrak{h}, C^\infty(G)).$$

(At this point I gave up. I couldn't see well, Calegari was moving quickly, and I wasn't really understanding. Sorry.)

14. Matt Emerton: Hecke operators and L-functions

$\mathcal{A} = \mathcal{A}(G(\mathbb{R})\backslash G(\mathbb{A}))$ is a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ representation. What does it look like? What symmetries do you have to break it up? The main thing is the action of $Z(\mathbb{G})$. Goal: decompose $\mathcal{A}$ with respect to $Z(\mathfrak{g})$.

Think of $\mathfrak{sl}_2$. Here $Z(\mathfrak{sl}_2) \simeq \mathbb{C}[T]$. A $\mathbb{C}[T]$-module looks like $\mathbb{C}[T]^r \oplus$torsion. On the free part, T shifts, and so it doesn't have a spectrum.

Let $R = \mathbb{C}[T_1, \ldots, T_r]$, M an R-module. All $m \in M$ are killed by some finite codimensional ideal. $SuppM \subset Spec(R)$ is a finite set of closed points.

$$\mathcal{A} = \bigoplus_{m \in \mathfrak{m}-spec(Z(\mathfrak{g}))} A[m^\infty] \tag{10}$$

Let $A \subset G$ be a maximal central torus split over $\mathbb{Q}$. Then A is the obstruction to $\Gamma\backslash G_\infty/K_\infty^0$ having finite volume. For example, if U' is a finite index subgroup of $\mathcal{O}_F^\times$, $U'\backslash(F^\times \otimes \mathbb{R})^{N=1}$ is compact. So one can put $\Gamma\backslash G_\infty/K_\infty^0 A_\infty^0$ ($A_\infty = A(\mathbb{R})$) to get finite volume.

Another decomposition of $\mathcal{A}$ is made by choosing $\chi : \mathbb{A}(\mathbb{R})^0 \to \mathbb{C}^\times$, and consider the χ-eigenspace $\mathcal{A}_\chi$ of $\mathcal{A}$. What does $\mathcal{A}_\chi$ look like? It won't, in general, be a summand of $\mathcal{A}$ as it will take some parts from each piece of (10).

It will turn out that any function that satisfies the automorphy conditions, except perhaps slowly increasing, which has a character χ under A_∞^0 and is "$L^p \mod A_\infty^0$" is slowly increasing, and so automorphic.

Let S be a maximal split torus quotient of G^{ab}. Then the map $A \to S$ obtained by the following compositions

$$A \hookrightarrow G \longrightarrow G^{ab} \longrightarrow S,$$

is an isogeny. "$L^p \mod A_\infty^0$" means that, by twisting by some character of $S(\mathbb{Q})\backslash S(\mathbb{A})$ to make χ unitary,

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})/A_\infty^0 K_f} |f|^p$$

makes sense.

In general, you can define *cusp forms*. Choose $P = MN$ a parabolic ($N = U$ of previous talks). These are cusps. (GL_2 example: $\mathrm{GL}_2/B \simeq \mathbb{P}(\mathbb{Q}) =$ cusps.)

Fix g. Then

$$f \mapsto c_f(g)(m) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(nmg)dn$$

gives a function on $M(\mathbb{A})$ that is actually automorphic on $M(\mathbb{Q})\backslash M(\mathbb{A})$ on the image of $g_\infty K_\infty g_\infty^{-1}$ in $M(\mathbb{R})$. For a given g, f, $c_f(g)$ is called the *constant term.*

Exercise 56. *See if you can reconcile this with the constant term of an Eisenstein series. This is how Langlands discovered his L-functions. (See Langlands Euler Products, Borel-Jacquet Proc. Symp. 33, and Langlands Proc. Symp. 33. Also, see an article of Li and Schewen... on vanishing of cohomology.)*

Definition 57. *f is* cuspidal *if the constant term $c_f(g)$ vanishes for every parabolic. (If you only want maximal terms to vanish, you only need to check for maximal parabolics (i.e. the Borels.)*

If f is cuspidal, f is rapidly decreasing and L^p for any p–in particular, L^2. This allows one to show that $\mathcal{A}_{\chi,cusp}$ is semisimple. This is in stark constrast to the Eisenstein series which are highly nonsemisimple.

Definition 58. *A representation V of $G(\mathbb{A})$ is* automorphic *if it is an irreducible subquotient of $\mathcal{A}$. It is* cuspidal *if it is an irreducible subquotient (actually it will be a sub) of $\mathcal{A}_{cusp}$.*

Such a representation V can always be obtained by taking X, the $G(\mathbb{A})$ orbit of a an element $x \in \mathcal{A}$, and then quotienting by a submodule Y. Therefore V is admissible and annihilated by an ideal $J \subset Z(\mathfrak{g})$ of finite codimension. Being admissible for the global group means that V^{K_f} is an admissible $(\mathfrak{g}, K_\infty)$-module for all K_f open in $G(\mathbb{A}_f)$.

From such a V on gets $V_\infty, V_2, V_3, \ldots$ such that V_∞ is an irreducible $(\mathfrak{g}, K_\infty)$-module, and V_p are irreducible admissible $G(\mathbb{Q}_p)$ representations such that for almost all p $V_p^{G(\mathbb{Z}_p)}$ is 1-dimensional. If v_p is a generator of all such, they can be used to form $\otimes V_p \simeq V$.

Suppose that $\dim V_p^{G(\mathbb{Z}_p)} = 1$. Then

$$\mathcal{H}_p^{sph} = \mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)]$$

acts by a character. Suppose that G is split over $\mathbb{Q}_p$. Then $\mathcal{H}_p^{sph} \simeq Rep(\widehat{G})$. $\widehat{G}$ is the Langlands dual group. For example,

G	GL_n	SL_n	$SO(2n)$	$SO(2n+1)$
$\widehat{G}$	GL_n	PGL_n	$SO(2n)$	Sp_{2n}

If G is not split over $\mathbb{Q}_p$ but unramified (i.e. $G(\mathbb{Z}_p)$ is a hyperspecial maximal compact) then $\mathcal{H}_p^{sph}$ is the $\widehat{G}$ conjugacy classes of $\widehat{G} \rtimes G_{\mathbb{Q}_p}^{ur} = {}^L G$ lying over $Frob_p \in G_{\mathbb{Q}_p}^{ur}$. More or less, this means choose $B \subset G_{\mathbb{Q}_p}$, take Galois element acting on B to get B'. Some $w \in W$ takes B' to B.

In short, any p where V_p is unramified gives $cl_p \in {}^L G$ lying over $Frob_p$.

$\mathrm{GL}_2(\mathbb{R})$: Consider the representation $\det^a \otimes \mathrm{Sym}^b \mathbb{C}^2$ where $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{\geq 0}$. Then $cl_p = \left(\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right)$ where α, β are roots of the Hecke polynomial $X^2 - T_p X + T_{p,p}$ which is the characteristic polynomial of cl_p.

If we choose any $\rho : {}^L G \to \mathrm{GL}_N(\mathbb{C})$, can compute the caracteristic polynomial of $\rho(cl_p)$. This is a polynomial with coefficients certain Hecke eigenvalues of $V^{G(\mathbb{Z}_p)}$. For example,

$$\mathrm{GL}_2 \hookrightarrow \mathrm{GL}_3 \qquad \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha^2 & & \\ & \alpha\beta & \\ & & \beta^2 \end{pmatrix},$$

which has characteristic polynomial:

$$X^3 - (\text{tr on } \mathbb{C}^3)X^2 + (\text{tr of } \bigwedge^2 \mathbb{C}^3)X - (\det \text{on} \mathbb{C}^3)$$

as element of GL_3. As element of GL_2 get $X^3 - \cdots$. Hecke eigenvalues remember all coefficients of the different characteristic polynomials.

Let S be the set of primes ramified for either V or G. Define

$$L_S(V, s, \rho) = \prod_{p \notin S} \det(Id - p^{-s}\rho(cl_p))^{-1}.$$

if V is automorphic this will converge for s sufficiently large.

Conjecture 59 (Langlands)**.** *$L_S(V, s, \rho)$ can be completed to $L(V, s, \rho)$ which has analytic continuation.*

Local Langlands: classsify the other representations at "bad" primes. (This is known for GL_n.)

Langlands' idea: $V \rightsquigarrow cl_p \in {}^L G$, $\rho \rightsquigarrow \rho(cl_p) \in \mathrm{GL}_N$. Does there exist W a representation of GL_N such that the correspdonding $\rho(cl_p)$ gives the same data? Suppose we have G, H groups (H quasi-split) and

$$\begin{array}{ccc} {}^L G & \xrightarrow{\psi} & {}^L H \\ & \searrow \quad \swarrow & \\ & G_{\mathbb{Q}} & \end{array}$$

If V is an automorphic representation of G, get $\psi(cl_p) \in {}^L H$. Then there should be W automorphic on H. For example, take GL_2 and B a quaternion algebra. The identity map ${}^L\mathrm{GL}_2 \to {}^L B$ for this idea is the Jacquet-Langlands correspondence.

Langlands' conjecture (functoriality) different though closely related to another conjecture: reciprocity. Their intersection is the Artin conjecture which is concerned with N-dimensional Artin representations

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C}) \times G_{\mathbb{Q}}.$$