Trilinear forms and subconvexity of the triple product L-function

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What is subconvexity?

Let $L(s,f)$ be an L-function.

- $$L(s,f) = \sum_{n\geq 1} \frac{a_n}{n^s} = \prod_p (1-\alpha_1(p)p^{-s})^{-1}\cdots(1-\alpha_d(p)p^{-s})^{-1}.$$
- There exist a gamma factor
$$\gamma(s,f) = \pi^{-ds/2}\prod_{j=1}^{d}\Gamma\left(\frac{s+t_i}{2}\right)$$
- There is an integer $N(f)$ called the conductor.
- Setting $\Lambda(s,f) = N(f)^{s/2}\gamma(s,f)L(f,s)$ there is a functional equation
$$\Lambda(f,s) = \varepsilon(f)\Lambda(\bar{f},1-s).$$

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What is subconvexity?

General methods allow one to show that if $f \in \mathcal{F}$ then

$$L(s, f) \ll [N_\infty(s)N(f)]^{\frac{1}{4}+\epsilon}.$$

where $N_\infty(s)$ depends on the values t_j.

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The triple product L-function: Classical formulation

Let $f, g, h \in S_k(\Gamma_0(N))$ be eigenforms. Then write

$$f(z) = \sum_{n=1}^{\infty} a_n(f) q^n.$$

We are interested in

$$L(s, f \times g \times h) = \sum_{n=1}^{\infty} \frac{a_n(f) a_n(g) a_n(h)}{n^s}.$$

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Representation theoretic point of view

Notation:

- F a number field, v a place of F, F_v the completed local field, $\mathcal{O}_v$ the ring of integers.
- $\mathbb{A} = \mathbb{A}_F = \prod' F_v$, the ring of adeles.
- For $i = 1, 2, 3$, let π_i be irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ (with trivial central character.)
- $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$.

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Subconvexity for the triple product L-function: Eigenvalue aspect

Idea: Fix π_1 and π_2 and vary π_3 is some way. We want to find a subconvexity bound for $L(\frac{1}{2}, \Pi)$.

Theorem (Bernstein-Reznikov)

Let $F = \mathbb{Q}$. Fix π_1, π_2 corresponding to Maass forms for $\mathrm{SL}_2(\mathbb{Z})$. There is a subconvexity bound for $L(\frac{1}{2}, \Pi)$ for π_3 corresponding to a level 1 Maass form of varying eigenvalue.

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Subconvexity for the triple product L-function: Level aspect

Relaxed conditions: Allow F to be any number field, π_1, π_2 to have nontrivial conductors and arbitrary ∞ type, and let $\pi_{3,\infty}$ to vary in a "bounded set."

Theorem (Venkatesh)

Suppose that the conductor of π_3 is a prime $\mathfrak{p}$ relatively prime to the conductors of π_1, π_2. For any $\varphi_i \in \pi_i$,

$$\left| \int_{[G]} \varphi_1(g)\varphi_2(g)\varphi_3(g)dg \right| \ll \|\varphi_1\|_4 \|\varphi_2\|_4 \|\varphi_3\|_2 N(\mathfrak{p})^{\epsilon - C}$$

for an explicit $C > 0$. ($N(\mathfrak{p})$ is the norm, $[G] = Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ and $\|\cdot\|_p$ is the L^p-norm.)

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Subconvexity for the triple product L-function: Level aspect

Conjecture (Venkatesh)

Let π_i be as above, φ_3 be the new vector. Then for $i = 1, 2$ there are finite collections $\mathcal{F}_i$ and $\varphi_i \in \mathcal{F}_i$ such that

$$L(\tfrac{1}{2}, \Pi) \ll N(\mathfrak{p})^{1+\epsilon} \left| \int_{[G]} \varphi_1(g)\varphi_2(g)\varphi_3(g)dg \right|^2 .$$

Combined with Venkatesh's theorem this would give subconvexity.

"Theorem" (W.)

The conjecture is true.

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Connection to trilinear forms

Let $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \in \Pi$. We want to say something about

$$J(\varphi) = \int_{[G]} \varphi(g)dg.$$

This is a *trilinear form* on Π.
Fact:

$$\dim \mathrm{Hom}_{B_{\mathbb{A}}^{\times}}(\Pi^B, \mathbb{C}) \leq 1.$$

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Local obstruction

Theorem (Prasad,Prasad-Loke)

Let $\pi_{i,v}$ $(i = 1, 2, 3)$ be admissible representations of $\mathrm{GL}_2(F_v)$. Let B_v be the division quaternion algebra over F_v and $\pi_{i,v}^{JL}$ the corresponding Jacquet-Langlands representation of $B_v^\times$. Then

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F_v)}(\Pi_v, \mathbb{C}) + \dim \mathrm{Hom}_{B_v^\times}(\Pi_v^{JL}, \mathbb{C}) = 1.$$

Which space is nonzero is determined by $\epsilon_v(\frac{1}{2}, \Pi_v)$.

If v is finite (infinite) then $\epsilon_v(\frac{1}{2}, \Pi_v)$ can be -1 only when $\pi_{i,v}$ is ramified (discrete series) for all $i = 1, 2, 3$.

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$L(\frac{1}{2}, \Pi)$ can't distinguish between quaternions

If $\Pi^{JL} \neq 0$ then $L(s, \Pi) = L(s, \Pi^{JL})$.

Theorem (Harris, Kudla)

Let Π be as above. Then $L(\frac{1}{2}, \Pi) \neq 0$ if and only if the global trilinear form

$$J : \Pi^B \to \mathbb{C} \qquad \varphi \mapsto \int_{[B^\times]} \varphi(b)db$$

is nonzero for some choice of B. (By Prasad, when such a B exists it is unique.)

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The correct theorem

So, we may need to replace G by $B^\times$.

Theorem (W.)

Let π_1, π_2 have fixed conductors $\mathfrak{n}_1, \mathfrak{n}_2$. Fix $\mathfrak{n}$. If π_3 has conductor $\mathfrak{n}\mathfrak{p}$, there exists a finite collection $\mathcal{B}$ of quaternion algebras and finite collections $\mathcal{F}_i^B \subset \pi_i^B$ for $B \in \mathcal{B}$ and $i = 1, 2$ such that

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Application to subconvexity

Let S_∞ be the set of real infinite places, and S_f be set of places dividing $\gcd(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n})$. Then by Prasad and Loke

$$\mathcal{B} = \{B \mid \Sigma_B \subset S_\infty \cup S_f\}.$$

In Venkatesh's case, $S_f = \emptyset$. So, for his theorem to imply subconvexity, there is a necessary and sufficient restriction on $\pi_{i,\infty}$. (Namely, there is a condition on the weights k_i for real place v such that $\pi_{i,v}$ are discrete series of weight k_i.)

If his theorem could be generalized to arbitrary quaternion algebras, with my theorem, this would give subconvexity unconditionally and more generally.

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Reformulation

It's easier to work with forms on $\Pi^B \otimes \widetilde{\Pi}^B$.

$$\dim \mathrm{Hom}_{B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times}(\Pi^B \otimes \widetilde{\Pi}^B, \mathbb{C}) \leq 1.$$

Example of an element:

$$I(\varphi \otimes \widetilde{\varphi}) = \int_{[B^\times]} \int_{[B^\times]} \varphi(b_1)\widetilde{\varphi}(b_2)\, db_1\, db_2$$

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By definition, there is

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for all $b \in B_v^\times$.

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for all $b \in B_v^\times$.

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Local trilinear forms

Again,

$$\dim \mathrm{Hom}_{B_v^\times \times B_v^\times}(\Pi_v^B \otimes \widetilde{\Pi}_v^B, \mathbb{C}) \leq 1.$$

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Normalization

Proposition (Ichino-Ikeda)

Whenever everything is unramified

$$I_v'(\varphi_v \otimes \widetilde{\varphi}_v) = \frac{L_v(\frac{1}{2}, \Pi_v)}{\zeta_v(2) L(1, \Pi, Ad)}.$$

$$I_v = \left(\frac{L_v(\frac{1}{2}, \Pi_v)}{\zeta_v(2) L(1, \Pi, \mathrm{Ad})} \right)^{-1} I_v'.$$

This gives a global form:

$$\varphi \otimes \widetilde{\varphi} = \bigotimes_v (\varphi_v \otimes \widetilde{\varphi}_v) \mapsto \prod_v I(\varphi_v \otimes \widetilde{\varphi}_v)$$

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Two global forms must differ by a constant

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- Bound growth of $L(1, \Pi, \mathrm{Ad})$.
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Further directions and applications

- Generalize Venkatesh's work to arbitrary B.
- Local matrix coefficients and trilinear forms in supercuspidal cases and on division quaternion algebra.
- Reprove Prasad's theorem on ϵ-factors 'analytically.'
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