Total positivity, SS 17

Exercise Sheet 1

Exercise 1.(Minor counting) Let $n \in \mathbb{N}$, $n \geq 1$. Consider the following two sets:

$$\mathcal{A} = \{(I, J) \mid I \subset [n], \quad J \subset [n], \quad I, J \neq \emptyset, \quad \#I = \#J\},$$

$$\mathcal{B} = \{(i_1, \ldots, i_n) \mid 1 \leq i_1 < i_2 < \ldots < i_n \leq 2n, \quad i_1 \neq n+1\}.$$

1. Construct a bijection between $\mathcal{A}$ and $\mathcal{B}$.
2. Deduce that $\#\mathcal{A} = \binom{2n}{n} - 1$.
3. Deduce the following identity:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Exercise 2. (Cauchy-Binet) Let $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 3 & 8 & 2 \\ 5 & 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 7 & 7 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. Apply Cauchy-Binet formula to evaluate $\det(AB)$.

Exercise 3. (Apply Cauchy-Binet to get Cauchy)

(1). Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \in \mathcal{M}_{2,3}(\mathbb{R})$. Use both definition and Cauchy-Binet formula to evaluate $\det(AA^T)$ and derive the following identity:

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 = \sum_{1 \leq i < j \leq 3} (a_i b_j - a_j b_i)^2.$$

(2). Deduce the Cauchy inequality in $\mathbb{R}^3$: for $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$,

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1 b_1 + a_2 b_2 + a_3 b_3)^2.$$

When the equality holds?

(3). Prove the general Cauchy inequality: for $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$,

$$\left(\sum_{j=1}^{n} a_j^2 \right) \left(\sum_{j=1}^{n} b_j^2 \right) \geq \left(\sum_{j=1}^{n} a_j b_j \right)^2.$$

Exercise 4. (Laplacian game) Evaluate the determinant of the following matrix by expanding along its second and third rows:

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}.$$

Exercise 5. (Dodgson condensation) Let $x_{i,j}$ $(1 \leq i,j \leq n)$ be n^2 variables, $\mathbb{K} = \mathbb{C}(x_{1,1}, x_{1,2}, \ldots, x_{n,n})$ be the field of rational functions in variables $x_{i,j}$, A be an $n \times n$ matrix whose (j,k)-entry is $x_{j,k}$. We define a sequence of matrices $(A^{(i)})_{i=0,\ldots,n}$, $A^i \in \mathcal{M}_{n-i+1}(\mathbb{K})$, as follows: we denote the (j,k) entry of $A^{(i)}$ by $a^{(i)}_{j,k}$,

(1). $A^{(0)}$ is an $(n+1) \times (n+1)$ matrix whose entries are all 1: $a^{(0)}_{j,k} = 1$;

(2). $A^{(1)} = A$: $a^{(1)}_{j,k} = x_{j,k}$;

(3). We inductively define the matrix $A^{(i+1)} \in \mathcal{M}_{n-i}(\mathbb{K})$ by:

$$a^{(i+1)}_{j,k} = \frac{\det \begin{bmatrix} a^{(i)}_{j,k} & a^{(i)}_{j,k+1} \\ a^{(i)}_{j+1,k} & a^{(i)}_{j+1,k+1} \end{bmatrix}}{a^{(i-1)}_{j+1,k+1}}.$$

The goal of this exercise is to prove the following Dodgson condensation theorem, which provides an efficient way to compute determinants for some matrices:

Theorem. (C. Dodgson) The 1×1 matrix $A^{(n)} = (\det A)$.

Step 1. Prove Dodgson condensation theorem for $n = 3$.

Step 2. Prove Dodgson condensation by induction on n and Jacobi's formula. (Hint: Assume that $A \in \mathcal{M}_{n+1}(\mathbb{K})$, what is $A^{(n)}$ by induction hypothesis?)

Application. Compute $\det A$ for the following matrix using Dodgson condensation theorem:

$$A = \begin{bmatrix} 2 & -3 & 1 & 2 & 5 \\ 4 & 1 & -2 & -3 & 2 \\ 5 & -4 & 2 & 2 & -3 \\ 3 & -1 & 5 & 2 & 1 \\ -4 & 1 & 5 & -1 & 2 \end{bmatrix}.$$

You should know this person, Charles Lutwidge Dodgson (1832-1898), who first stated and proved the Dodgson condensation theorem. He has a pen name Lewis Carroll: yes he is the author of Alice's Adventures in Wonderland!

We finish this exercise by the following poem of Lewis Carroll:

I often wondered when I cursed,
Often feared where I would be –
Wondered where she'd yield her love
When I yield, so will she,
I would her will be pitied!
Cursed be love! She pitied me...

A mathematical way to look at this poem is via matrices (try to find a symmetry):

$$\begin{bmatrix} \text{I} & \text{often} & \text{wondered} & \text{when} & \text{I} & \text{cursed} \\ \text{Often} & \text{feared} & \text{where} & \text{I} & \text{would} & \text{be} \\ \text{Wondered} & \text{where} & \text{she'd} & \text{yield} & \text{her} & \text{love} \\ \text{When} & \text{I} & \text{yield} & \text{so} & \text{will} & \text{she} \\ \text{I} & \text{would} & \text{her} & \text{will} & \text{be} & \text{pitied!} \\ \text{Cursed} & \text{be} & \text{love!} & \text{She} & \text{pitied} & \text{me...} \end{bmatrix}.$$