Total positivity, SS 17

Exercise Sheet 2

to be discussed on 18.05.2017

This sheet contains five regular exercises and one optional exercise.

Exercise 1. (4×4 matrices) Let $A = (a_{i,j}) \in \mathcal{M}_4(\mathbb{R})$ be a 4×4 matrix.

1. To determine the total positivity of A using Fekete criterion, which minors are necessary?
2. To determine the total positivity of A using Gasca-Peña criterion, which minors are necessary?
3. Give (with detailed proof) a totally positive matrix $M \in \mathcal{M}_4(\mathbb{R})$.

Exercise 2. (A familiar matrix) Evaluate the determinant of the following Hilbert matrix:

$$\begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{bmatrix}.$$

Exercise 3. (Small pertubation) Let $\mathbf{E}_{i,j}$ be the matrix whose (i, j) entry is 1 and all other entries are zero. Let $A \in \mathcal{M}_{n,m}(\mathbb{R})$ be a totally positive matrix. Prove that for $t > 0$, $A + t\mathbf{E}_{1,1}$ is totally positive.

Exercise 4. (Restricted Gauß elimination) Let $A \in \mathcal{M}_{n,m}(\mathbb{R})$ be a totally positive matrix, and A' be the matrix obtained by adding a positive multiple of a row (column) to the preceding or the succeeding row (column).

1. Prove that A' is totally positive.
2. How about adding a positive multiple of a row (column) to an arbitrary row (column)?

Exercise 5. (Totally positive kernel) Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be real numbers satisfying $x_1 < x_2 < \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$. Let $A = (a_{i,j}) \in \mathcal{M}_n(\mathbb{R})$ be the matrix defined by

$$a_{i,j} = \exp(x_i y_j).$$

Prove that A is totally positive. (Hint: apply the method of proving the total positivity of van der Monde matrices.)

Optional Exercise. (will not be discussed in the exercise class)
This exercise propose a method to compute the power sum of natural numbers using Taylor formula: for $r \in \mathbb{N}$, we denote

$$S(n,r) = 1^r + 2^r + 3^r + 4^r + \cdots + n^r.$$

We know that $S(n,1) = \frac{n(n+1)}{2}$ and $S(n,2) = \frac{1}{6}n(n+1)(2n+1)$.

(1) Consider the linear map $D : \mathbb{R}[X] \to \mathbb{R}[X]$ defined by: $P(X) \mapsto \frac{d}{dX}P(X)$. Prove that $\exp(D) : \mathbb{R}[X] \to \mathbb{R}[X]$,

$$\exp(D)(P(X)) = P(X) + \frac{dP(X)}{dX} + \frac{1}{2!}\frac{d^2P(X)}{dX^2} + \cdots + \frac{1}{n!}\frac{d^nP(X)}{dX^n} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} D^n(P(X))$$

is a linear map.

(2) Using Taylor expansion formula, prove that for $P(X) \in \mathbb{R}[X]$, $\exp(D)(P(X)) = P(X+1)$.

(3) We define the Bernoulli numbers $B_n \in \mathbb{R}$ by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

and consider the Bernoulli polynomial

$$B_n(X) = \sum_{i=0}^{n} \binom{n}{i} B_i X^{n-i}, \quad \text{where} \quad \binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

Evaluate B_0, B_1, B_2 and B_3, then deduce $B_0(X)$, $B_1(X)$, $B_2(X)$ and $B_3(X)$.

(4) Compare $B_2(n)$, $B_3(n)$ with $S(n,1)$, $S(n,2)$.

(5) Acting

$$D = (\exp(D) - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} D^n$$

on X^r and apply (3), prove that $rX^{r-1} = B_r(X+1) - B_r(X)$.

(6) Prove that $B_r(0) = B_r$, and deduce that

$$S(n,r) = \frac{1}{r+1}(B_{r+1}(n+1) - B_{r+1}).$$