Total positivity, SS 17

Exercise Sheet 4

to be discussed on 22.06.2017

This sheet contains five regular exercises.

Exercise 1. (Down to earth factorisation)

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 8 & 10 \\ 3 & 12 & 21 \end{bmatrix}$.

1. Prove that A is totally positive.
2. Determine the positive parameters $t_1, \cdots, t_9 > 0$ such that

$$A = y_2(t_1)y_1(t_2)y_2(t_3)h_1(t_4)h_2(t_5)h_3(t_6)x_1(t_7)x_2(t_8)x_1(t_9).$$

Exercise 2. (Factorisation revisited) Let

$$\sigma = \binom{123456}{546231} \in \mathfrak{S}_6.$$

Write σ into a product of transpositions s_1, s_2, s_3, s_4 and s_5.

Exercise 3. (Play with length function) Let $\ell : \mathfrak{S}_n \to \mathbb{N}$ be the length function. Prove the following statements:

1. for $1 \leq a \neq b \leq n$, $\ell((a,b)) = 2|b-a| - 1$;
2. for $\sigma \in \mathfrak{S}_n$, $\ell(\sigma) = \ell(\sigma^{-1})$;
3. let $\sigma \in \mathfrak{S}_n$ and $\sigma = u_1 \cdots u_{n-1}$ is its decomposition in Corollary 3.11 where $u_k \in A_k$. Then $\ell(\sigma) = \ell(u_1) + \cdots + \ell(u_{n-1})$.

Exercise 4. (Move, move, move)
Show that $s_1s_2s_3s_2s_1$, $s_1s_3s_2s_3s_1$ and $s_3s_1s_2s_1s_3$ are reduced words for the same permutation $\sigma \in \mathfrak{S}_4$. Determine all other reduced decompositions of σ.

Exercise 5. (Length function revisited)
Let $\sigma \in \mathfrak{S}_n$ and w_0 be the longest element (*i.e.* element of maximal length) in $\mathfrak{S}_n$. Prove that

$$\ell(\sigma^{-1}w_0) = \binom{n}{2} - \ell(\sigma).$$