Total positivity, SS 17

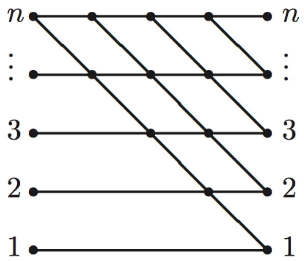
Exercise Sheet 6

to be discussed on 20.07.2017

This sheet contains three regular exercises.

Exercise 1. (Half chicken)

Consider the following network (N, w) where all edges have weight 1: the boundary vertices, which connects to $1, \cdots, n$, are not drawn; the horizontal edges are oriented to the right, and the slanted edges are oriented from upper-left to down-right.



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1. Is the network (N, w) totally connected? Why?
2. Find the boundary measurement matrix $X(N, w)$.
3. Is the matrix $X(N, w)$ triangular totally positive? Is the matrix $X(N, w)$ totally non-negative? Give your argument.

Exercise 2. (Entire chicken)

For $n \geq 3$, prove that

1. the directed graph $\Gamma_{0,n}$ is totally connected;
2. the directed graph $\Gamma_{0,n}$ satisfies property (I).

Exercise 3. (Grid lattices and symmetric functions)

Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition with $\lambda_r > 0$ and $n \geq r$. We consider the following network $(\mathcal{L}_\lambda, w)$ defined in $\mathbb{R}^2$ as follows:

- The vertices are points (i, j) such that $i, j \in \mathbb{N}$, $1 \leq i \leq n$ and $0 \leq j \leq n + \lambda_1 - 1$, as well as $(0, n-k)$, $(n+1, n+\lambda_k - k)$ for $k = 1, \cdots, n$.
- The edges are defined as follows: for a vertex (i, j) with $1 \leq i \leq n$ and $0 \leq j \leq n + \lambda_1 - 1$, if $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$ are vertices, then the directed edges (the arrows go from source to target) are:

$$e_{i,j}^{i+1,j} : (i, j) \to (i+1, j), \quad e_{i+1,j}^{i+1,j+1} : (i+1, j) \to (i+1, j+1),$$

$$e_{i,j}^{i,j+1} : (i,j) \to (i,j+1), \quad e_{i,j+1}^{i+1,j+1} : (i,j+1) \to (i+1,j+1).$$

Then we add the following directed edges:

- $e_{0,0}^{1,0}, e_{0,1}^{1,1} \cdots, e_{0,n-1}^{1,n-1}$;
- $e_{n,n+\lambda_j-j}^{n+1,n+\lambda_j-j}$ for $j = 1, 2, \cdots, n$.

- The source set $A = \{A_1, \cdots, A_n\}$ where $A_i = (0, n-i)$, the sink set $B = \{B_1, \cdots, B_n\}$ where $B_j = (n+1, n+\lambda_j - j)$ (recall that if $j > r$ then $\lambda_j = 0$).
- The weight function w on edges taking values in $\mathbb{Z}[x_1, \cdots, x_n]$ is defined as follows:

$$w(e_{i,j}^{i,j+1}) = x_i, \quad w(e_{i+1,j}^{i+1,j+1}) = x_{i+1},$$

  and all other edges have weight 1.

Let $X(\mathcal{L}_\lambda, w)$ be the associated boundary measurement matrix.

1. Prove that the (i,j)-entry of $X(\mathcal{L}_\lambda, w)$ is the complete symmetric function $h_{\lambda_j - j + i}(x_1, \cdots, x_n)$.
2. Prove that $\det X(\mathcal{L}_\lambda, w) = s_\lambda(x_1, \cdots, x_n)$ is the Schur function.