General Linear Groups, WS 18/19

Exercise Sheet 1

Exercise 1. For two sets X and Y, denote by $\mathcal{F}(X,Y)$ the set of functions between X and Y.

1. If X and Y are G-sets, show that $\mathcal{F}(X,Y)$ is a G-set via: for $f \in \mathcal{F}(X,Y)$, $g \in G$ and $x \in X$,
$$(g \cdot f)(x) := g \cdot f(g^{-1} \cdot x).$$
2. Let X, Y be two G-sets. Prove that $\mathcal{F}_G(X,Y) = \mathcal{F}(X,Y)^G$.

Exercise 2. Let G be a group, $H \subseteq G$ be a subgroup and G/H be the coset space.

1. Prove that $\rho : G \to \mathrm{Bij}(G/H)$, for $g, a \in G$, $\rho(g)(aH) := gaH$, defines a G-action on G/H.
2. Is this action transitive?
3. Is this action free? If not, determine the stabilizer $\mathrm{Stab}_G(gH)$; if yes, prove it.
4. Prove that this action if faithful if and only if
$$\bigcap_{g \in G} gHg^{-1} = \{e_G\}.$$

Exercise 3. For $n \geq 2$ and $1 \leq d \leq n-1$, let $\mathrm{Gr}_d(\mathbb{C}^n)$ denote the set of d-dimensional $\mathbb{C}$-vector subspaces in $\mathbb{C}^n$. Let

$$\mathrm{P}_d := \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in \mathrm{GL}_d(\mathbb{C}),\ B \in \mathcal{M}_{d,n-d}(\mathbb{C}),\ C \in \mathrm{GL}_{n-d}(\mathbb{C}) \right\}.$$

Prove that there exists a bijection

$$\mathrm{GL}_n(\mathbb{C})/\mathrm{P}_d \cong \mathrm{Gr}_d(\mathbb{C}^n).$$

The set $\mathrm{Gr}_d(\mathbb{C}^n)$ is called a Graßmann variety, or a Graßmannian.

Exercise 4. Let G be a group and $H \subseteq G$ be a subgroup. The normalizer of H in G, denoted by $\mathcal{N}_G(H)$, is defined by:
$$\mathcal{N}_G(H) := \{g \in G \mid gHg^{-1} = H\}.$$

1. Prove that H is a normal subgroup in $\mathcal{N}_G(H)$.
2. Show that $\mathcal{N}_{\mathrm{GL}_n(\mathbb{C})}(B_n(\mathbb{C})) = B_n(\mathbb{C})$.
3. Determine the flag $V_\bullet \in \mathcal{F}_n(\mathbb{C})$ having stabilizer $B_n(\mathbb{C})$ under the $\mathrm{GL}_n(\mathbb{C})$-action.

Exercise 5. (Complex structures on a real vector space)

Let $n \geq 1$ be a natural number. We denote

$$\mathcal{C} = \{M \in \mathcal{M}_n(\mathbb{R}) \mid M^2 = -\mathbf{I}_n\}.$$

Elements in $\mathcal{C}$ are called complex structures on $\mathbb{R}^n$.

1. Show that $\mathbb{R}^n$ admits a complex structure if and only if n is even.
2. In this exercise we set $n = 2$. Let $\mathrm{tr} : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ be the trace function.
   (a) Prove that $\mathcal{C} \subseteq \ker(\mathrm{tr})$.
   (b) Consider the map
   $$\varphi : \mathbb{R}^3 \to \ker(\mathrm{tr}), \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$
   determine the pre-image of $\mathcal{C}$.
   (c) Let $\mathcal{H} := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid b > 0 \right\}$ be the Poincaré upper half plane. Fix $\mathbf{v} := \begin{pmatrix} a \\ b \end{pmatrix} \in \mathcal{H}$. Prove that there exists $M \in \mathcal{C}$ such that its first column is $\mathbf{v}$. This gives a map $\mathcal{H} \to \mathcal{C}$. Prove that it is a homeomorphism (continuous bijection, whose inverse map is also continuous) to its image, then determine the image.
3. Let M be a complex structure on $\mathbb{R}^{2n}$. Prove that
   $$\mathbb{C} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad (a + bi, \mathbf{v}) \mapsto a\mathbf{v} + bM\mathbf{v}$$
   defines a $\mathbb{C}$-vector space structure on $\mathbb{R}^{2n}$. We will call this $\mathbb{C}$-vector space structure "associated to M".
4. Let $\mathbf{v}_1, \cdots, \mathbf{v}_d \in \mathbb{R}^{2n}$. Prove that $\{\mathbf{v}_1, \cdots, \mathbf{v}_d\}$ forms a basis of the $\mathbb{C}$-vector space structure associated to M if and only if $\{\mathbf{v}_1, \cdots, \mathbf{v}_d, M\mathbf{v}_1, \cdots, M\mathbf{v}_d\}$ is a basis of the $\mathbb{R}$-vector space $\mathbb{R}^{2n}$. Then deduce the dimension of the $\mathbb{C}$-vector space structure on $\mathbb{R}^{2n}$ associated to M.
5. Verify that the map $\varphi_M : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \mathbf{v} \mapsto M\mathbf{v}$ is a linear map with respect to the $\mathbb{C}$-vector space structure on $\mathbb{R}^{2n}$ associated to M on both sides. Then deduce that $\mathcal{C} = \mathrm{GL}_{2n}(\mathbb{R}) \cdot J_n$, where
   $$J_n = \begin{pmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{pmatrix} \in \mathcal{C}.$$
   Determine the matrix of φ_M in a basis of the $\mathbb{C}$-vector space $\mathbb{R}^{2n}$ associated to M.
6. Use the matrix J_n to give an injective map $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{R})$.
7. Prove that there exists a bijection $\mathcal{C} \cong \mathrm{GL}_{2n}(\mathbb{R})/\mathrm{GL}_n(\mathbb{C})$.