General Linear Groups, WS 18/19

Exercise Sheet 3

Exercise 1. (Draw something!)

Draw the Hasse diagram of the dominance order on $\mathcal{P}(7)$. Study the orbits and the orbit closures of the $G = GL_7(\mathbb{C})$ conjugation action on the nilpotent cone $\mathscr{N}_7 \subset \mathcal{M}_7(\mathbb{C})$: for each of the orbit, give the Jordan normal form and the Young diagram.

Exercise 2. (Not always possible to glue)

Let $A, \widetilde{A}$ be two matrices in $\mathcal{M}_4(\mathbb{C})$:

$$A := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{A} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1. Compute the minimal polynomials $\mu_A(t)$ and $\mu_{\widetilde{A}}(t)$.
2. Show that A and $\widetilde{A}$ are not in the same $GL_n(\mathbb{C})$-orbit.
3. Compute the Jordan-Dunford-Chevalley decomposition of $A = D + N$ and $\widetilde{A} = \widetilde{D} + \widetilde{N}$.
4. Show that N and $\widetilde{N}$ are in the same nilpotent orbit.
5. Write $\widetilde{D}$ and $\widetilde{N}$ as polynomials in $\widetilde{A}$.

Exercise 3. (Diagonalisable vs semi-simple)

Let $A \in \mathcal{M}_n(\mathbb{C})$. We call a subspace $W \subseteq \mathbb{C}^n$ A-stable, if $A(W) \subseteq W$. Prove that $A \in \mathcal{M}_n(\mathbb{C})$ is diagonalisable, if and only if for any A-stable subspace W of $\mathbb{C}^n$, that exists an A-stable subspace $W' \subseteq \mathbb{C}^n$ such that $\mathbb{C}^n = W \oplus W'$.

Exercise 4. Let $M \in GL_n(\mathbb{C})$. Assume that there exists $k \geq 1$ such that M^k is diagonalisable. Prove that M is diagonalisable. (Hint: apply lemma of kernel to $M^k - \lambda I_n$.)

Exercise 5. (Harish-Chandra isomorphism)

Let $\mathbb{C}[\mathcal{M}_n(\mathbb{C})]$ denote the ring of polynomial functions on $\mathcal{M}_n(\mathbb{C})$: that is to say, functions $f : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ given by a polynomial in entries of a matrix in $\mathcal{M}_n(\mathbb{C})$. Let $\mathscr{T}_n \subset \mathcal{M}_n(\mathbb{C})$ denote the set of diagonal matrices and $\mathbb{C}[\mathscr{T}_n]$ the ring of polynomial functions on $\mathscr{T}_n$.

The group $\mathfrak{S}_n$ acts on $\mathscr{T}_n$ by permuting the diagonal elements; the group $GL_n(\mathbb{C})$ acts on $\mathbb{C}[\mathcal{M}_n(\mathbb{C})]$ by: for $P \in GL_n(\mathbb{C})$, $A \in \mathcal{M}_n(\mathbb{C})$ and $f \in \mathbb{C}[\mathcal{M}_n(\mathbb{C})]$,

$$(P \cdot f)(A) = f(PAP^{-1}).$$

Let $\mathbb{C}[\mathcal{M}_n(\mathbb{C})]^{GL_n(\mathbb{C})}$ and $\mathbb{C}[\mathscr{T}_n]^{\mathfrak{S}_n}$ denote the set of invariants.

1. Prove that both $\mathbb{C}[\mathcal{M}_n(\mathbb{C})]^{\mathrm{GL}_n(\mathbb{C})}$ and $\mathbb{C}[\mathscr{T}_n]^{\mathfrak{S}_n}$ are rings.
2. Prove that the ring $\mathbb{C}[\mathscr{T}_n]^{\mathfrak{S}_n}$ is isomorphic to Λ_n, the ring of symmetric polynomials.
3. Notice that $\mathscr{T}_n \subset \mathcal{M}_n(\mathbb{C})$. Prove that the restriction of a function induces a well-defined ring homomorphism
$$\psi : \mathbb{C}[\mathcal{M}_n(\mathbb{C})]^{\mathrm{GL}_n(\mathbb{C})} \to \mathbb{C}[\mathscr{T}_n]^{\mathfrak{S}_n}.$$
4. Prove that ψ is injective. (Hint: $\mathscr{D}_n$ is dense in $\mathcal{M}_n(\mathbb{C})$.)
5. Admitting that the elementary symmetric functions generate the ring Λ_n, prove that ψ is an isomorphism.