General Linear Groups, WS 18/19

Exercise Sheet 4

Exercise 1. Complete the proof of Theorem 1.68 in the lecture.

Exercise 2. (Linear algebra is a game of restating a notion in millions of ways)

Let $N \in \mathcal{M}_n(\mathbb{C})$. Prove that the following statements are equivalent:

1. there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that $PNP^{-1} = J_{(n)}$ (Jordan block);
2. the matrix N satisfies $N^n = 0$ but $N^{n-1} \neq 0$;
3. $\mathrm{rank}(N) = n - 1$ and $\chi_N(t) = t^n$;
4. N is nilpotent and $\chi_N(t) = \mu_N(t)$;
5. N is nilpotent and $\dim \ker(N^2) = 2$;
6. N is not invertible, if $NA = AN$ for $A \in \mathcal{M}_n(\mathbb{C})$, then A is invertible or nilpotent.

If this is the case, we call N regular nilpotent (see the definition of regular nilpotent orbit in the lecture).

Exercise 3. Prove that: if both V and W are finite dimensional, then the canonical ring homomorphism

$$\phi : \mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W)$$

is an isomorphism.

Exercise 4.

1. Prove that for three vector spaces V_1, V_2 and V_3, the tensor product satisfies the following associativity: there exists an isomorphism of vector spaces
$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3), \quad (v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3).$$
2. Prove that for two vector spaces V_1 and V_2, $V_1 \otimes V_2 \to V_2 \otimes V_1$, $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ is a vector space isomorphism.
3. (Optional) Prove the following isomorphism cher à Henri Cartan: for three $\mathbb{K}$-vector spaces U, V and W,
$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(V, \mathrm{Hom}(U, W)).$$

Exercise 5. (From complex to real)

Part I

The goal of this part is to show the following statement: for $A \in \mathcal{M}_n(\mathbb{R})$, under the conjugation action,

$$\mathrm{GL}_n(\mathbb{R}) \cdot A = (\mathrm{GL}_n(\mathbb{C}) \cdot A) \cap \mathcal{M}_n(\mathbb{R}).$$

1. Let $P = Q + iR \in \mathrm{GL}_n(\mathbb{C})$ where $Q, R \in \mathcal{M}_n(\mathbb{R})$ and $i^2 = -1$. Show that the function $D : \mathbb{C} \to \mathbb{C}$, $z \mapsto \det(Q + zR)$, is a non-zero polynomial. Deduce that there exists $t \in \mathbb{R}$ such that $Q + tR$ is invertible.

2. For $A, B \in \mathcal{M}_n(\mathbb{R})$, show that the following statements are equivalent:

   - There exists $P \in \mathrm{GL}_n(\mathbb{R})$ such that $B = PAP^{-1}$.
   - There exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that $B = PAP^{-1}$.

3. Conclude.

Part II

Let $\mathscr{D}_n(\mathbb{R})$ denote the set of diagonalisable matrices in $\mathcal{M}_n(\mathbb{R})$. We study the closure $\overline{\mathscr{D}_n(\mathbb{R})} \subseteq \mathcal{M}_n(\mathbb{R})$.

1. Give a matrix $A \in \mathcal{M}_2(\mathbb{R})$ which is not in $\overline{\mathscr{D}_2(\mathbb{R})}$.

2. Let $(A_k)_{k\geq 1} \subset \mathcal{M}_n(\mathbb{R})$ be a sequence of diagonalisable matrices over $\mathbb{R}$ converging to A. Show that the characteristic polynomial of A factorises into a product of linear polynomials over $\mathbb{R}$.

3. A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called triangularisable, if there exists $P \in \mathrm{GL}_n(\mathbb{R})$ such that PAP^{-1} is an upper triangular matrix. Show that a triangularisable matrix is a limit of diagonalisable matrices in $\mathcal{M}_n(\mathbb{R})$.

4. Show that $\overline{\mathscr{D}_n(\mathbb{R})}$ coincides with the triangularisable matrices in $\mathcal{M}_n(\mathbb{R})$.