General Linear Groups, WS 18/19

Exercise Sheet 5

Exercise 1. Let $V = \mathbb{C}^3$ be the standard representation of $\mathfrak{S}_3$.

1. Decompose $V \otimes V \otimes V$ into a direct sum of irreducible representations of $\mathfrak{S}_3$.
2. (Optional) Find such a decomposition for $V^{\otimes n}$.

Exercise 2. Let G be a finite group. We define the right regular representation of G, $\rho^{ger} : G \to \mathrm{GL}(\mathbb{C}(G))$, in the following way: for $h \in G$,

$$\rho^{ger}(h)\left(\sum_{g\in G} \lambda_g g\right) = \sum_{g\in G} \lambda_g g h^{-1}.$$

1. Show that this defines a representation of G.
2. Show that the left regular representation and the right regular representation of G are isomorphic as G-representations.

Exercise 3. Let G be a finite group, $\mathrm{Irr}_{\mathbb{C}}(G) = \{S_1, \cdots, S_t\}$ and for $i = 1, \cdots, t$, $\chi_i := \chi_{S_i}$. Prove the following orthogonality relations:

1. For $g \in G$,
$$\sum_{i=1}^{t} \chi_i(g)\overline{\chi_i(g)} = \frac{\#G}{\#\mathcal{C}},$$
where $\mathcal{C}$ is the conjugacy class containing g.
2. Let $g, h \in G$ be in different conjugacy classes. Then
$$\sum_{i=1}^{t} \chi_i(g)\overline{\chi_i(h)} = 0.$$

As a consequence, you have proved that the columns in the character table are orthogonal.

Exercise 4. Let G be a finite group. Recall that for a representation V of G and a function $\alpha : G \to \mathbb{C}$, we have defined

$$\psi_{\alpha,V} := \sum_{g\in G} \alpha(g)\rho_V(g) \in \mathrm{Hom}(V, V).$$

Show that if $\psi_{\alpha,V} \in \mathrm{Hom}_G(V, V)$ then $\alpha \in \mathrm{CF}(G)$.

Exercise 5. (Fourier transformation in representation theory) Let G be a finite group with group ring $\mathbb{C}(G)$. Let $\mathcal{F}(G)$ denote the $\mathbb{C}$-vector space of functions on the set G. We denote $\mathrm{Irr}_{\mathbb{C}}(G) = \{S_1, \cdots, S_t\}$.

1. Show that $\mathcal{F}(G)$ is a ring via the following definition of the multiplication $*$ (called the convolution product): for $\phi, \psi \in \mathcal{F}(G)$,

$$(\phi * \psi)(h) := \sum_{g \in G} \phi(g)\psi(g^{-1}h).$$

2. Show that

$$\xi : \mathcal{F}(G) \to \mathbb{C}(G), \quad \phi \mapsto \sum_{g \in G} \phi(g) g$$

is a ring homomorphism.

3. Let $(V, \rho_V) \in \mathrm{rep}_{\mathbb{C}}(G)$ and $\phi \in \mathcal{F}(G)$. We define the Fourier transformation $\widehat{\phi}(\rho_V) \in \mathrm{Hom}(V, V)$ in the following way:

$$\widehat{\phi}(\rho_V) := \sum_{g \in G} \phi(g)\rho_V(g).$$

Show that for $\phi, \psi \in \mathcal{F}(G)$, $\widehat{\phi * \psi}(\rho_V) = \widehat{\phi}(\rho_V) \circ \widehat{\psi}(\rho_V)$.

4. Prove the Fourier inversion formula

$$\phi(g) = \frac{1}{\#G} \sum_{i=1}^{t} \dim(S_i) \mathrm{Tr}(\rho_{S_i}(g^{-1}) \circ \widehat{\phi}(\rho_{S_i})).$$

5. Prove the following Plancherel formula: for $\phi, \psi \in \mathcal{F}(G)$,

$$\sum_{g \in G} \phi(g^{-1})\psi(g) = \frac{1}{\#G} \sum_{i=1}^{t} \dim(S_i) \mathrm{Tr}(\widehat{\phi}(\rho_{S_i}) \circ \widehat{\psi}(\rho_{S_i})).$$