General Linear Groups, WS 18/19

Exercise Sheet 6

Exercise 1. Let $\lambda \in \mathcal{P}(n)$, $\lambda = (k^{n_k}, (k-1)^{n_{k-1}}, \cdots, 1^{n_1})$, where r^{n_r} stands for there are n_r copies of r. (For example, $(3,3,2,1,1,1,1,1) = (3^2, 2^1, 1^5)$.) Let $\mathcal{C}_\lambda$ be the conjugacy class in $\mathfrak{S}_n$ associated to λ. Show that

$$\#\mathcal{C}_\lambda = \frac{n!}{k^{n_k} \cdots 2^{n_2} 1^{n_1} n_k! \cdots n_1!}.$$

Exercise 2. We study representations of $\mathfrak{S}_5$.

1. Find $\#\mathrm{Irr}_{\mathbb{C}}(\mathfrak{S}_5)$.
2. Find the dimensions of finite dimensional irreducible representations of $\mathfrak{S}_5$.
3. Find the character of the 6-dimensional irreducible representation of $\mathfrak{S}_5$.

Exercise 3. Let $V \in \mathrm{Irr}_{\mathbb{C}}(\mathfrak{S}_n)$. Show that $V^* \cong V$ as representations of $\mathfrak{S}_n$.

Exercise 4. Let G and H be two finite groups and $\varphi : G \to H$ be a surjective group homomorphism.

1. Show that φ induces a map $\varphi^* : \mathrm{rep}_{\mathbb{C}}(H) \to \mathrm{rep}_{\mathbb{C}}(G)$, $\rho_V \mapsto \rho_V \circ \varphi$.
2. Show that φ^* induces a map $\varphi^* : \mathrm{Irr}_{\mathbb{C}}(H) \to \mathrm{Irr}_{\mathbb{C}}(G)$.
3. Show that φ^* is an injective map.

Exercise 5.

Part I

Let G be a finite group. Recall that a representation (ρ_V, V) of G is called faithful, if ρ_V is injective. The goal of this exercise is to prove the following statement: any finite dimensional irreducible representation of G is a subrepresentation in some tensor power of V.

1. Let $V, W \in \mathrm{rep}_{\mathbb{C}}(G)$. Show the following identity

$$(*) \qquad \sum_{n=0}^{+\infty} \langle \chi_V^n, \chi_W \rangle t^n = \frac{1}{\#G} \sum_{C \in \mathcal{C}(G)} \#C \frac{\overline{\chi_W(C)}}{1 - t\chi_V(C)}.$$

2. Show that the right hand side of $(*)$ is non-zero. (Hint: look at $e \in G$.)
3. Assume that V is faithful and $W \in \mathrm{Irr}_{\mathbb{C}}(G)$. Show that if W is not a subrepresentation of any $V^{\otimes k}$ ($k \geq 1$), then the left hand side of $(*)$ is zero.
4. Conclude.

Part II

Let G be a finite group and $V \in \mathrm{rep}_{\mathbb{C}}(G)$ be a faithful representation of G. For $W \in \mathrm{Irr}_{\mathbb{C}}(G)$, we let r_W^V denote the smallest natural number k such that W is a subrepresentation of $V^{\otimes k}$. According to Part I, this number is well-defined.

1. Show that for any finite group G, its left regular representation $\mathbb{C}(G)$ is faithful. For any $W \in \mathrm{Irr}_{\mathbb{C}}(G)$, find $r_W^{\mathbb{C}(G)}$.

2. For $G = \mathfrak{S}_3$ and $\mathfrak{S}_4$, find a finite dimensional faithful irreducible representation V of G, and compute r_W^V for $W \in \mathrm{Irr}_{\mathbb{C}}(G)$.