Degenerations of Flag varieties

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Oberwolfach.

What is a

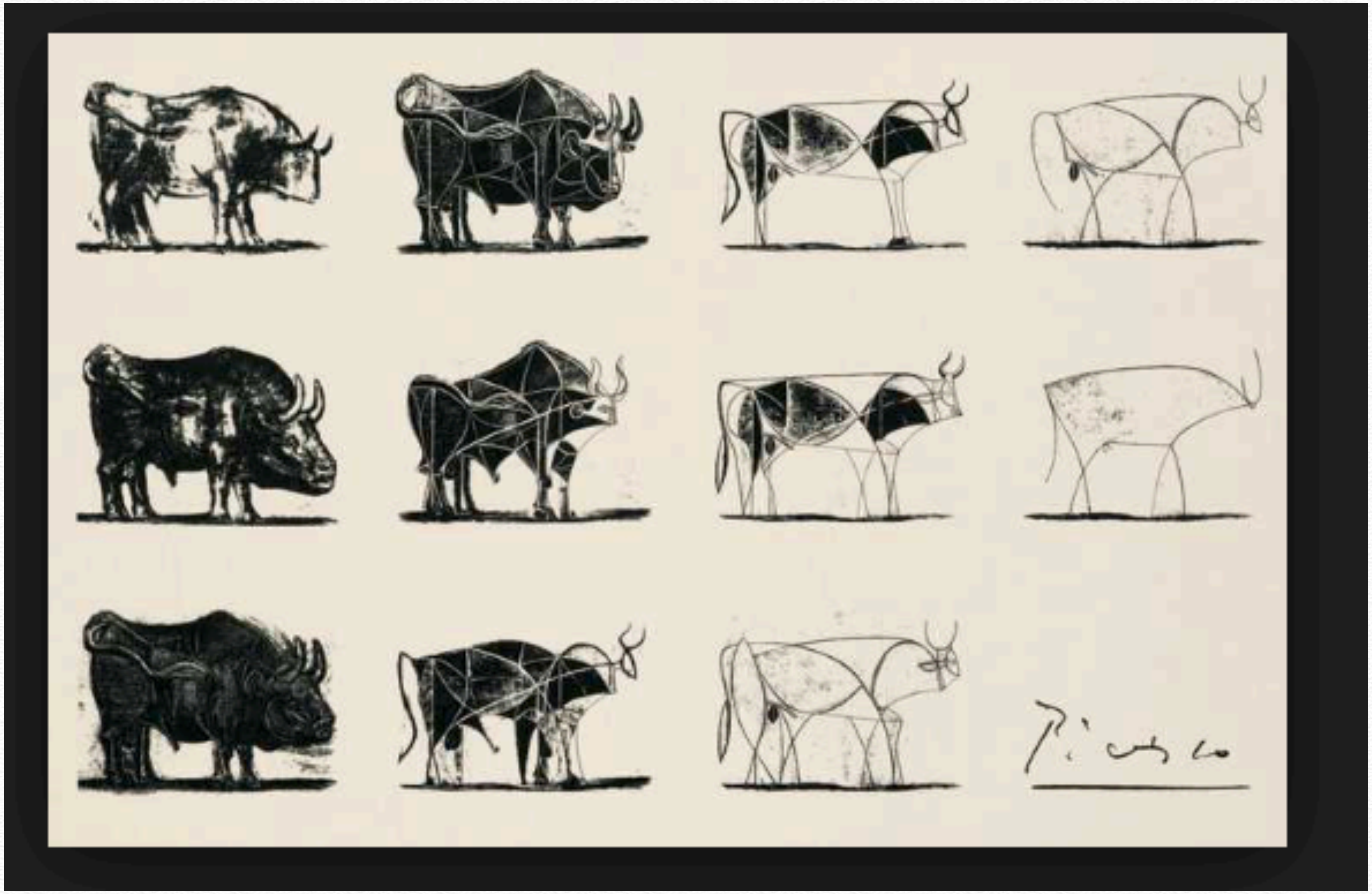
DEGENERATION ?

↓

DGNERATN

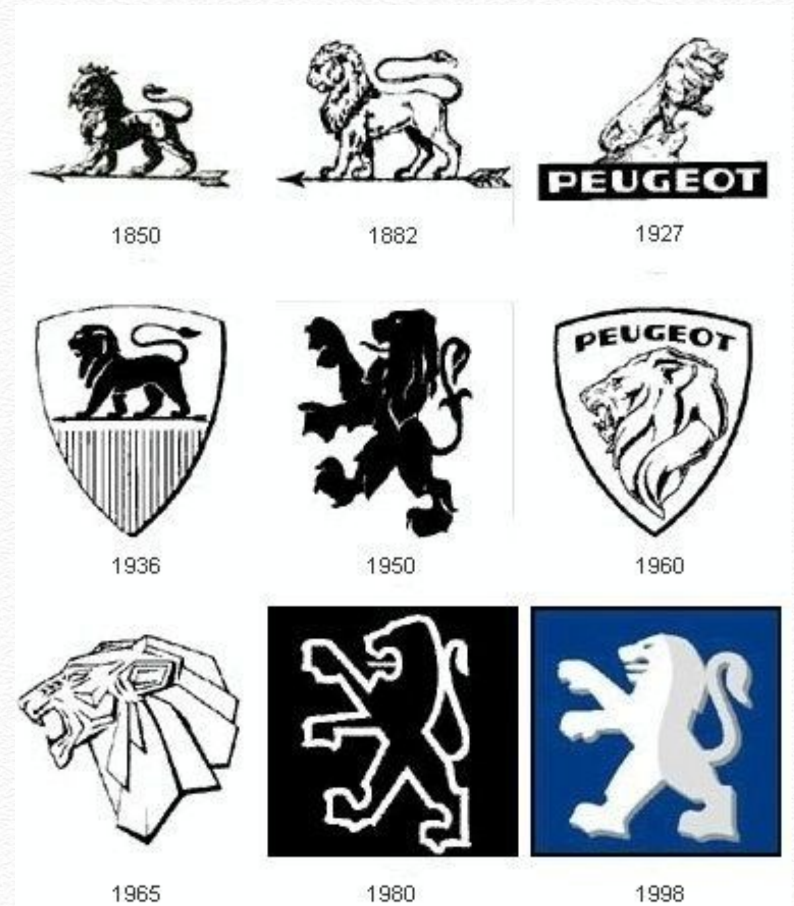
Ex: TE QCK BRON FOX JMP OVR TE LZY DOG.

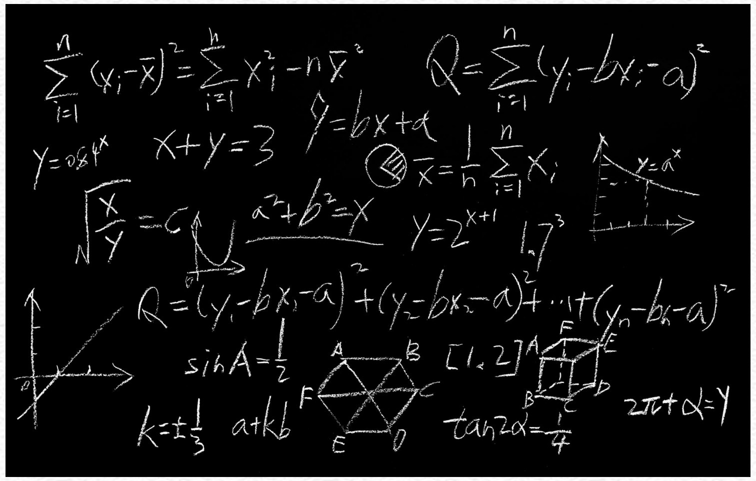
Degenerations in ART*



*. ALGEBRA & REPRESENTATION THEORY

Degeneration is TREND !





How about

Degenerations in MATHs?

Let's say projective varieties.

Are we DONE?

Every Projective Variety $\xrightarrow{\text{T. Hibi}}$ can be DEGENERATE!

Every Projective Variety $\xrightarrow{\text{M. Reineke}}$ is a QUIVER GRASSMANNIAN!

<u>RULE</u>:

Need to restrict to some particular projective varieties.

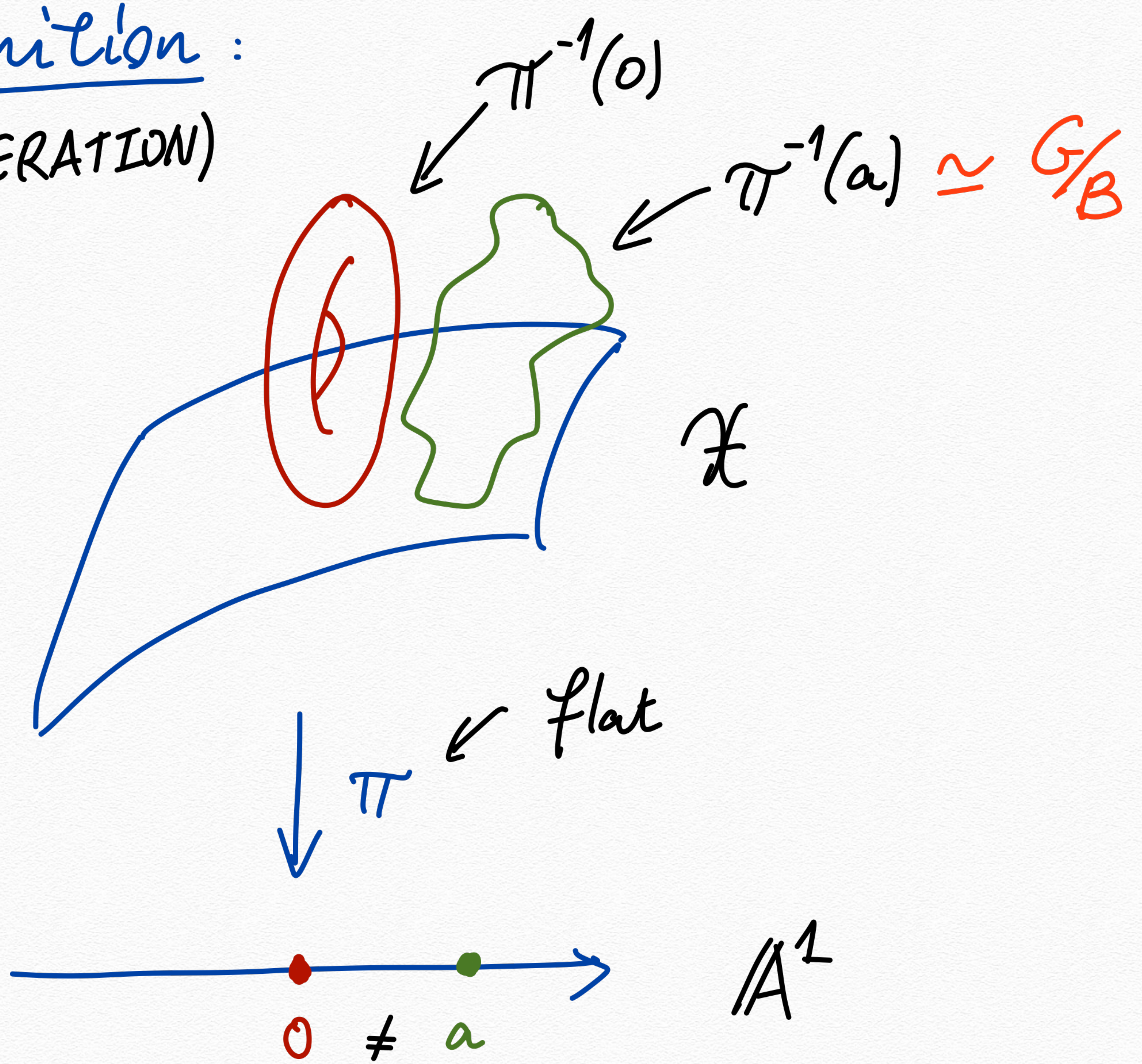
In this talk:

$$G/B \ (G/P) \hookrightarrow \mathbb{P}(V_\lambda)$$

Borel Parabolic regular

Definition:

(DEGENERATION)



$$\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \mathbb{A}^1 \end{array}$$

▷ If $\pi^{-1}(0)$ is toric

$\rightsquigarrow$ toric degeneration

▷ If $\pi^{-1}(0)$ a (reduced) union of torics

$\rightsquigarrow$ semi-toric degeneration

<u>DISCLAIMER</u>: We are <u>not</u> symplectic geometers.

Relation to basis problem:

Degeneration problem is closely related to the basis problem of V_λ^*.

Homogeneous coordinate ring

$$\mathbb{C}[G/B] \simeq \bigoplus_{n \in \mathbb{N}} V_{n\lambda}^*$$

algebra structure = basis + multiplication.

$\mathfrak{g} = \mathrm{Lie}(G)$ simple, of finite type

(A). "Canonical type" basis.

(dual) canonical, global crystal, semicanonical, KLR, ...

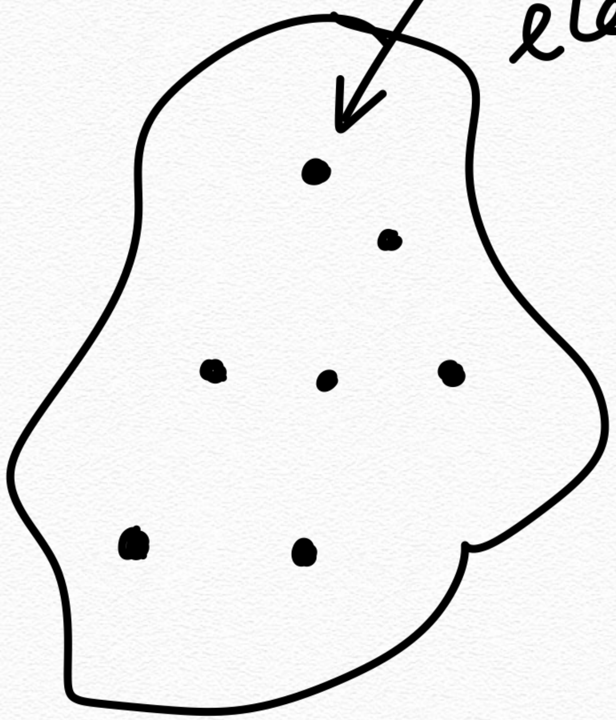
(B). Standard monomial basis

(C). F-F-L-V basis (A, C, G)

(D). Theta basis (A, D, E)

Basis:

basis elements



Vector space

V

this is an eye

observe!

Parameterisation of basis

e.g.

$e_1, e_2, e_3 \ldots$

Parameterisations:

(A).

Lattice points in

fix $\underline{w_0}$!

canonical type basis V_λ $\longrightarrow$ Lusztig polytopes $\mathcal{L}_{\underline{w_0}}(\lambda)$

(Lusztig)

$\longrightarrow$ String polytopes $Q_{\underline{w_0}}(\lambda)$

(Littelmann, Berenstein-Zelevinski)

$\longrightarrow$ Nakashima-Zelevinski polytopes $\mathcal{NZ}_{\underline{w_0}}(\lambda)$

(Kashiwara, Nakashima-Zelevinski)

Toric degeneration of $G/B \hookrightarrow \mathbb{P}(V_\lambda)$ to the toric variety associated to

- $Q_{\underline{w}_0}(\lambda)$ Caldero, Kaveh.
- $NZ_{\underline{w}_0}(\lambda)$ Fujita-Naito
- $\mathcal{L}_{\underline{w}_0}(\lambda)$ Fourier-Littelmann-F.

<u>Framework</u>: Newton-Okounkov bodies

(B). L-S: Lakshmibai-Seshadri

Standard monomial basis $\nearrow$ Lattice points in some "generalized polytopes"

$\updownarrow$

L-S paths

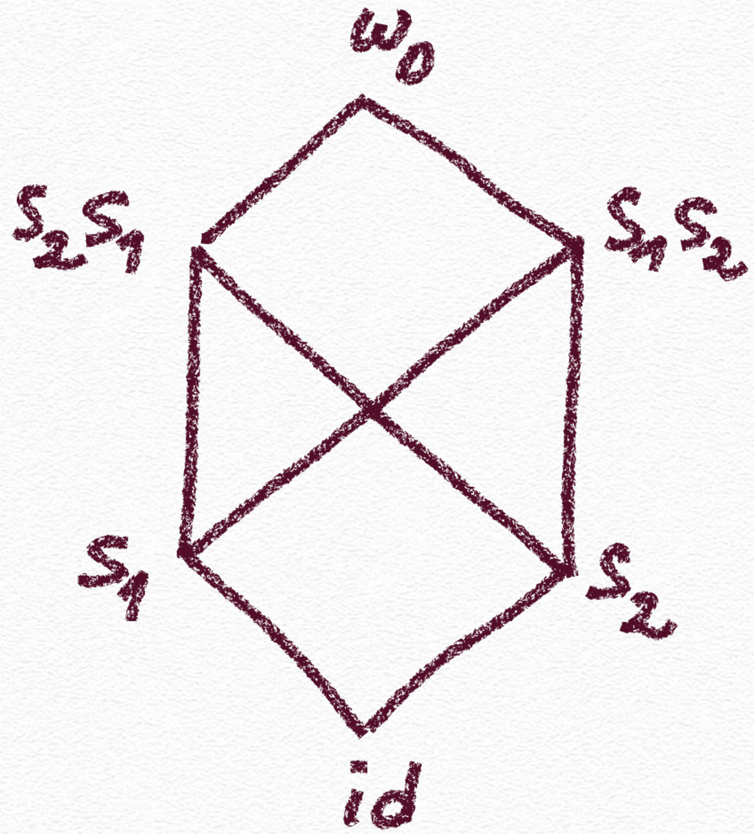
(Lakshmibai-Seshadri, Littelmann)

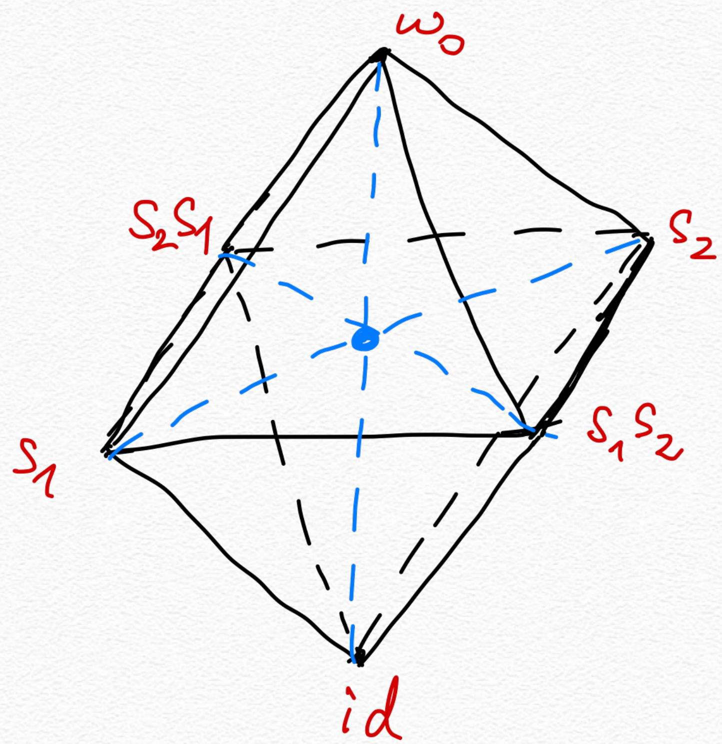
"Newton-Okounkov"

Semi-toric degeneration of $G/B \hookrightarrow \mathbb{P}(V_\lambda)$

into "semi-toric variety associated to the order complex of the Bruhat poset".

{ Gr(k,n) case (Littelmann-F)
{ Schuberts in Kac-Moody (Chirivì-Littelmann-F).





(c) type A, C

FFLV basis $\longrightarrow$ FFLV polytopes $FFLV(\lambda)$

(very easily-defined polytopes)

Toric degeneration of $G/B \hookrightarrow \mathbb{P}(V_\lambda)$ to the toric variety associated to $FFLV(\lambda)$.

- Feigin-Fourier-Littelmann.
- (type A) Kiritchenko

(d) Theta basis:

Hope to say something in future.

Goal of this talk:

Explain recent results around

FFLV basis & Degenerations

Q:

Why I choose this topic?

10 years of FFLV basis

arXiv: 1002.0674
3, Feb, 2010