General Linear Groups, WS 18/19

Partial solution

Solution to Sheet 1, Exercise 4.

The part (1) is follows from the definition. We start with the part (3).

Recall the standard flag $V_\bullet^{std} := (V_1^{std}, \cdots, V_{n-1}^{std})$ where $V_i^{std} = \mathrm{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_i\}$. Assume that $V_\bullet = (V_1, \cdots, V_{n-1})$ is a flag such that for any $b \in B_n(\mathbb{C})$, $b \cdot V_\bullet = V_\bullet$, we show that $V_\bullet = V_\bullet^{std}$. This implies that the flag having $B_n(\mathbb{C})$ as stabilizer is exactly $V_\bullet^{std}$.

We start by showing that $V_1 = \mathrm{span}\{\mathbf{e}_1\}$. Assume that $V_1 = \mathrm{span}\{\mathbf{v}_1\}$, $\mathbf{v}_1 = \lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n$. Let $b = (b_{i,j}) \in B_n(\mathbb{C})$ be given by $b_{i,j} = 1$ if and only if $i \leq j$. Then

$$b\mathbf{v}_1 = (\lambda_1 + \cdots + \lambda_n)\mathbf{e}_1 + (\lambda_2 + \cdots + \lambda_n)\mathbf{e}_2 + \cdots + \lambda_n\mathbf{e}_n.$$

Since $b\mathbf{v}_1 \in V_1$, $\lambda_2 = \cdots = \lambda_n = 0$. This implies $V_1 = V_1^{std}$.

Assume for $\ell = 1, \cdots, k-1$, $V_\ell = V_\ell^{std}$. We show that $V_k = V_k^{std}$. Since $V_{k-1}^{std} = V_{k-1} \subset V_k$, we can assume that $V_k = \mathrm{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_{k-1}, \mathbf{v}_k\}$ where $\mathbf{v}_k = \lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n$. Similarly,

$$b\mathbf{v}_k = (\lambda_1 + \cdots + \lambda_n)\mathbf{e}_1 + \cdots + (\lambda_k + \cdots + \lambda_n)\mathbf{e}_k + (\lambda_{k+1} + \cdots + \lambda_n)\mathbf{e}_{k+1} + \cdots + \lambda_n\mathbf{e}_n \in V_k.$$

Since $b\mathbf{v}_k \in V_k$, we ca assume that

$$b\mathbf{v}_k = \mu_1\mathbf{e}_1 + \cdots + \mu_{k-1}\mathbf{e}_{k-1}\mu_k\mathbf{v}_k = (\mu_1 + \mu_k\lambda_1)\mathbf{e}_1 + \cdots + (\mu_{k-1} + \mu_k\lambda_{k-1})\mathbf{e}_{k-1} + \mu_k\lambda_k\mathbf{e}_k + \cdots + \mu_k\lambda_n\mathbf{e}_n.$$

If $\lambda_n \neq 0$ then $\mu_k = 1$. Consider the coefficient of $\mathbf{e}_{n-1}$ gives a contradiction, which implies $\lambda_n = 0$. Continue this argument shows that $\lambda_{k+1} = \cdots = \lambda_n = 0$, hence $\mathbf{v}_k \in V_k^{std}$ and $\mathbf{e}_k \in V_k$. This terminates the proof.

By (1) we know that $B_n(\mathbb{C}) \subseteq \mathcal{N}_{\mathrm{GL}_n(\mathbb{C})}(B_n(\mathbb{C}))$. It suffices to show that if $gB_n(\mathbb{C})g^{-1} = B_n(\mathbb{C})$ then $g \in B_n(\mathbb{C})$.

In the lecture we have shown that $\mathrm{Stab}_{\mathrm{GL}_n(\mathbb{C})}(V_\bullet^{std}) = B_n(\mathbb{C})$, by Exercise 1.11, $\mathrm{Stab}(gV_\bullet^{std}) = gB_n(\mathbb{C})g^{-1} = B_n(\mathbb{C})$. From (3) this implies $gV_\bullet^{std} = V_\bullet^{std}$, and $g \in B_n(\mathbb{C})$ (see the discussion after Proposition 1.21 in the lecture).

Solution to Sheet 2, Exercise 4 (1).

We show that if for any $k > 0$, $\mathrm{Tr}(A^k) = 0$ then $A \in \mathcal{M}_n(\mathbb{C})$ is nilpotent.

Assume that $\lambda_1, \cdots, \lambda_r$ are all distinct non-zero eigenvalues of A with multiplicities $m_1, \cdots, m_r$. The assumption $\mathrm{Tr}(A^k) = 0$ implies that

$$m_1\lambda_1 + m_2\lambda_2 + \cdots + m_r\lambda_r = 0,$$
$$\cdots$$
$$m_1\lambda_1^r + m_2\lambda_2^r + \cdots + m_r\lambda_r^r = 0.$$

It means that $(m_1, \cdots, m_r)$ is a non-zero solution of the linear system

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_r x_r = 0,$$

$$\cdots$$

$$\lambda_1^r x_1 + \lambda_2^r x_2 + \cdots + \lambda_r^r x_r = 0.$$

By Cramer's rule, it implies that the matrix $\begin{bmatrix} \lambda_1 & \cdots & \lambda_r \\ \vdots & & \vdots \\ \lambda_1^r & \cdots & \lambda_r^r \end{bmatrix}$ has zero determinant, but as a van der Monde determinant, it has determinant

$$\lambda_1 \cdots \lambda_r \prod_{1 \le i < j \le r} (\lambda_i - \lambda_j) \neq 0.$$

This contradiction implies that there is no non-zero eigenvalue of A, and hence A is nilpotent.