Convex polytopes in algebraic combinatorics, WS 17/18
Exercise Sheet 3

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to be handed in on 20, November, 2017 in the lecture.

Exercise 1. Let $P \subset \mathbb{R}^N$ be a polytope and $\mathbf{x} \in \mathbb{R}^N$ such that $\mathbf{x} \notin P$. Show that there exists $\alpha \in (\mathbb{R}^N)^*$ and $\lambda \in \mathbb{R}$ such that $\alpha(\mathbf{y}) < \lambda$ for any $\mathbf{y} \in P$ but $\alpha(\mathbf{x}) > \lambda$. If $0 \in P$ then we can choose $\lambda = 1$.

Exercise 2. Compute the volume of the crosspolytope C_N^Δ.

Exercise 3.

1. Let $\mathbf{v}_1, \cdots, \mathbf{v}_n \in \mathbb{R}^N$. Show that the dimension of the polytope $P = \mathrm{conv}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is the rank of the following matrix A minus 1:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}.$$

2. Deduce that the permutahedron Π_N has dimension N.

Exercise 4. Compute and draw K° for axis parallel rectangles K in $\mathbb{R}^2$ with opposite vertices $\mathbf{v}_1, \mathbf{v}_2$ given by:

1. $\mathbf{v}_1 = (0,0)$ and $\mathbf{v}_2 = (M,1)$ for $M > 0$ large;
2. $\mathbf{v}_1 = (-\varepsilon, -\varepsilon)$ and $\mathbf{v}_2 = (M,1)$ for $M > 0$ large and $\varepsilon > 0$ small;
3. $\mathbf{v}_1 = (\varepsilon, \varepsilon)$ and $\mathbf{v}_2 = (M,1)$ for $M > 0$ large and $\varepsilon > 0$ small.

Study the limit shape when $\varepsilon \to 0$, $M \to +\infty$.

Exercise 5. Let $p(z) \in \mathbb{C}[z]$ be a polynomial with complex coefficients in one variable with roots $r_1, \cdots, r_d$, then the roots of the derivative $p'(z)$ of $p(z)$ are contained in the convex hull $\mathrm{conv}\{r_1, \cdots, r_d\}$.