Convex polytopes in algebraic combinatorics, WS 17/18
Exercise Sheet 4

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to be handed in on 4, December, 2017 in the lecture.

Exercise 1. Let $P \subset \mathbb{R}^N$ be a full dimensional polytope. Show that $\text{int}(P) \neq \emptyset$.

Exercise 2. Let $P \subset \mathbb{R}^N$ and $Q \subset \mathbb{R}^M$ be two polytopes.

1. Show that the non-empty faces of $P \times Q$ are exactly the products of the faces of P with the faces of Q.
2. Deduce the f-vector and f-polynomial of $P \times Q$ in terms of the f-vectors and f-polynomials of P and Q.

Exercise 3.

1. Let P be a polytope of dimension $d-1$ and $\text{bipyr}(P)$ is the bipyramid over P. Prove that:
   - for $0 \leq k \leq d-2$, $f_k(\text{bipyr}(P)) = 2f_{k-1}(P) + f_k(P)$;
   - $f_{d-1}(\text{bipyr}(P)) = 2f_{d-2}(P)$.
2. Deduce the f-polynomial of the crosspolytope C_N^{Δ}.

Exercise 4. For σ, $\tau \in \mathfrak{S}_N$, show that the segment $\mathcal{S}_{X^\sigma, X^\tau}$ (X^σ is the permutation matrix associated to σ) connecting X^σ and X^τ is an edge of the Birkhoff polytope B_N if and only if $\sigma^{-1}\tau$ is a cycle.

Exercise 5. Assume for the moment that $f(P) = (15, 34, 28, 9)$ is the f-vector of a polytope P. Answer the following questions:

1. What is the dimension of P?
2. (optional) Is P simple or simplicial?
3. Could P be a prism over another polytope?
4. Could P be a pyramid over another polytope?

(optional) Study the assumption: Is $f(P)$ really an f-vector of a polytope P?